Homework sheet 1: SYSTEMS OF LINEAR EQUATIONS
Year 2011-2012

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1. Express in your own words the next elementary row operation to perform in order to continue with the solving process of the following linear systems.

a)
$$\begin{array}{rrrrr} x_1 & +4x_2 & -2x_3 & +8x_4 = & 12 \\ & x_2 & -7x_3 & +2x_4 = & -4 \\ & & 5x_3 & -x_4 = & 7 \\ & & x_3 & +3x_4 = & -5 \end{array}$$

b)
$$\begin{array}{rrrrr} x_1 & -3x_2 & +5x_3 & -2x_4 = & 0 \\ & x_2 & +8x_3 & = & -4 \\ & & 2x_3 & = & 3 \\ & & & x_4 = & 1 \end{array}$$

2. Consider the following matrix to be an augmented matrix of a linear system. Express in your own words the next elementary row operation to perform in the solving process of the system.

$$\begin{bmatrix} 1 & 8 & 2 & -7 \\ 0 & 1 & -1 & 9 \\ 0 & 4 & 5 & 0 \end{bmatrix}$$

3. Determine the consistency of the linear system whose augmented matrix has been transformed into the following matrix by elementary row operations.

$$\begin{bmatrix} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

4. Check if $(3, 4, -2)$ is a solution of the following system.

$$\begin{array}{rrrr} 5x_1 & -x_2 & +2x_3 = & 7 \\ -2x_1 & +6x_2 & +9x_3 = & 0 \\ -7x_1 & +5x_2 & -3x_3 = & -7 \end{array}$$

5. Assume that the augmented matrix of a linear system has been transformed into the following matrix by elementary row operations. Continue the solving process and describe the solution set of the original system.

a) $\begin{bmatrix} 1 & 2 & 0 & -3 & -9 \\ 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ c) $\begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

6. Solve the following linear systems:

$$a)\left\{\begin{array}{rcr} x_1-5x_2+4x_3 & = & -3\\ 2x_1-7x_2+3x_3 & = & -2\\ -2x_1+x_2+7x_3 & = & -1 \end{array}\right. \qquad b)\left\{\begin{array}{rrrr} & x_2 & +4x_3= & -5\\ x_1 & +3x_2 & +5x_3= & -2\\ 3x_1 & +7x_2 & +7x_3= & 6 \end{array}\right.$$

$$c)\left\{\begin{array}{rrrr} x_1 & & -3x_3= & 8\\ 2x_1 & +2x_2 & +9x_3= & 7\\ & x_2 & +5x_3= & -2 \end{array}\right.$$

7. Study if the following systems are consistent:

$$a)\left\{\begin{array}{rrrrcr} x_1 & & -2x_3 & & = & -1\\ & x_2 & & -x_4 & = & 2\\ & -3x_2 & +2x_3 & & = & 0\\ -4x_1 & & & +7x_4 & = & -5 \end{array}\right. \qquad b)\left\{\begin{array}{rrrrcr} x_1 & & +3x_3 & & = & 2\\ & x_2 & & -3x_4 & = & 3\\ & -2x_2 & +3x_3 & +2x_4 & = & 1\\ 3x_1 & & & +7x_4 & = & -5 \end{array}\right.$$

$$c)\left\{\begin{array}{rrrrcr} x_1 & & & -2x_4 & = & -3\\ & 2x_2 & +2x_3 & & = & 0\\ & & x_3 & +3x_4 & = & 1\\ -2x_1 & +3x_2 & +2x_3 & +x_4 & = & 5 \end{array}\right.$$

8. Determine the value(s) of h such that the following augmented matrices describe consistent linear systems.

$$\left[\begin{array}{rrr} 1 & 4 & -2\\ 3 & h & -6 \end{array}\right] \quad \left[\begin{array}{rrr} 2 & -6 & -3\\ -4 & 12 & h \end{array}\right]$$

9. Determine the values of h and k such that the following system is consistent.

$$\begin{array}{rrcl} 2x_1 & -x_2 & = & h\\ -6x_1 & +3x_2 & = & k \end{array}$$

10. Do the three straight lines $x_1-4x_2=1$, $2x_1-x_2=-3$, $-x_1-3x_2=4$ have the same intersection point? Explain your answer.

