Homework sheet 3: REAL VECTOR SPACES
Year 2011-2012

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1. Let W be the set of all vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$, where $s,\ t \in \mathbb{R}$.

   Show that W is a subspace of $\mathbb{R}^4$.

2. Let W be the set of all vectors of the form shown, where a, b and c represent arbitrary real numbers. In each case, either find a set S of vectors that spans W or give an example to show that W is not a vector space.

   $$\text{a)}\ \begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix} \qquad \text{b)}\ \begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix}$$

3. Let W be the union of the first and third quadrants in the xy-plane. That is, let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$. If $u \in W$ and α is any scalar, is αu in W? Why? Find specific vectors u, v in W such that $u+v$ is not in W. This is enough to show that W is not a vector space.

4. Show that the given set W is a vector space or find a specific example to the contrary.

   $$\text{a)}\ \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{c} a+3b=c \\ b+c+a=d \end{array} \right\} \qquad \text{b)}\ \left\{ \begin{bmatrix} b-5d \\ 2b \\ 2d+1 \\ d \end{bmatrix} : b,d\ \text{ reales} \right\}$$

5. Let $\mathbb{P}_n$ be the set of all polynomials of degree less or equal to n. Determine if the following subsets of $\mathbb{P}_n$ are subspaces or not.

   a) The set of all polynomials of the form $p(t) = at^2$, where $a \in \mathbb{R}$.

   b) The set of all polynomials of the form $p(t) = a + t^2$, where $a \in \mathbb{R}$.

6. Determine if the set H of all matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is a subspace of $\mathrm{M}_{2\times 2}$, where $a, b, d \in \mathbb{R}$.

7. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

   a) Is $\mathbf{w}$ in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

   b) How many vectors are in $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

   c) Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

   d) Is $\mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ 14 \end{pmatrix}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

8. Determine if $w = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$ is in $\operatorname{Nul} A$, where $A = \begin{pmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{pmatrix}$.

9. Find an explicit expression, in terms of the generating vectors, of the null space of the following matrices:

$$\text{(a)} \begin{pmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

10. Find $\operatorname{Nul} A$ for the matrix $A = \begin{pmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{pmatrix}$.

11. Assume that $A \sim B$, use this information to find $\operatorname{Nul} A$ and $\operatorname{Col} A$.

$$A = \begin{pmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

12. Find the matrix A such that $\operatorname{Col} A = \left\{ \begin{pmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{pmatrix} \mid b, c, d \in \mathbb{R} \right\}$.

13. Let $A = \begin{pmatrix} -6 & 12 \\ -3 & 6 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Determine if $\mathbf{w}$ is in $\operatorname{Col} A$ and also in $\operatorname{Nul} A$.

14. Determine if the following transformations are linear or not:

   a) $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$.

b) $T(x_1, x_2) = (x_2, 0, x_1 + x_2, 0)$.

c) $T(x_1, x_2) = (x_2, 1, x_1 + x_2, 0)$.

d) $T(x_1, x_2, x_3) = (x_3, x_2)$.

e) $T(x_1, x_2) = x_1 x_2$.

f) $T(x_1, x_2, x_3) = x_1 + x_2 + x_3$.

15. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(x) = Ax$ where $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Find the images under T of $u = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $v = \begin{pmatrix} a \\ b \end{pmatrix}$.

16. Let A be a 6×5 matrix. What must be a and b in order to define $T\colon \mathbb{R}^a \to \mathbb{R}^b$ by $T(x) = Ax$?

17. How many rows and columns must a matrix A have in order to define a mapping T from $\mathbb{R}^5$ into $\mathbb{R}^7$ by the rule $T(x) = Ax$?

18. Let T be a linear transformation given by $T(x) = Ax$. Find x whose image under T is b, where
$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 0 & 3 & -8 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 5 \\ 9 \\ 4 \end{bmatrix}.$$

19. Let
$$A = \begin{bmatrix} 1 & 2 & -7 & 5 \\ 0 & 1 & -4 & 0 \\ 1 & 0 & 1 & 6 \\ 2 & -1 & 6 & 8 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 9 \\ 5 \\ 0 \\ -9 \end{bmatrix}.$$

a) Find all the vectors $x \in \mathbb{R}^4$ such that are mapped to zero under the transformation $x \to Ax$

b) Is b in the range of the linear transformation defined by A?

