Bachelor in Statistics and Business
Universidad Carlos III de Madrid

Mathematical Methods II
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Homework sheet 6: THE SINGULAR VALUE DECOMPOSITION

Year 2011-2012

Solutions

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1. Determine which of the matrices are symmetric:

$$\begin{pmatrix} 3 & 5 \\ 5 & -7 \end{pmatrix}, \quad \begin{pmatrix} -3 & 5 \\ -5 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 8 & 3 \\ 8 & 0 & -2 \\ 3 & -2 & 0 \end{pmatrix}, \quad \begin{pmatrix} -6 & 2 & 0 \\ 0 & -6 & 2 \\ 0 & 0 & -6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}.$$

Solution: Symmetric, nonsymmetric, symmetric, nonsymmetric, nonsymmetric (respectively).

2. Determine which of the matrices are orthogonal. If they are orthogonal, find the inverse.

$$P_1 = \begin{pmatrix} 0{,}6 & 0{,}8 \\ 0{,}8 & -0{,}6 \end{pmatrix}, \quad \begin{pmatrix} -5 & 2 \\ 2 & 5 \end{pmatrix}, \quad P_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

Solution: Orthogonal, no ortoghonal, ortoghonal, no ortoghonal (respectively).

$$P_1^{-1} = P_1^T = \begin{pmatrix} 0{,}6 & 0{,}8 \\ 0{,}8 & -0{,}6 \end{pmatrix}, \quad P_2^{-1} = P_2^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

3. Orthogonally diagonalize the following matrices, giving P and D. For some matrices the eigenvalues are given to make computations easier.

Solution:

(a) $\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$ $\quad D = \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix}$ $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

(b) $\begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}$ $\quad D = \begin{pmatrix} 25 & 0 \\ 0 & -25 \end{pmatrix}$ $P = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ $\quad D = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ $P = \begin{pmatrix} -1/\sqrt{5} & 4/(3\sqrt{5}) & -2/3 \\ 2/\sqrt{5} & 2/(3\sqrt{5}) & -1/3 \\ 0 & 5/(3\sqrt{5}) & 2/3 \end{pmatrix}$

Eigenvalues : $7, -2$.

(d) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ $\quad D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 0 & 1 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$

Eigenvalues : $2, 0$.

4. Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Verify that 2 is an eigenvalue of A and v is an eigenvector. Then orthogonally diagonalize the matrix A.

   Solution (without computing the characteristic polynomial): v is an eigenvector of eigenvalue 5 and 2 is an eigenvalue of multiplicity 2.

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad P = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}.$$

5. Let $A = \begin{pmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{pmatrix}$, $v_1 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Verify that v_1 and v_2 are eigenvectors of A. Then orthogonally diagonalize the matrix A.

   Solution: v_1 is an eigenvector of A of eigenvalue 10 and v_2 is an eigenvector of A with eigenvalue 1. The eigenvalue 10 has geometric multiplicity equal to 1 and the eigenvalue 1 has geometric multiplicity 2.

$$D = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} -2/3 & 1/\sqrt{2} & 1/(3\sqrt{2}) \\ 2/3 & 1/\sqrt{2} & -1/(3\sqrt{2}) \\ 1/3 & 0 & 4/(3\sqrt{2}) \end{pmatrix}.$$

6. Find the singular values of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}, \quad \begin{pmatrix} -5 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \sqrt{6} & 1 \\ 0 & \sqrt{6} \end{pmatrix}, \quad \begin{pmatrix} \sqrt{3} & 2 \\ 0 & \sqrt{3} \end{pmatrix}.$$

   Solution: $3, 1$; $\quad 5$; $\quad 3, 2$; $\quad 3, 1$.

7. Let $A = \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix}$. Find a unit vector x at which Ax has maximum length. Find the following matrix norms of A: $\|A\|_1$, $\|A\|_\infty$, $\|A\|_2$.

