Bachelor in Statistics and Business — Mathematical Methods II
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Name — Student number

Solution: Quiz 1 (Unit 1 and 2)
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Instructions

- You have 90 minutes to answer the quiz.
- Marks per question are given in bold. **Total marks: 10 points**
- The mark of this quiz is part of the 40% corresponding with the continuous evaluation of this course.
- Write your name in all the sheets.
- It is not allowed to use lecture notes, scientific calculators or cellphones during the exam.

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1. Discuss the following systems of linear equations. Whenever is possible solve them by means of the **reduced echelon form**. (Do not forget to specify the row operations, to identify basic and free variables.)

(a) $$\left\{\begin{array}{rrrcl} 6x_1 & +4x_2 & +16x_3 & = & 0 \\ 2x_1 & +4x_2 & +8x_3 & = & 0 \\ -2x_1 & -2x_2 & -6x_3 & = & 0 \end{array}\right.$$ **[1]**

Solution: The augmented matrix of the system is the following one:

$$\begin{pmatrix} 6 & 4 & 16 & 0 \\ 2 & 4 & 8 & 0 \\ -2 & -2 & -6 & 0 \end{pmatrix}$$

Now we do elementary row operations to obtain a matrix in echelon form equivalent to the augmented matrix of the system. Then we will discuss the system.

$$\begin{pmatrix} 6 & 4 & 16 & 0 \\ 2 & 4 & 8 & 0 \\ -2 & -2 & -6 & 0 \end{pmatrix} \underset{\substack{R_1/2 \to R_1 \\ R_2/2 \to R_2 \\ R_3/2 \to R_3}}{\sim} \begin{pmatrix} 3 & 2 & 8 & 0 \\ 1 & 2 & 4 & 0 \\ -1 & -1 & -3 & 0 \end{pmatrix} \underset{\substack{-3R_2 + R_1 \to R_2 \\ 3R_3 + R_1 \to R_3}}{\sim}$$

$$\begin{pmatrix} 3 & 2 & 8 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \underset{R_2 - 4R_3 \to R_3}{\sim} \begin{pmatrix} 3 & 2 & 8 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Discussion of the system: The matrix in echelon form has two pivots, that is, two basic variables and one free variable, x_3. As the last column does not have a pivot and there are less pivots than the number of unknowns, the system is consistent with infinitely many solutions.

To solve the system, as requested in the exercise, we use the matrix in reduced echelon form equivalent to the augmented matrix of the system. To obtain the matrix in reduced echelon form we have to make zeros in all the entries of in the pivot column above the pivot.

$$\begin{pmatrix} 3 & 2 & 8 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{R_2+2R_1 \to R_1}{\sim} \begin{pmatrix} 6 & 0 & 12 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\substack{R_1/6 \to R_1 \\ R_2/(-4) \to R_2}}{\sim} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution of the system is

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3. \end{aligned}$$

where $x_3 \in \mathbb{R}$.

(b) $\left\{ \begin{array}{rrrrrr} x_1 & & +x_3 & & = & 3 \\ 2x_1 & +x_2 & -x_3 & +x_4 & = & 5 \\ 2x_1 & & -x_3 & +x_4 & = & -1 \\ & -x_2 & -x_3 & & = & -1 \end{array} \right.$ **[1]**

Solution: The augmented matrix of the system is the following one:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 & 5 \\ 2 & 0 & -1 & 1 & -1 \\ 0 & -1 & -1 & 0 & -1 \end{pmatrix}$$

Now let us do elementary row operations to obtain a matrix in echelon form equivalent to the augmented matrix:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 & 5 \\ 2 & 0 & -1 & 1 & -1 \\ 0 & -1 & -1 & 0 & -1 \end{pmatrix} \underset{\substack{-2R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3}}{\sim} \begin{pmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -3 & 1 & -1 \\ 0 & 0 & -3 & 1 & -7 \\ 0 & -1 & -1 & 0 & -1 \end{pmatrix} \underset{R_2+R_4 \to R_4}{\sim}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -3 & 1 & -1 \\ 0 & 0 & -3 & 1 & -7 \\ 0 & 0 & -4 & 1 & -2 \end{pmatrix} \underset{4R_3-3R_4 \to R_4}{\sim} \begin{pmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -3 & 1 & -1 \\ 0 & 0 & -3 & 1 & -7 \\ 0 & 0 & 0 & 1 & -22 \end{pmatrix}$$

Discussion of the system: The matrix in echelon form has four pivots, that is, as many as unknowns. Hence there are four basic variables. Moreover, the last column does not have any pivot. Thus the system is consistent with a unique solution.