20. Use a rectangular coordinate system to plot
$$u = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$$
and their images under the given transformation T defined by the rule $T(x) = Ax$. Describe geometrically what T does to each vector $x \in \mathbb{R}^2$.
$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

21. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation that maps $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ into $\begin{pmatrix} 7 \\ -3 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ into $\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$. Use the fact that T is linear to find the images under T of $3u$, $-2v$ and $3u - 2v$.

22. Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

   a) Find the matrix associated with that transformation and provide the analytical expression of T.

   b) Compute $T\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

   c) Compute $T^{-1}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

   d) Is T injective? Is it onto?

23. Determine whether the following set of vectors are bases for $\mathbb{R}^3$. Of the sets that are not bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

   a) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$

   b) $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} -8 \\ 5 \\ 4 \end{pmatrix}.$

   c) $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$

24. Find a basis for the null spaces of the following matrix $\begin{pmatrix} 1 & 1 & -2 & 1 & 5 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -8 & 0 & 16 \end{pmatrix}$.

25. Find a basis for $\mathrm{Span}\{v_1, \ldots, v_5\}$ if

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 6 \\ -1 \\ 2 \\ -1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 5 \\ -3 \\ 3 \\ -4 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 0 \\ 3 \\ -1 \\ 1 \end{pmatrix}.$$

26. Decide if the following vectors are linearly independent:

$$\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -6 \end{bmatrix}.$$

27. Decide if the columns in the following matrix are linearly independent:

$$A = \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & 1 & 3 \end{bmatrix}.$$

28. Let

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 9 \\ -1 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} -2 \\ -6 \\ h \end{bmatrix}.$$

For what values of h

a) $v_3 \in \text{Span}[v_1, v_2]$?

b) v_1, v_2 and v_3 are linearly independent?

29. Find a basis for the set of vectors in $\mathbb{R}^2$ in the line $y = 5x$.

30. Let $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 4 \\ -2 \\ -5 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 4 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 5 \\ -6 \\ -14 \end{pmatrix}$. It can be verified that $2\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = 0$. Use this information to find a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. (Remark: The solution is not unique).

31. Fill in the blank space in the following sentence: "If A is an $m \times n$ matrix, then the columns of A are linearly independent if and only if A has ________________ pivot columns".

32. Find a basis and the dimension of the following subspaces:

a) $$\left\{ \begin{pmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

b) $$\left\{ \begin{pmatrix} 2a \\ -4b \\ -2a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

c) $\left\{ \begin{pmatrix} p+2q \\ -p \\ 3p-q \\ p+q \end{pmatrix} \ : \ p, q \in \mathbb{R} \right\}$.

d) $\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \ : a-3b+c=0, \ b-2c=0, \ 2b-c=0 \right\}$.

e) $\left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \ : a-3b+c=0 \right\}$.

33. Find the dimension of the subspace spanned by the vectors:

$$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -6 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 5 \\ 5 \end{pmatrix}.$$

34. Determine the dimension of the subspaces $\operatorname{Nul} A$ and $\operatorname{Col} A$ in the following cases:

a) $A = \begin{pmatrix} 1 & 2 & -4 & 3 & -2 & 6 & 6 \\ 0 & 0 & 0 & 1 & 0 & -3 & 7 \\ 0 & 0 & 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

b) $A = \begin{pmatrix} 3 & 2 \\ -6 & 5 \end{pmatrix}$.

c) $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

35. Assume that the matrices A and B are row equivalent. Without additional computations determine the rank of A, the dimension of $\ker A$. Find a basis of $\operatorname{Col} A$ and $\ker A$.

a) $A = \begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

b) $A = \begin{pmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 6 \\ 1 & -2 & 1 & 0 & 2 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & -13 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

36. If a 7×5 matrix A has rank 2, find the dimension of $\operatorname{Nul} A$ and the rank of A^T.

37. Assume that a 6×8 matrix A has 4 pivots. What is the dimension of $\operatorname{Nul} A$? Is $\operatorname{Col} A = \mathbb{R}^4$? Justify your answers.

