FULL SOLUTION TO EXERCISES OF LU FACTORIZATION IN HOMEWORK SHEET 2

Course 2011-2012

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21. Find an LU factorization of the matrices:

$$A = \begin{pmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{pmatrix}$$

Solution:

We start by using row operations to obtain a matrix in echelon form associated with A.

First, we have to get zeros in the entries a_{21} and a_{31} of the matrix A, then in the entry a_{32}.

$$\begin{pmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{pmatrix} \underset{\substack{2R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3}}{\sim} \begin{pmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & 14 \end{pmatrix} \underset{5R_2 + R_3 \to R_3}{\sim} \begin{pmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{pmatrix} = U.$$

The final matrix is the matrix U of the LU factorization.

NOTE: To successfully compute the LU factorization the replacement row operations used to obtain a matrix in echelon form from A must be as follows

$$\lambda R_i + \mathbf{R_j} \to \mathbf{R_j}, \quad \text{with } \lambda \in \mathbb{R}. \tag{1}$$

Then the elementary matrices associated with these row operation always have $1's$ in the entries of the main diagonal.

To compute L there are two different methods:

1. The row operation we just did in the first step above can be written in terms of the following elementary matrix:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

The row operations in the second step above to get the echelon matrix are contained in the following elementary matrix:

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}.$$

Observe that $E_2(E_1A) = U$:

$$\begin{aligned} E_2(E_1A) &= E_2\left[\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}\begin{pmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & 14 \end{pmatrix} \\ &= \begin{pmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{pmatrix} = U. \end{aligned}$$

The inverse of L is precisely E_2E_1:

$$L^{-1} = E_2E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 13 & 5 & 1 \end{pmatrix}.$$

To compute L we must compute the inverse of the inverse of L, that is $(L^{-1})^{-1} = L$, as follows

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 13 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & 1 & -13 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \overbrace{\begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{matrix}}^{L=} \\ 0 & 0 & 1 & \end{array}\right).$$

Then the LU factorization of A is:

$$\begin{pmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{pmatrix}\begin{pmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{pmatrix}.$$

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2. The second method to find L consists of looking carefully the results obtained in the process to get the echelon matrix associated with A.

$$\begin{pmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{pmatrix} \underset{\substack{2R_1+\mathbf{R_2} \to \mathbf{R_2} \\ 3R_1+\mathbf{R_3} \to \mathbf{R_3}}}{\sim} \overbrace{\begin{pmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & 14 \end{pmatrix}}^{B=} \underset{5R_2+\mathbf{R_3} \to \mathbf{R_3}}{\sim} \begin{pmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{pmatrix}.$$

Remember that the matrix L has size 3×3 in this exercise and it is always a lower triangular matrix with 1's in the main diagonal:

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}.$$

To determine the unknown entries we do as follows:

- The entries l_{21} and l_{31} are given by a_{21} and a_{31} divided by $-5 = a_{11}$.
- The entry l_{32} comes from dividing b_{32} by $-2 = b_{22}$.

As a result, we have $L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{pmatrix}$, as in the previous method.

NOTE: This method is faster than the previous one to find L.

$$C = \begin{pmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{pmatrix} \text{ (}\mathbf{Lay\ 2.5.16}\text{)}$$

Solution 1:

In this exercise, we find the LU factorization using the second method described above. Note that C is not a square matrix, but 5×3. In spite of that, we can still compute the LU factorization of C.

First, note that U will be a 5×3 matrix, as C, and L has size 5×5. Remember that L is always a square matrix.

Before computations, we already know that L looks like:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 & 0 \\ l_{41} & l_{42} & l_{43} & 1 & 0 \\ l_{51} & l_{52} & l_{53} & l_{54} & 1 \end{pmatrix}$$

To find an equivalent matrix to C in echelon from remember to only use replacement row operations that look like (1).

