Bachelor in Statistics and Business — Mathematical Methods II
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Homework sheet 5: ORTHOGONALITY AND LEAST-SQUARES PROBLEM

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1. Let $u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, $w = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$, $x = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$. Compute the following quantities: $w \cdot w$, $x \cdot w$, $\dfrac{x \cdot w}{w \cdot w}$, $\dfrac{1}{u \cdot u}u$, $\dfrac{x \cdot w}{x \cdot x}x$, $||x||$.

2. Let $v = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$, $w = \begin{bmatrix} 8/3 \\ 2 \end{bmatrix}$. Find a unit vector in the direction of each given vector.

3. Find the distance between $x = \begin{pmatrix} 0 \\ -5 \\ 2 \end{pmatrix}$ and $z = \begin{pmatrix} -4 \\ -1 \\ 8 \end{pmatrix}$.

4. Let $u = [5, -6, 7]^T$ and W be the subset of all vectors x in $\mathbb{R}^3$ such that $u \cdot x = 0$. Give a geometric interpretation of W and show that W is a subspace.

5. Let $u = (u_1, u_2, u_3)$. Explain why $u \cdot u \geq 0$. When is $u \cdot u = 0$?

6. Let $v = \begin{pmatrix} a \\ b \end{pmatrix}$. Describe the set H of vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ that are orthogonal to v.

7. Determine which set of vectors are orthogonal:

$$S_1 = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix} \right\}.$$

8. Show that $u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ is an orthogonal basis for $\mathbb{R}^3$. Then express the vector $x = [5, -3, 1]^T$ as a linear combination of the $u's$.

9. Compute the orthogonal projection of $y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ onto the line through $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and the origin.

10. Let $y = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $u = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$. Write y as the sum of a vector in span$\{u\}$ and a vector orthogonal to u.

11. Let $y = \begin{pmatrix} -3 \\ 9 \end{pmatrix}$, $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Compute the distance from y to the line through u and the origin.

12. Determine which of the following sets of vectors are orthonormal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

$$S_1 = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\}, \quad S_2 = \left\{ \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix} \right\},$$

$$S_3 = \left\{ \begin{pmatrix} \frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} \right\}.$$

13. Let $\{v_1, v_2\}$ be an orthogonal set of nonzero vectors and let c_1, c_2 be any nonzero scalars. Show that $\{c_1\, v_1, c_2\, v_2\}$ is also an orthogonal set.

14. Let W be the subspace spanned by the vectors $u's$. Write y as the sum of a vector in W and a vector orthogonal to W.

   a) $y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}$

   b) $y = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}$

15. Let W be the subspace spanned by the vectors u_1 and u_2. Find the closest point to y in the subspace W.

$$y = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -3 \\ -1 \\ -1 \\ 5 \end{bmatrix}$$

16. Let $y = [7, 9]^T$, $u_1 = [1/\sqrt{10}, -3/\sqrt{10}]^T$, and $W = \text{Gen}\{u_1\}$.

   a) Let U be the 2×1 matrix whose only column is u_1. Compute $U^T U$ and $U U^T$.

b) Compute $\text{proy}_W y$ and $(UU^T)y$.

17. Let $u_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $u_2 = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$, $u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Show that u_1 and u_2 are orthogonal but that u_3 is not orthogonal to u_1 or u_2. Show that u_3 is not in the subspace W spanned by u_1 and u_2. Use this fact to construct a nonzero vector v in $\mathbb{R}^3$ that is orthogonal to u_1 and u_2.

18. Use the Gram-Schmidt process to produce an orthonormal basis for the subspace W spanned by the following vectors:

$$\text{(a)} \quad \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ -7 \end{pmatrix} \quad \text{(b)} \quad \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 14 \\ -7 \end{pmatrix} \quad \text{(c)} \quad \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix}.$$

19. Use the Gram-Schmidt process to find an orthogonal basis for the column space of each of the following matrices:

$$\text{(a)} \quad \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix} \quad \text{(b)} \quad \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}.$$

20. Find a least-squares solution of $Ax = b$ by constructing the normal equations for $\hat{x}$ and solving for $\hat{x}$. In the first two exercises compute the least-squares error associated with the least-squares solution found.

$$a)\ A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$$

$$b)\ A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$$

$$c)\ A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$$

21. Find the orthogonal projection of b onto the column space of A and find a least-squares solution of $Ax = b$.

a) $A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$

22. Describe all least-squares solutions of the system: $\begin{cases} x + y = 2 \\ x + y = 4 \end{cases}$.

23. Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits the given data points:

    a) $(0, 1), (1, 1), (2, 2), (3, 2)$.

    b) $(2, 3), (3, 2), (5, 1), (6, 0)$.

24. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be the data that must be fit by a least-squares line. Show that the normal equations have a unique solution if and only if the data include at least two data points with different x-coordinates.

25. A certain experiment produces the data (1,1.8), (2,2.7), (3,3.4), (4,3.8), (5,3.9). Describe the model that produces a least-squares fit of these points by a function of the form $y = \beta_1 x + \beta_2 x^2$.

Additional exercises: D. C. Lay "Linear algebra and its applications", 2012.

- **Sections 6.1-6.6.**