38. If the null space of a 8×7 matrix A has dimension 5, what is the dimension of the column space of A?

39. Let A be a 7×5 matrix. What is the largest possible rank of A? And if A is a 5×7 matrix? Justify your answers.

40. Let A be a 7×5 matrix. What is the smallest possible dimension of the null space of A?

41. Assume that a nonhomogeneous system $Ax = b$ with 6 equations and 8 unknowns has 2 free variables. Is it possible to change some of the values in the column vector b so that the system is inconsistent?

42. Let A be a $m \times n$ matrix and $b \in \mathbb{R}^m$. What are the relative rank values of the matrices A and $\begin{pmatrix} A & b \end{pmatrix}$ so that the system $Ax = b$ is consistent? (*Rouché-Frobenius' Theorem*).

43. Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ be bases for a vector space V and suppose $b_1 = -2c_1 + 4c_2$, $b_2 = 3c_1 - 6c_2$. Find the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Find $[x]_{\mathcal{C}}$ for $x = 2b_1 + 3b_2$.

44. Let $\mathcal{A} = \{a_1, a_2, a_3\}$ and $\mathcal{D} = \{d_1, d_2, d_3\}$ be bases for a vector space V and suppose $P = [[d_1]_{\mathcal{A}}\,[d_2]_{\mathcal{A}}\,[d_3]_{\mathcal{A}}]$. Which of the following equations is satisfied by P for all $x \in V$?

$$\text{(a) } [x]_{\mathcal{A}} = P\,[x]_{\mathcal{D}} \quad \text{(b) } [x]_{\mathcal{D}} = P\,[x]_{\mathcal{A}}$$

45. Let $\mathcal{D} = \{d_1, d_2, d_3\}$ and $\mathcal{F} = \{f_1, f_2, f_3\}$ be bases for a vector space V and suppose $f_1 = 2d_1 - d_2 + d_3$, $f_2 = 3d_2 + d_3$, $f_3 = -3d_1 + 2d_3$. Find the change-of-basis matrix from $\mathcal{F}$ to $\mathcal{D}$. Find $[x]_{\mathcal{D}}$ for $x = f_1 - 2f_2 + 2f_3$.

46. Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ be bases for $\mathbb{R}^2$. Find the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ and from $\mathcal{C}$ to $\mathcal{B}$ if

$$a)\ b_1 = \begin{pmatrix} -1 \\ 8 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ -7 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$b)\ b_1 = \begin{pmatrix} 6 \\ -12 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 3 \\ 9 \end{pmatrix}.$$

47. Find the coordinate vectors $[x]_{\mathcal{B}}$ of x in the basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ in the following cases:

$$a)\ b_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 5 \\ -6 \end{pmatrix}, \quad x = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

$b)$ $b_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad x = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}.$

48. Find the change-of-basis matrix, $P_{\mathcal{B}_c,\mathcal{B}}$ from bases $\mathcal{B}$ to the standard bases $\mathcal{B}_c$ of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively:

$$\text{(a) } \mathcal{B} = \left\{ \begin{pmatrix} 2 \\ -9 \end{pmatrix}, \begin{pmatrix} 1 \\ 8 \end{pmatrix} \right\}, \quad \text{(b) } \mathcal{B} = \left\{ \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} 8 \\ -2 \\ 7 \end{pmatrix} \right\}.$$

49. The vectors $v_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$, $v_3 = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$ span $\mathbb{R}^2$, but do not form a basis. Find two different ways to express vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as a linear combination of v_1, v_2 and v_3.

Additional exercises: D. C. Lay "Linear algebra and its applications", 2012.

- **Sections 4.1-4.2**
- **Sections 1.8-1.9.**
- **Sections 4.3-4.7.**