Homework sheet 2: MATRICES AND DETERMINANTS
Year 2011-2012

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1. Let

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix},$$
$$D = \begin{pmatrix} 3 & 5 \\ -1 & 4 \end{pmatrix}, \quad E = \begin{pmatrix} -5 \\ 3 \end{pmatrix}.$$

Compute each of the following matrix sum or product if it is defined. If an expression is undefined, explain why.

(a) -2A (b) B-2A (c) AC (d) CD
(e) A+2B (f) 3C-E (g) CB (h) EB.

2. Let $A = \begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}$. Compute $3I_2 - A$ and $(3I_2)A$.

3. Compute the following products AB in two ways: (1) by the definition, where Ab_1 and Ab_2 are computed separately, and (2) by the row-column rule for computing AB.

(a) $A = \begin{pmatrix} 3 & 4 \\ 5 & 0 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}.$

(b) $A = \begin{pmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}.$

4. If a matrix A is 5×3 and the product AB is 5×7, what is the size of B?

5. How many rows does B have if BC is a 3×4 matrix?

6. Let $A = \begin{pmatrix} 3 & -4 \\ -5 & 1 \end{pmatrix}$ y $B = \begin{pmatrix} 7 & 4 \\ 5 & k \end{pmatrix}$. What value(s) of k, if any, will make $AB = BA$?

7. Let $A = \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 8 & 4 \\ 5 & 5 \end{pmatrix}$, $C = \begin{pmatrix} 5 & -2 \\ 3 & 1 \end{pmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.

8. Let $A = \begin{pmatrix} 2 & -6 \\ -1 & 3 \end{pmatrix}$. Find a 2×2 matrix B, whose both columns are different from zero, and such that $AB = 0$.

9. Let A be a $m \times p$ matrix and B be a $p \times n$ matrix.

(a) If the first and third rows of A are equal, what can you say about the rows of AB? Why?

(b) If the second column of B is zero, what can you say about the second column of AB? Why?

(c) If the third column of B is the sum of the first two columns, what can you say about the third column of AB? Why?

10. If $A = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$ y $AB = \begin{pmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{pmatrix}$, determine the first and second columns of B.

11. Let $u = \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix}$ and $v = \begin{pmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{pmatrix}$.

(a) Compute $u^T v$, $v^T u$, uv^T and vu^T.

(b) If $u, v \in \mathbb{R}^n$, how are $u^T v$ and $v^T u$ related? How are uv^T and vu^T related?

12. Find the inverses of the matrices:

$$(a) \begin{pmatrix} 8 & 6 \\ 5 & 4 \end{pmatrix}, \quad (b) \begin{pmatrix} 3 & 2 \\ 7 & 4 \end{pmatrix}, \quad (c) \begin{pmatrix} 8 & 5 \\ -7 & -5 \end{pmatrix}.$$

13. Find the inverse of the matrix $\begin{pmatrix} -4 & -5 \\ 5 & 6 \end{pmatrix}$ and use it to solve the system:

$$\begin{aligned} -4x_1 - 5x_2 &= -3\,, \\ 5x_1 + 6x_2 &= 1\,. \end{aligned}$$

14. Let

$$A = \begin{bmatrix} 5 & 13 \\ 3 & 8 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad b_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(a) Find A^{-1}, and use it to solve the four equations

$$Ax = b_1\,, \quad Ax = b_2\,, \quad Ax = b_3\,, \quad Ax = b_4\,.$$

(b) The four equations in part (a) can be solved by the same set of row operations, since the coefficient matrix is the same in each case. Solve the four equations in part (a) by row reducing the augmented matrix $[A\ b_1\ b_2\ b_3\ b_4]$.

15. Find the inverses of the matrices

$$A = \begin{pmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 5 & 0 \\ -2 & -7 & 6 \\ 1 & 3 & -4 \end{pmatrix},$$

if they exist.

16. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Show that if $ad - bc = 0$, then the equation $Ax = 0$ has more than one solution. Why does this imply that A is not invertible? (*Hint:* First, consider $a = b = 0$. Then, if a and b are not both zero, consider the vector $x = \begin{pmatrix} -b \\ a \end{pmatrix}$.)

(b) Find a formula for A^{-1} if $ad - bc \neq 0$.

17. Let $A = \begin{pmatrix} -1 & -5 & -7 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{pmatrix}$. Determine the second column of A^{-1} without computing the other columns.

