Homework sheet 6: THE SINGULAR VALUE DECOMPOSITION

Year 2011-2012

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1. Determine which of the matrices are symmetric:

$$\begin{pmatrix} 3 & 5 \\ 5 & -7 \end{pmatrix}, \quad \begin{pmatrix} -3 & 5 \\ -5 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 8 & 3 \\ 8 & 0 & -2 \\ 3 & -2 & 0 \end{pmatrix}, \quad \begin{pmatrix} -6 & 2 & 0 \\ 0 & -6 & 2 \\ 0 & 0 & -6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}.$$

2. Determine which of the matrices are orthogonal. If they are orthogonal, find the inverse.

$$\begin{pmatrix} 0{,}6 & 0{,}8 \\ 0{,}8 & -0{,}6 \end{pmatrix}, \quad \begin{pmatrix} -5 & 2 \\ 2 & 5 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

3. Orthogonally diagonalize the following matrices, giving P and D. For some matrices the eigenvalues are given to make computations easier.

(a) $\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ Eigenvalues : $7, -2$.

(d) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ Eigenvalues : $2, 0$.

4. Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Verify that 2 is an eigenvalue of A and v is an eigenvector. Then orthogonally diagonalize the matrix A.

5. Let $A = \begin{pmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{pmatrix}$, $v_1 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Verify that v_1 and v_2 are eigenvectors of A. Then orthogonally diagonalize the matrix A.

6. Find the singular values of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}, \quad \begin{pmatrix} -5 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \sqrt{6} & 1 \\ 0 & \sqrt{6} \end{pmatrix}, \quad \begin{pmatrix} \sqrt{3} & 2 \\ 0 & \sqrt{3} \end{pmatrix}.$$

7. Let $A = \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix}$. Find a unit vector x at which Ax has maximum length. Find the matrix norm of A.

8. Plot the image of the unit sphere under the linear transformations given by each of the following matrices:

$$\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

9. Plot the image of the unit sphere under the linear transformations given by each of the following matrices and determine if the matrices are orthogonal:

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

10. Find a singular value decomposition (SVD) of the following matrices:

(a) $\begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}$ **(Lay: 7.4.5)** (b) $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ **(Lay: 7.4.6)**

(c) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$ **(Lay: 7.1.12)** (f) $\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

11. Find the norm for each of the matrices in the exercise 10.

12. Plot the image of the unit sphere under the linear transformations given by the matrices (b), (c), (f) in exercise 10.

13. Find the Moore-Penrose pseudoinverse matrix of $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & -2 \\ 1 & 2 & -1 \end{pmatrix}$.

14. Find the Moore-Penrose pseudoinverse matrix of $A = \begin{pmatrix} -3 & 0 \\ 3 & -3 \\ 0 & 3 \end{pmatrix}$. Use $A^\dagger$ to solve the least-square problems of $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

15. Solve the least-square problem of $Ax = b$ by using the Moore-Pensore pseudoinverse matrix of A, where $A = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$.

Additional exercises: D. C. Lay “Linear algebra and its applications”, 2012.

- **Section 7.1.**
- **Section 7.4.**