To solve the system we continue doing elementary row operations to obtain the matrix in reduced echelon form equivalent to the augmented matrix of the system.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -3 & 1 & -1 \\ 0 & 0 & -3 & 1 & -7 \\ 0 & 0 & 0 & 1 & -22 \end{pmatrix} \underset{\substack{-R_2+R_4 \to R_2 \\ -R_3+R_4 \to R_3}}{\sim} \begin{pmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & -1 & 3 & 0 & -21 \\ 0 & 0 & 3 & 0 & -15 \\ 0 & 0 & 0 & 1 & -22 \end{pmatrix} \underset{\substack{-3R_1+R_3 \to R_1 \\ -R_2+R_3 \to R_2}}{\sim}$$

$$\begin{pmatrix} -3 & 0 & 0 & 0 & -24 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 3 & 0 & -15 \\ 0 & 0 & 0 & 1 & -22 \end{pmatrix} \underset{\substack{R_1/(-3) \to R_1 \\ R_3/3 \to R_3}}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -22 \end{pmatrix}$$

The solution of the system is:

$$\begin{aligned} x_1 &= 8, \\ x_2 &= 6, \\ x_3 &= -5, \\ x_4 &= -22. \end{aligned}$$

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2. Let $\begin{pmatrix} 1 & 3 & k \\ 4 & h & 8 \end{pmatrix}$ be the augmented matrix of a system. Determine h and k such that the solution set of this system **[2]**

(a) is empty;

(b) has a unique solution;

(c) contains infinitely many solutions.

Solution:

Firstly, let us obtain a matrix in echelon form equivalent to the given one:

$$\begin{pmatrix} 1 & 3 & k \\ 4 & h & 8 \end{pmatrix} \underset{-4R_1+R_2 \to R_2}{\sim} \begin{pmatrix} 1 & 3 & k \\ 0 & h-12 & 8-4k \end{pmatrix}$$

(a) As the given matrix is the augmented matrix of the system, the solution set is empty if and only if the last column has a pivot. Thus,

$$\begin{aligned} h - 12 = 0 \quad &\text{and} \quad 8 - 4k \neq 0, \\ h = 12 \quad &\text{and} \quad k \neq 2. \end{aligned}$$

(b) That system has a unique solution if and only if the coefficient matrix has two pivots, that is, as many as unknowns. Thus

$$\begin{aligned} h-12 \neq 0 \quad &\text{and} \quad k \in \mathbb{R}, \\ h \neq 12 \quad &\text{and} \quad k \in \mathbb{R}. \end{aligned}$$

(c) That system has infinitely many solutions if and only if the coefficient matrix has less pivots than two and the last column of the augmented matrix does not have a pivot. Hence,

$$\begin{aligned} h-12=0, \quad &\text{and} \quad 8-4k=0, \\ h=12 \quad &\text{and} \quad k=2. \end{aligned}$$

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3. Compute the following determinants developing by a row or a column:

(a) $\begin{vmatrix} 1 & 2 & -1 \\ 4 & 0 & 3 \\ 6 & 4 & 1 \end{vmatrix}$. **[0.5]**

Solution:

Let us compute the determinant as a cofactor expansion across the second row:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & -1 \\ 4 & 0 & 3 \\ 6 & 4 & 1 \end{vmatrix} &= 4(-1)^{2+1} \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} + 3(-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 6 & 4 \end{vmatrix} \\ &= -4(2+4) - 3(4-12) = -24+24 = 0. \end{aligned}$$

(b) $\begin{vmatrix} 9 & 1 & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 9 & 0 & 3 & 9 & 0 \\ 6 & 0 & 0 & 7 & 0 \end{vmatrix}$. **[0.5]**

Solution:

Let us compute the determinant as a cofactor expansion across the second column:

$$\begin{vmatrix} 9 & 1 & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 9 & 0 & 3 & 9 & 0 \\ 6 & 0 & 0 & 7 & 0 \end{vmatrix} = 1(-1)^{1+2} \begin{vmatrix} 9 & 9 & 9 & 2 \\ 4 & 0 & 5 & 0 \\ 9 & 3 & 9 & 0 \\ 6 & 0 & 7 & 0 \end{vmatrix}$$

Now as a cofactor expansion across the fourth column:

$$= (-1)2(-1)^{1+4} \begin{vmatrix} 4 & 0 & 5 \\ 9 & 3 & 9 \\ 6 & 0 & 7 \end{vmatrix}$$

Finally, as a cofactor expansion across the second column:

$$= \quad 2\cdot 3(-1)^{2+2}\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} = 6(28-30) = -12.$$

4. Determine if the following matrices are invertible or not. If they are invertible, compute the inverse of the matrix.

   (a) $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 5 \end{pmatrix}$ (only use row operations to solve this exercise). **[1]**

   Solution:

   Consider the matrix obtained by taking the matrix A and the identity matrix of size 3×3. We do elementary row operations to obtain the identity matrix in the left half of the matrix.