18. Assume that the matrices are partitioned conformably for block multiplication. Compute the products shown:

(a) $\begin{pmatrix} I & 0 \\ E & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

(b) $\begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

(c) $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$.

(d) $\begin{pmatrix} I & 0 \\ -X & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

19. Find formulas for X, Y, Z in terms of A, B and C in the following equalities:

(a) $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ X & Y \end{pmatrix} = \begin{pmatrix} 0 & I \\ Z & 0 \end{pmatrix}$.

(b) $\begin{pmatrix} X & 0 & 0 \\ Y & 0 & I \end{pmatrix} \begin{pmatrix} A & Z \\ 0 & 0 \\ B & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$.

20. The inverse of $\begin{pmatrix} I & 0 & 0 \\ C & I & 0 \\ A & B & I \end{pmatrix}$ is $\begin{pmatrix} I & 0 & 0 \\ Z & I & 0 \\ X & Y & I \end{pmatrix}$. Find X, Y and Z.

21. Find an LU factorization of the matrices:

$$A = \begin{pmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 & -1 & 5 \\ 3 & 7 & -2 & 9 \\ -2 & -3 & 1 & -4 \\ -1 & 6 & -1 & 7 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{pmatrix}$$

22. Solve the equation $Ax = b$ by using the LU factorization given for A.

(a) $A = \begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}$, $b = \begin{pmatrix} -7 \\ 5 \\ 2 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

(b) $A = \begin{pmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

(c) $A = \begin{pmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

23. Compute the following determinants:

$$\text{(a)} \begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}, \qquad \text{(b)} \begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix}, \qquad \text{(c)} \begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix}.$$

24. Compute the following elementary matrices and identify them with a row operation when they multiply a matrix.

$$\text{(a)} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{vmatrix}, \qquad \text{(b)} \begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{vmatrix}, \qquad \text{(c)} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}.$$

25. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a 2×2 matrix and a real number α. Find a formula that relates $\det[\alpha \cdot A]$ with α and $\det A$.

26. Find the determinants by row reduction to echelon form:

$$\text{(a)} \begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix}, \qquad \text{(b)} \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix}.$$

27. Combine the methods of row reduction and cofactor expansion to compute the following determinants:

$$\text{(a)} \begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix} \qquad \text{(b)} \begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix}.$$

28. Find the following determinants where $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$

$$\text{(a)} \begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}, \qquad \text{(b)} \begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}, \qquad \text{(c)} \begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix}.$$

29. Let A be a $n \times n$ square matrix.

(a) If A is invertible, then show that $\det A^{-1} = \dfrac{1}{\det A}$

(b) If α is a real number, find a formula for $\det(\alpha \cdot A)$.

(c) Let B be also a $n \times n$ square matrix. Show that $\det(A \cdot B) = \det(B \cdot A)$, even though, in general, $A \cdot B \neq B \cdot A$.

(d) If P is a $n \times n$ square invertible matrix. Show that $\det(P \cdot A \cdot P^{-1}) = \det A$.

(e) Let U be a $n \times n$ square matrix such that $U^T \cdot U = I$. Show that $\det U = \pm 1$.

(f) If $\det A^4 = 0$, is A invertible?

30. Let A and B be 3×3 matrices, with $\det A = 4$ and $\det B = -3$. Use properties of determinants to compute:

(a) $\det(A \cdot B)$ (b) $\det(A^{-1})$ (c) $\det(5\,A)$ (d) $\det(A^3)$ (e) $\det(B^T)$.

31. Use Cramer's rule to compute the solutions of the following systems:

$$\text{(a)} \left\{ \begin{array}{rrcr} 4x_1 & +x_2 & = & 6, \\ 5x_1 & +2x_2 & = & 7. \end{array} \right. \qquad \text{(b)} \left\{ \begin{array}{rrrcr} 2x_1 & +x_2 & +x_3 & = & 4, \\ -x_1 & & +2x_3 & = & 2, \\ 3x_1 & +x_2 & +3x_3 & = & -2. \end{array} \right.$$

32. Use determinants to decide if the following matrices are invertible or not. Compute the adjugate of the given matrix and give the inverse of the matrix, whenever is possible:

$$\text{(a)} \begin{pmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{(c)} \begin{pmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{pmatrix}.$$

Additional exercises: D. C. Lay "Linear algebra and its applications", 2012.

- **Sections 2.1-2.5 for matrices.**
- **Sections 3.1-3.3 for determinants.**