Homework sheet 4: EIGENVALUES AND EIGENVECTORS. DIAGONALIZATION

Year 2011-2012

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1. Is $\lambda = -3$ an eigenvalue of $\begin{pmatrix} -1 & 4 \\ 6 & 9 \end{pmatrix}$? Why or why not?

2. Is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 5 & 2 \\ 3 & 6 \end{pmatrix}$? If so, find the eigenvalue.

3. Is $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{pmatrix}$? If so, find the eigenvalue.

4. Find a basis for the eigenspace corresponding to each listed eigenvalue:

   a) $A = \begin{pmatrix} -4 & 2 \\ 3 & 1 \end{pmatrix}$, $\lambda = -5$.

   b) $A = \begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix}$, $\lambda = 3, 7$.

   c) $A = \begin{pmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{pmatrix}$, $\lambda = 3$.

5. Find the eigenvalues and eigenvectors of the following matrices:

   (a) $\begin{pmatrix} 3 & -1 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{pmatrix}$

6. List the real eigenvalues, repeated according to their multiplicities, of the following matrices:

   (a) $\begin{pmatrix} 5 & 5 & 0 & 2 \\ 0 & 2 & -3 & 6 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 0 & 3 & 6 & 0 \\ 2 & 3 & 3 & -5 \end{pmatrix}$

7. It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 4$ is two-dimensional.

$$A = \begin{pmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

8. Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}. (*Hint:* Suppose a nonzero x satisfies $Ax = \lambda x$.)

9. Let $A = PDP^{-1}$. Compute A^4 if $P = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

10. Use the factorization $A = PDP^{-1}$ to compute A^k, where k represents an arbitrary positive integer, if

$$A = \begin{pmatrix} a & 0 \\ 2(a-b) & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

11. The matrix A is factored in the form PDP^{-1}. Find the eigenvalues of A and a basis for each eigenspace.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -3 & 1 & 9 \\ -1 & 0 & 3 \end{pmatrix}$$

12. Diagonalize the following matrices and give the matrices P and D in the factorization $A = PDP^{-1}$, if possible.

a) $\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

c) $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{pmatrix}$

f) $\begin{pmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{pmatrix}$

13. A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why?

14. A is a 7×7 matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspace is three-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

15. Show that if A is both diagonalizable and invertible, then so is A^{-1}.

16. Determine if the application matrix of the following linear transformations is diagonalizable.

    a) $F(x, y) = (2y, x - y)$,
    b) $G(x, y) = (2x - y, x)$,
    c) $H(x, y, z) = (2(x + z), x - z, x + 3z)$.

17. Determine a basis for $\mathbb{R}^3$ such that the application matrix of the above linear transformation $H(x, y, z) = (2(x + z), x - z, x + 3z)$ in that basis is diagonal.

Additional exercises: D. C. Lay "Linear algebra and its applications", 2012.

- **Sections 5.1-5.4.**