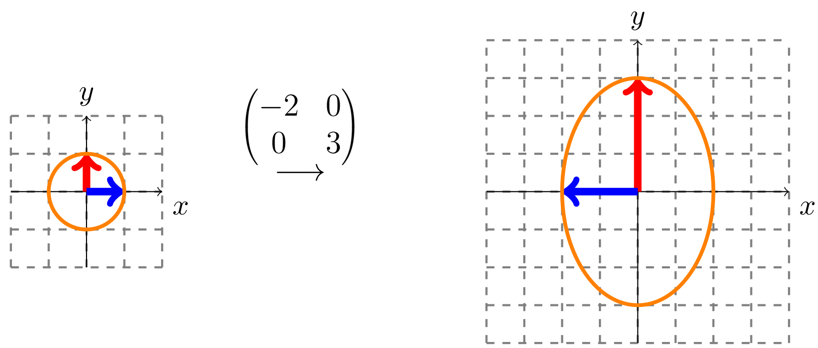
Solution: A unit vector x such that Ax has maximum length is $x = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$.

$$\|A\|_1 = 4, \quad \|A\|_\infty = 4, \quad \|A\|_2 = 3.$$

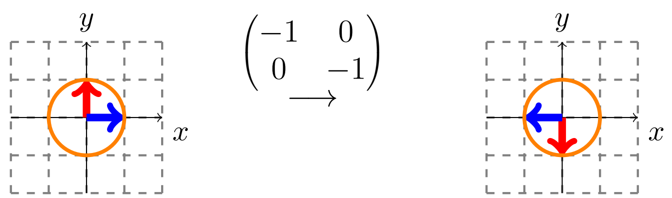
8. Plot the image of the unit sphere under the linear transformations given by each of the following matrices:

$$\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

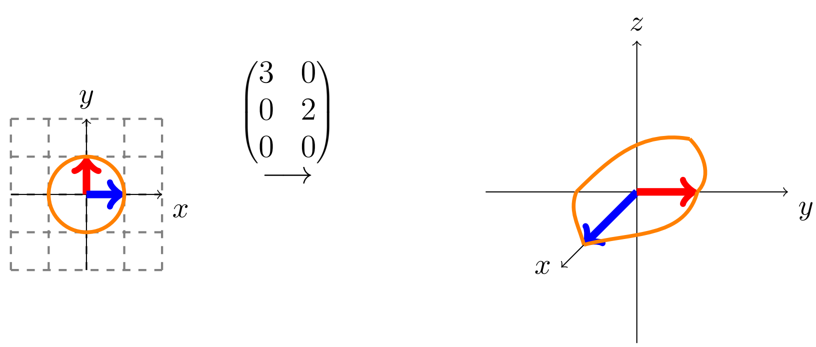
Solution:



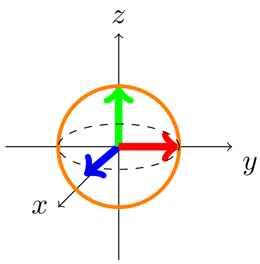
The image of the sphere unit of $\mathbb{R}^2$ under the above matrix is the ellipse centered at $(0,0)$ with semiaxis of length 3 and 2.



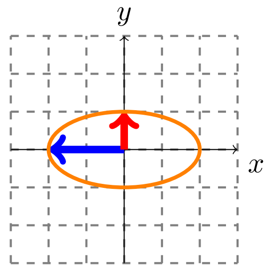
The image of the unit sphere of $\mathbb{R}^2$ under the above matrix is the same sphere unit, but the matrix applies a rotation of 180 degrees to the vectors for the standard basis.



The image of the unit sphere of $\mathbb{R}^2$ under the above matrix is the ellipse contained in the plane XY with center at $(0,0,0)$ and semiaxis of length 3 and 2.



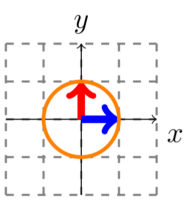
$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow$$



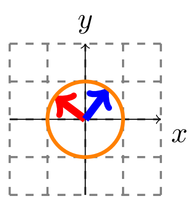
The image of the unit sphere of $\mathbb{R}^3$ under the above matrix is the ellipse in $\mathbb{R}^2$ with center at $(0,0)$ and semiaxis of length 2 and 1.

