MATHEMATICAL METHODS II
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**Full solution to some singular value decomposition exercises in HW-6**
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10. Find a singular value decomposition (SVD) of the following matrices:

(b) $A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$

**First find the eigenvalues of the matrix $A^TA$:**

$$A^TA = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$

As the matrix is diagonal, the eigenvalues are the numbers in the main diagonal, that is, 4 and 1.

**The singular values of $A$ are the square roots of the eigenvalues of $A^TA$**, that is, 2 and 1. We always name the singular values in a decreasing order. Thus $\sigma_1 = 2$ and $\sigma_2 = 1$.

In a singular value decomposition $U\Sigma V^T$ of $A$, $\Sigma$ **always has the same size as $A$ and it contains the singular values**. In this case,

$$\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

**To set up $V$ we have to find the eigenvectors of $A^TA$**, these vectors are called *right singular vectors of $A$*. **The matrix $V$ is an orthogonal square matrix of size $m \times m$, where $m$ is equal to the number of columns of $A$.**

$$\text{For } \lambda_1 = 4, \quad A^TA - 4\mathrm{I}_2 = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix} \qquad \begin{cases} a = a \\ -3b = 0, \quad b = 0 \end{cases}$$

$$V(4) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} = \text{span}\{v_1\}.$$

$$\text{For } \lambda_2 = 1, \quad A^TA - 1\mathrm{I}_2 = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{cases} 3a = 0 \quad a = 0 \\ b = b \end{cases}$$

$$V(1) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \text{span}\{v_2\}.$$

As $v_1$ and $v_2$ are unit vectors corresponding with different eigenvalues, they form the orthogonal matrix $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we need.

**To set up $U$ we first note that it is an orthogonal square matrix of size $n \times n$, where $n$ is the number of rows of $A$.** Moreover, **the vectors $u$'s coming from the singular values of $A$ must satisfy** $Av_i = \sigma_i u_i$. Thus,

$$\begin{aligned} u_1 &= \frac{1}{\sigma_1} A v_1 = \frac{1}{2} \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \\ u_2 &= \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \end{aligned}$$

The vectors we obtain following this process are always unit vectors and orthogonal. As in this case $U$ has size $2 \times 2$, we have enough vectors $u$ to construct $U$:

$$U = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The columns of $U$ are called *left singular vectors of* $A$.

Hence a singular value decomposition of $A$ is

$$\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} = U\Sigma V^T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(f) $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

Find the eigenvalues of the matrix $A^T A$:

$$A^T A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}.$$

As $A^T A$ is a diagonal matrix, the eigenvalues are the values in its main diagonal, that is, 4 and 0.

The singular values of $A$ are the square root of the nonzero eigenvalues of $A^T A$. In this case there is only one: $\sigma_1 = \sqrt{4} = 2$.

The size of $\Sigma$ is the same as $A$. It contains a submatrix of size $1 \times 1$ with the singular value and the remaining entries are zero.

$$\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

To set up $V$ we have to find the eigenvectors of $A^T A$. Note that $V$ must be an orthogonal matrix of size $2 \times 2$, the number 2 corresponds with the number of columns of $A$.

$$\text{For } \lambda_1 = 4, \quad A^T A - 4\mathrm{I}_2 = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix} \qquad \left\{ \begin{array}{l} -4a = 0 \quad a = 0 \\ b = b \end{array} \right.$$

$$V(4) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \operatorname{span}\{v_1\}.$$

$$\text{For } \lambda_2 = 0, \quad A^T A - 0\mathrm{I}_2 = A^T A = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \qquad \left\{ \begin{array}{l} a = a \\ 4b = 0 \quad b = 0 \end{array} \right.$$

$$V(0) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} = \operatorname{span}\{v_2\}.$$

As the vectors have norm one and they are orthogonal because they come from different eigenvalues,

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

To set up $U$ we can only use the expression $Av_1 = \sigma_1 u_1$ for the only singular value we found. Note that $U$ must be an orthogonal matrix of size $3 \times 3$, where 3 is the number of rows of $A$.

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{2} \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We only have one vector $u$, we need two more to construct $U$ and they must be unit vectors and be an orthogonal set of vectors. **Here we can proceed in two different ways:**

$a)$ Find the orthogonal space to $u_1$, that is, find $w = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ such that the inner product $w \cdot u_1$ is zero.

$w{\cdot}u_1 = a = 0$, b and c are free variables, $w \in \{u_1\}^\perp = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Note that the spanning set of the orthogonal space to $u_1$ has only unit vectors and with $u_1$ form an orthonormal set of vectors.

**If the three vectors we obtain here are not orthogonal we use the Gram-Schmidt to orthogonalize them so that we can construct** $U$.

Thus, we have $U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

A singular value decomposition of $A$ is

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = U\Sigma V^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

$b)$ Another alternative method is to find the eigenvectors of $AA^T$ that do not correspond with the eigenvalue 4 associated with $u_1$.

$$AA^T = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $AA^T$ are 0 and 4. From 4 we already have the eigenvector. Let us find the eigenspace of eigenvalue 0:

$$AA^T - 0\mathrm{I}_3 = AA^T = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \left\{ \begin{array}{ll} 4a = 0 & a = 0 \\ b = b & \\ c = c & \end{array} \right.$$

Thus, $V(0) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. These vectors are the same as before, they are already unit vectors and orthogonal. The singular value decomposition we obtained is the same as in (1)

Remember that if the total number of vectors $u$'s found using any of the two methods are not ortoghonal, we will have to use the Gram-Schmidt process to orthogonalize them. If you follow the two methods mentioned above, there is no space of invention as happened in the class.