The first column of C divided by $2 = c_{11}$ determine the unknown entries in the first column of L, then

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & l_{32} & 1 & 0 & 0 \\ -3 & l_{42} & l_{43} & 1 & 0 \\ 4 & l_{52} & l_{53} & l_{54} & 1 \end{pmatrix}$$

We make zeros below the pivot of the first row of C, that is c_{11}:

$$\begin{pmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{pmatrix} \begin{array}{c} \sim \\ 2R_1 + R_2 \to R_2 \\ -\frac{3}{2}R_1 + R_3 \to R_3 \\ 3R_1 + R_4 \to R_4 \\ -4R_1 + R_5 \to R_5 \end{array} \begin{pmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 14 & -10 \\ 0 & -14 & 10 \\ 0 & 21 & -15 \end{pmatrix}$$

The second column of the obtained matrix divided by -7, the value of the pivot position in the second row of the resulting matrix, determines the unknown entries of the second column of L, then

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & -2 & 1 & 0 & 0 \\ -3 & 2 & l_{43} & 1 & 0 \\ 4 & -3 & l_{53} & l_{54} & 1 \end{pmatrix}.$$

We continute obtaining an equivalent matrix to C in echelon form:

$$\begin{pmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 14 & -10 \\ 0 & -14 & 10 \\ 0 & 21 & -15 \end{pmatrix} \begin{array}{c} \sim \\ 2R_2 + R_3 \to R_3 \\ -2R_2 + R_4 \to R_4 \\ 3R_2 + R_5 \to R_5 \end{array} \begin{pmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = U.$$

As there are no more pivots, the remaining unknown elements in L are just zero:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & -2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 1 \end{pmatrix}.$$

Thus the LU factorization of C is:

$$\begin{pmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & -2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution 2:
An equivalent matrix to C in echelon form is obtained as follows:

$$\begin{pmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{pmatrix} \underset{\substack{2R_1+R_2\to R_2 \\ -\frac{3}{2}R_1+R_3\to R_3 \\ 3R_1+R_4\to R_4 \\ -4R_1+R_5\to R_5}}{\sim} \begin{pmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 14 & -10 \\ 0 & -14 & 10 \\ 0 & 21 & -15 \end{pmatrix} \underset{\substack{2R_2+R_3\to R_3 \\ -2R_2+R_4\to R_4 \\ 3R_2+R_5\to R_5}}{\sim} \begin{pmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = U.$$

The elementary matrices are always square and they have as many columns and rows as the given matrix C. The set of replacement row operations made above are contained in the following elementary matrices:

$$L^{-1} = E_2E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 5/2 & 2 & 1 & 0 & 0 \\ -1 & -2 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \end{pmatrix}.$$

Exercise: Check that the inverse of $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 5/2 & 2 & 1 & 0 & 0 \\ -1 & -2 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \end{pmatrix}$ **is the matrix** L **obtained in solution 1.**

Exercise: Find the LU **factorization of** $\begin{pmatrix} 1 & 4 & -1 & 5 \\ 3 & 7 & -2 & 9 \\ -2 & -3 & 1 & -4 \\ -1 & 6 & -1 & 7 \end{pmatrix}$ **using the both methods described above.**

Solution: $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 4 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

22. Solve the equation $Ax = b$ by using the LU factorization given for A.

(b) $A = \begin{pmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Solution:

As we have the LU factorization of A we can rewrite $Ax = b$ as $L(Ux) = b$. Let us denote Ux by y and then we solve the following two systems of equations to find x:

$$Ly = b \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}.$$

From here we have

$$\left\{ \begin{array}{rrrcr} y_1 & & & = & 2 \\ -y_1 & +y_2 & & = & -4 \\ 2y_1 & & +y_3 & = & 6 \end{array} \right. \qquad \left\{ \begin{array}{lcl} y_1 & = & 2 \\ y_2 & = & -4 + y_1 = -4 + 2 = -2 \\ y_3 & = & 6 - 2y_1 = 6 - 2(2) = 2 \end{array} \right.$$

Now let us solve the system:

$$Ux = y \Leftrightarrow \begin{pmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}.$$

From here we obtain:

$$\left\{ \begin{array}{rrrcr} 4x_1 & +3x_2 & -5x_3 & = & 2 \\ & -2x_2 & +2x_3 & = & -2 \\ & & 2x_3 & = & 2 \end{array} \right. \qquad \left\{ \begin{array}{lcl} x_1 & = & \dfrac{2 + 5x_3 - 3x_2}{4} = \dfrac{2+5-6}{4} = \dfrac{1}{4} \\ x_2 & = & \dfrac{-2 - 2x_3}{-2} = \dfrac{-2-2}{-2} = 2 \\ x_3 & = & 1 \end{array} \right.$$

Then the solution of $Ax = b$ is $(1/4, 2, 1)$.

Exercise: Solve the equation $Ax = b$ by using the LU factorization given for A.

(a) $A = \begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}$, $b = \begin{pmatrix} -7 \\ 5 \\ 2 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Solution: $(3, 4, -6)$.

(c) $A = \begin{pmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution: $(-1, 3, 3)$**.**