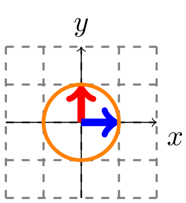
9. Plot the image of the unit sphere under the linear transformations given by each of the following matrices and determine if the matrices are orthogonal:

   Solution: Both matrices are orthogonal. Hence, these matrices rotate the vectors without modifying the norm. Then the image of the sphere unit in $\mathbb{R}^2$ under these transformations is the sphere unit, but the vectors have been rotated. Note the pictures:

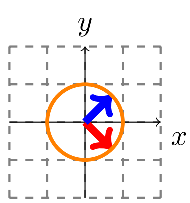


$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \longrightarrow$$





$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \longrightarrow$$



10. Find a singular value decomposition (SVD) of the following matrices:

   Solution: $\mathrm{A} = \mathrm{U}\Sigma\mathrm{V}^{\mathrm{T}}$

   (a) $\begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

11. Find the norm for each of the matrices in the exercise 10 using the norm 1, norm 2 and norm infinity.

    Solution: Euclidean norm: 3, 2, 2, $\sqrt{2}$, $\sqrt{3}$, 2, respectively.

    Norm one: 3, 2, 3, 1, 3, 2, respectively.

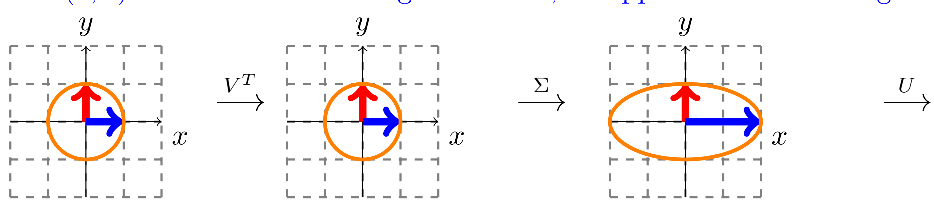
    Norm infinity: 3, 2, 2, 2, 2, 2, respectively.

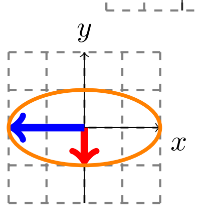
12. Plot the image of the unit sphere under the linear transformations given by the matrices (b), (c), (f) in exercise 10.

    Solution:

    (b) $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = U\Sigma V^T$

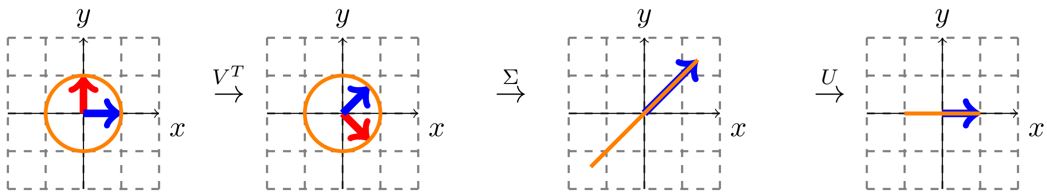
    The image of the sphere unit in $\mathbb{R}^2$ under the matrix in (b) is the ellipse in $\mathbb{R}^2$ centered at $(0,0)$ with semiaxis of length 2 and 1, as appears in the last figure below.





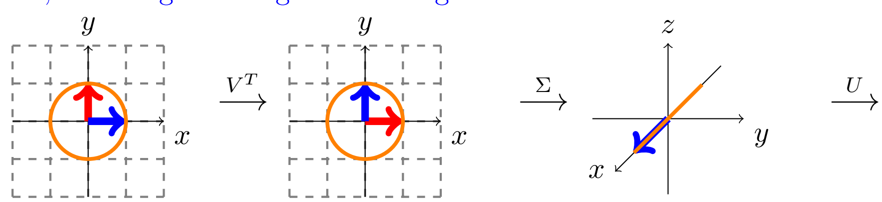
    (c) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = U\Sigma V^T$

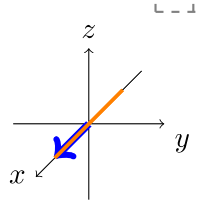
    The image of the sphere unit in $\mathbb{R}^2$ by the matrix in (c) is the degenerate ellipse in $\mathbb{R}^2$ centered at $(0,0)$ with semiaxis of length 1 and 0, as appears in the figure below. In other words, it is a segment of length 2 on the axis OX centered at $(0,0)$.