11. Find the elementary row operation that transforms the first matrix into the second one and in the other way round.

$$a)\left[\begin{array}{rrr} 0 & -2 & 5\\ 1 & 4 & -7\\ 3 & -1 & 6 \end{array}\right], \left[\begin{array}{rrr} 1 & 4 & -7\\ 0 & -2 & 5\\ 3 & -1 & 6 \end{array}\right]$$

$$b)\left[\begin{array}{rrr} 1 & 3 & -4\\ 0 & -2 & 6\\ 0 & -5 & 9 \end{array}\right], \left[\begin{array}{rrr} 1 & 3 & -4\\ 0 & 1 & -3\\ 0 & -5 & 9 \end{array}\right]$$

c) $\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 3 & 5 & 1 \\ 3 & -4 & 7 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 3 & 5 & 1 \\ 0 & 2 & 7 & -7 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & -3 & 9 & 5 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

12. Give three different augmented matrices whose solution sets only contains $x_1 = -2$, $x_2 = 1$, $x_3 = 0$.

13. Determine which matrices are in reduced echelon form, which others are in echelon form (but not reduced) and which others are not in row echelon form.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

14. Row reduce the following matrices to reduced echelon form, determine the pivot columns and the pivots in both the original matrices and in the final matrices.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

15. Find the general solution of linear systems whose augmented matrices are the following ones:

a) $\begin{bmatrix} 1 & 5 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 10 \end{bmatrix}$ d) $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ 6 & 8 & -4 & 0 \end{bmatrix}$

16. Determine the values of h and k such that the following linear system (a) has no solutions, (b) has a unique solution and (c) has infinitely many solutions.

$$\begin{aligned} x_1 + hx_2 &= 1 \\ 2x_1 + 3x_2 &= k \end{aligned}$$

17. Suppose that a system of linear equations has a 3×5 coefficient matrix with three pivot columns. Is the system consistent? Why or why not?

18. A system with more equations than unknowns is called *overdetermined.* One with more unknowns than equations is called *underdetermined.* Do consistent overdetermined systems exist? Reason your answer and provide an example if they do exist. Is it possible to find an underdetermined linear system that has a unique solution?

19. Reason if the following statements are true or false. (To reason means that theorems or suitable results will be used to prove the validity of the statements and counterexamples must be provided if any statement is false).

1. If a matrix B is obtained from another matrix A by means of elementary row operations, then A can be obtained from B by means of elementary row operations.
2. Each matrix is row equivalent to a unique matrix in row echelon form.
3. To multiply all the elements in a row by a constant is an elementary row operation.
4. If the augmented matrices of two systems of equations are row equivalent, then both systems have the same solution set.
5. Any system with n linear equations and n unknowns has at most n solutions.
6. If a system of linear equations has two different solutions, then it has infinitely many solutions.
7. If a system of linear equations does not have free variables, then it has a unique solution.
8. An overdetermined system cannot have a unique solution.
9. An underdetermined system cannot have a unique solution.
10. If all the columns of the coefficient matrix of a consistent system are pivot columns, then the systems has a unique solution.
11. A system of linear equations has infinitely many solutions if and only if at least one column in the coefficient matrix does not contain a pivot position
12. A consistent system of linear equations has infinitely many solutions if and only if at least one column in the coefficient matrix does not contain a pivot position.
13. An inconsistent system of linear equations sometimes has a unique solution.
14. A 5×7 matrix cannot have a pivot position in every row.
15. A 6×5 matrix cannot have a pivot position in every row.

20. Find a polynomial of degree 2, $p(t) = a_0 + a_1 t + a_2 t^2$, whose graph passes through the following points in the plane: $(1, 12), (2, 15), (3, 16)$. Such a polynomial is called an *interpolating polynomial* for those points.

21. Each of the following equations determines a plane in $\mathbb{R}^3$. Do the two planes intersect? If so, describe their intersection.

$$\begin{aligned} x_1 + 4x_2 - 5x_3 &= 0 \\ 2x_1 - x_2 + 8x_3 &= 9 \end{aligned}$$

22. Determine if the following systems have a nontrivial solution. Try to use as few row operations as possible.

a) $$\begin{array}{rrrl} 2x_1 & -5x_2 & +8x_3 = & 0 \\ -2x_1 & -7x_2 & +x_3 = & 0 \\ 4x_1 & +2x_2 & +7x_3 = & 0 \end{array}$$

b) $$\begin{array}{rrrl} x_1 & -3x_2 & +7x_3 = & 0 \\ -2x_1 & +x_2 & -4x_3 = & 0 \\ x_1 & +2x_2 & +9x_3 = & 0 \end{array}$$

c) $$\begin{array}{rrrl} -3x_1 & +5x_2 & -7x_3 = & 0 \\ -6x_1 & +7x_2 & +x_3 = & 0 \end{array}$$

d) $$\begin{array}{rrrl} -5x_1 & +7x_2 & +9x_3 = & 0 \\ x_1 & -2x_2 & +6x_3 = & 0. \end{array}$$