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 5 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \sim \\ R_1 - R_2 \to R_2 \\ R_1 + R_3 \to R_3 \end{matrix} \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 3 & 6 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} \sim \\ -3R_2 + R_3 \to R_3 \end{matrix} \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & -2 & 3 & 1 \end{pmatrix}$$

   As there is a row of zeros in the first half of the matrix, the matrix A is not invertible.

   (b) $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ (do not use row operations to solve this exercise). **[1]**

Solution:

First we compute the determinant of the matrix B.

$$\det B = 1 - 1 - 1 = -1 \neq 0$$

As the determinant is different from zero, the matrix is invertible. Let us compute the cofactor matrix having in mind that $c_{ij} = (-1)^{i+j}A_{ij}$.

$$c_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \qquad c_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1, \qquad c_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1,$$

$$c_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1, \qquad c_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, \qquad c_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1,$$

$$c_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1, \qquad c_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1, \qquad c_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0.$$

Then the cofactor matrix is $C = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$.

Thus the inverse matrix of B is:

$$B^{-1} = \frac{1}{\det B} C^T = \frac{1}{-1} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

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5. Determine all the matrices that commute, with respect to the product, with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
[1.5]

Solution:

A general matrix of size 2×2 is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.$$

The 2×2-matrices that commute with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ must satisfy

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

That is,

$$\begin{pmatrix} a & a+b \\ c & a+d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}.$$

Two matrices are equal if all the entries are equal, that is,

$$\left.\begin{array}{l} a = a + c \\ a + b = b + d \\ c = c \\ c + d = d \end{array}\right\} \Rightarrow \begin{array}{l} c = 0 \\ a = d \\ c = c \\ c = 0 \end{array}$$

Thus all the matrices that commute with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

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6. Answer the following questions and justify your answers.

(a) Describe the solution set of a system with 2 equations and 3 unknowns whose coefficient matrix has two pivots. **[0.25]**

Solution: The coefficient matrix has two pivots, that is, less than the number of unknowns of the system. Moreover, as there are only two equations in the system, the last column of the augmented matrix will never have a pivot. Thus the system is consistent with infinitely many solutions.

$$\begin{pmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \bullet & * & * & * \\ 0 & 0 & \bullet & * \end{pmatrix}$$

(b) Give an example of an inconsistent overdetermined system of linear equations. **[0.25]**

Solution: For instance, three parallel straight lines in $\mathbb{R}^2$:

$$\begin{cases} x+y & = & 0 \\ x+y & = & 2 \\ x+y & = & 3 \end{cases}$$

(c) Show that the partitioned matrix $A = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right)$ satisfies that $A^2 = I$. This kind of matrices are called involutive. **[0.25]**

Solution:

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right) = \left(\begin{array}{c|c} I^2 & 0 \\ \hline 0 & (-I)^2 \end{array}\right) = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) = I.$$

(d) If the determinant of $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is equal to 2, find the determinant of $\begin{pmatrix} a & 5b \\ c & 5d \end{pmatrix}$. **[0.25]**

Solution: Using the properties of the determinant we have:

$$\begin{vmatrix} a & 5b \\ c & 5d \end{vmatrix} = 5 \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 5 \cdot 2 = 10.$$

(e) Let C and D be square $n \times n$ matrices such that $\det C = 3$ and $\det D = 2$. Compute the determinant of $C^T D^2 C^{-1}$. **[0.25]**

Solution: Due to the properties of the determinants:

$$\det(C^T) = \det C = 3, \quad \det(D^2) = (\det D)^2 = 4, \quad \det(C^{-1}) = \frac{1}{\det C} = \frac{1}{3}.$$

Then

$$\det(C^T D^2 C^{-1}) = \det(C^T)\det(D^2)\det(C^{-1}) = 3 \cdot 4 \cdot \frac{1}{3} = 4.$$

(f) Let U be a square $n \times n$ matrix such that $U^T U = \mathrm{I}_n$. Show that $\det U = \pm 1$. **[0.25]**

Solution: Take determinants to both sides of the equality $U^T U = \mathrm{I}_n$ and use the properties of the determinants:

$$\begin{aligned} \det(U^T U) &= \det(I_n), \\ \det(U^T)\det U &= 1, \\ (\det U)^2 &= 1, \\ \det U &= \pm\sqrt{1} = \pm 1. \end{aligned}$$