(f) $\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = U\Sigma V^T.$

The image of the sphere unit in $\mathbb{R}^2$ for the matrix (f) is the ellipsoid degenerate in $\mathbb{R}^3$ centered at $(0,0,0)$ with semiaxis of length 2, 0 and 0, as appears below. In other words, the image is a segment of length 4 on the axis OX and centered at $(0,0,0)$.





13. Find the Moore-Penrose pseudoinverse of $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & -2 \\ 1 & 2 & -1 \end{pmatrix}$.

Solution: A singular value decomposition of A is given by

$$A = U\Sigma V^T = \begin{pmatrix} 1/2 & 1/\sqrt{10} & -4/\sqrt{26} & -3/2\sqrt{65} \\ 1/2 & 2/\sqrt{10} & 3/\sqrt{26} & -1/2\sqrt{65} \\ 1/2 & -2/\sqrt{10} & 1/\sqrt{26} & -9/2\sqrt{65} \\ 1/2 & -1/\sqrt{10} & 0 & 13/2\sqrt{65} \end{pmatrix} \begin{pmatrix} \sqrt{20} & 0 & 0 \\ 0 & \sqrt{10} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 0 & -2\sqrt{5} \\ 2/\sqrt{5} & 0 & 1\sqrt{5} \\ 0 & 1 & 0 \end{pmatrix}.$$

The Moore-Penrose pseudoinverse matrix of A is given by

$$A^\dagger = V\Sigma^\dagger U^T = V \begin{pmatrix} 1/\sqrt{20} & 0 & 0 & 0 \\ 0 & 1/\sqrt{10} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} U^T = \begin{pmatrix} 1/20 & 1/20 & 1/20 & 1/20 \\ 1/10 & 1/10 & 1/10 & 1/10 \\ 1/10 & 2/10 & -2/10 & -1/10 \end{pmatrix}.$$

14. Find the Moore-Penrose pseudoinverse matrix of $A = \begin{pmatrix} -3 & 0 \\ 3 & -3 \\ 0 & 3 \end{pmatrix}$. Use $A^\dagger$ to

solve the least-square problems of $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Solution: A singular value decomposition of A is given by

$$A = U\Sigma V^T = \begin{pmatrix} -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 3\sqrt{3} & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The Moore-Penrose pseudoinverse matrix of A is given by

$$A^\dagger = V\Sigma^\dagger U^T = V \begin{pmatrix} 1/3\sqrt{3} & 0 & 0 \\ 0 & 1/3 & 0 \end{pmatrix} U^T = \begin{pmatrix} -2/9 & 1/9 & 1/9 \\ -1/9 & -1/9 & 2/9 \end{pmatrix}.$$

A least-square solution is obtained from $A^\dagger b = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$.

15. Solve the least-square problem of $Ax = b$ by using the Moore-Pensore pseudoinverse of A, where $A = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$.

Solution: First find a singular value decomposition of A:

$$A = U\Sigma V^T = \begin{pmatrix} 1/3 & -2/\sqrt{5} & 2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} 1.$$

The Moore-Penrose pseudoinverse matrix is given by

$$A^\dagger = V\Sigma^\dagger U^T = V \begin{pmatrix} 1/3 & 0 & 0 \end{pmatrix} U^T = \frac{1}{9} \begin{pmatrix} 1 & 2 & -2 \end{pmatrix}.$$

The least-square solution is given by $A^\dagger b = \dfrac{14}{9}$.

Additional exercises: D. C. Lay "Linear algebra and its applications", 2012.

- **Section 7.1.**
- **Section 7.4.**