23. Write the solution set of the given homogeneous systems in parametric vector form.

a) $$\begin{array}{rrrl} x_1 & +3x_2 & +x_3 = & 0 \\ -4x_1 & -9x_2 & +2x_3 = & 0 \\ & -3x_2 & -6x_3 = & 0 \end{array}$$

b) $$\begin{array}{rrrl} x_1 & +3x_2 & -5x_3 = & 0 \\ x_1 & +4x_2 & -8x_3 = & 0 \\ -3x_1 & -7x_2 & +9x_3 = & 0 \end{array}$$

24. Describe all solutions of the following homogeneous linear systems in parametric vector form, where the **coefficient matrix** is row equivalent to the given matrix.

a) $\begin{pmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{pmatrix}$ b) $\begin{pmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{pmatrix}$

c) $\begin{pmatrix} 3 & -9 & 6 \\ -1 & 3 & -2 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{pmatrix}$

e) $\begin{pmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

25. Suppose a nonhomogenous linear system ($Ax = b$) has a solution. Explain why the solution is unique precisely when the homogeneous linear system with the same coefficient matrix as the previous system ($Ax = 0$) has only the trivial solution.

26. Suppose A is the 3×3 zero matrix, that is, with all zero entries. Describe the solution set of the homogeneous linear system with coefficient matrix A ($Ax = 0$).

27. Answer the following questions:

- does the equation $Ax = 0$ have a nontrivial solution?,
- does the equation $Ax = b$ have at least one solution for every non zero b?,

for the following coefficient matrices:

a) A is a 3×3 matrix with three pivot positions.

b) A is a 3×3 matrix with two pivot positions.

c) A is a 3×2 matrix with two pivot positions.

d) A is a 2×4 matrix with two pivot positions.

28. Suppose an economy has only two sectors. Goods and Services. Each year, Goods sells 80 % of its output to Services and keeps the rest, while Services sells 70 % of its output to Goods and retains the rest. Find equilibrium prices for the annual outputs of the Goods and Services sectors that make each sector's income match its expenditures.

29. Consider an economy with three sectors, Chemical & Metals, Fuels & Power, and Machinery. Chemicals sells 30 % of its output to Fuels and 50 % to Machinery and retains the rest. Fuels sells 80 % of its output to Chemicals and 10 % to Machinery and retains the rest. Machinery sells 40 % to Chemicals and 40 % to Fuels and retains the rest.

a) Construct the exchange table for this economy.

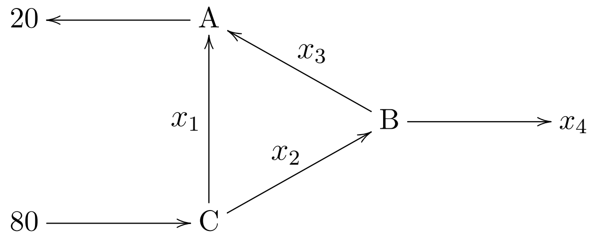
b) Develop a system of equations that leads to prices at which each sector's income matches its expenses. Then write the augmented matrix that can be row reduced to find these prices.

30. Balance the following chemical equations, that appear unbalanced, using linear systems.

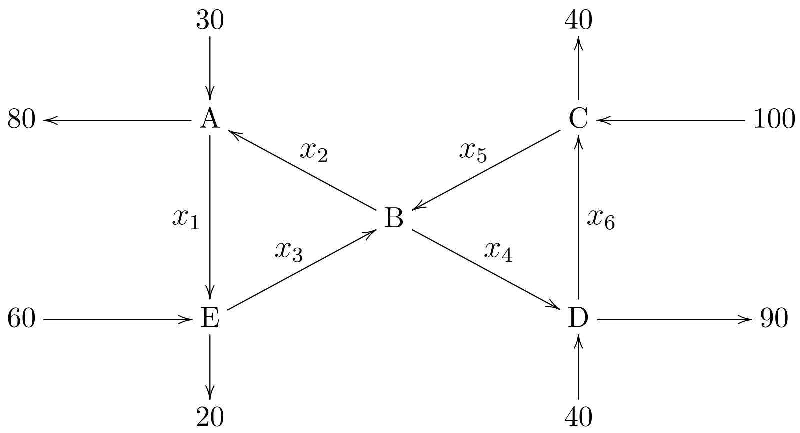
Boron sulfide reacts violently with water to form boric acid and hydrogen sulfide gas (the smell of rotten eggs).

$$\mathrm{B_2S_3 + H_2O \rightarrow H_3BO_3 + H_2S}.$$

31. Find the general flow pattern of the network shown in the figure. Assuming that the flows are all nonnegative, what is the largest possible value for x_3?



32. Find the general flow pattern in the network shown in the figure. Assuming that the flow must be in the directions indicated, find the minimum flows in the branches denoted by x_2, x_3, x_4 and x_5?



Additional exercises: D. C. Lay "Linear algebra and its applications", 2012.

- **Sections 1.1-1.2.**
- **Section 1.5.**
- **Section 1.6.**