Name ____________________ Student number ____________________

Solution: Quiz 2 (Units 3 and 4)

November 9, 2011

Instructions

- You have 90 minutes to answer the quiz.
- Marks per question are given in bold. **Total marks: 10 points**
- The mark of this quiz is part of the 40% corresponding with the continuous evaluation of this course.
- Write your name in all the sheets.
- It is not allowed to use lecture notes, scientific calculators or cellphones during the exam.

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1. Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 2 \\ -1 \\ 5 \end{pmatrix}$, $v_4 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ be vectors in $\mathbb{R}^4$.

   (a) Are the vectors v_1, v_2, v_3 and v_4 linearly independent? Why? **[1]**

   (b) Find a basis for the subspace spanned by v_1, v_2, v_3 and v_4. **[0.5]**

Solution: (a) We take the matrix whose columns are the vectors v_1, v_2, v_3 and v_4. We compute the equivalent matrix in row echelon form to identify the dependences among all the columns.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 2 \\ -1 & 0 & -1 & 1 \\ 2 & 3 & 5 & 1 \end{pmatrix} \underset{\substack{R_1+R_3 \to R_3 \\ -2R_1+R_4 \to R_4}}{\sim} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \underset{\substack{-2R_3+R_2 \to R_3 \\ 2R_4+R_2 \to R_4}}{\sim} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix has two pivto columns, less than the number of vectors given. Hence the vectors are linearly dependent.

(b)A basis for the space spanned by v_1, v_2, v_3 and v_4 is given by the columns in the original matrix corresponding with the pivot columns in the matrix in row echelon form, that is, v_1 and v_2. Thus, $\text{span}\{v_1, v_2, v_3, v_4\} = \text{span}\{v_1, v_2\}$.

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2. Let $A = \begin{pmatrix} -2 & 4 & 6 \\ -1 & 2 & 3 \\ 2 & -4 & -6 \end{pmatrix}$ and $w = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$.

   (a) Find a basis for the subspaces $\text{Col}\, A$ and $\text{Nul}\, A$. **[1]**

   (b) Is w in $\text{Nul}\, A$? Is w in $\text{Col}\, A$? **[0.5]**

Solution: (a) Let us compute the row echelon form of the matrix A to find a basis for both the null space and the column space:

$$\begin{pmatrix} -2 & 4 & 6 \\ -1 & 2 & 3 \\ 2 & -4 & -6 \end{pmatrix} \underset{\substack{R_1 - 2R_2 \to R_2 \\ R_1 + R_3 \to R_3}}{\sim} \begin{pmatrix} -2 & 4 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underset{R_1/(-2) \to R_1}{\sim} \begin{pmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The column space of A is spanned by the columns in the matrix A that correspond with the pivots columns in the matrix in row echelon form, that is, $\text{Col}\, A = \text{span}\left\{\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}\right\}$.

The null space of A is given by the solutions to the system $Ax = 0$. From the augmented matrix of the system

$$\begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ we have } \begin{cases} x &= 2y + 3z \\ y &= y \\ z &= z \end{cases}$$

Thus, $\text{Nul}\, A = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}\right\}$.

(b) To check if $w = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ lies in Nul$A$ we just have to compute

$$Aw = \begin{pmatrix} -2 & 4 & 6 \\ -1 & 2 & 3 \\ 2 & -4 & -6 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2+4-6 \\ 1+2-3 \\ -2-4+6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, $w \in \text{Nul}\, A$.

To see if w lies in Col A we must solve the system $Ax = w$. We use the same elementary row operations as in the first part, but fot the augmented matrix of the system $(A\ w)$.

$$\begin{pmatrix} -2 & 4 & 6 & -1 \\ -1 & 2 & 3 & 1 \\ 2 & -4 & -6 & -1 \end{pmatrix} \underset{\substack{R_1 - 2R_2 \to R_2 \\ R_1 + R_3 \to R_3}}{\sim} \begin{pmatrix} -2 & 4 & 6 & -1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -2 \end{pmatrix} \underset{\frac{R_1}{-2} \to R_1}{\sim} \begin{pmatrix} 1 & -2 & -3 & 1/2 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

As the last column of the augmented matrix of the systems is a pivot column, the system $Ax = w$ is inconsistent. Thus w is not in Col A.

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3. Let $\mathcal{B}_1 = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ and $\mathcal{B}_2 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be bases for $\mathbb{R}^2$.

   (a) Find the change-of-basis matrix from $\mathcal{B}_1$ to the standard basis $\mathcal{B}_c$ for $\mathbb{R}^2$ and from $\mathcal{B}_2$ to the standard basis $\mathcal{B}_c$ for $\mathbb{R}^2$. **[0.5]**

   (b) Find the change-of-basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. **[1]**

   (c) Find the coordinate vectors $[x]_{\mathcal{B}_2}$ of $[x]_{\mathcal{B}_1} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ in the basis $\mathcal{B}_2$. **[0.5]**

Solution: (a) Note that $\mathcal{B}_1 = \left\{ \overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{[b_1]_{\mathcal{B}_c}}, \overbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^{[b_2]_{\mathcal{B}_c}} \right\}$. The change-of-basis matrix from $\mathcal{B}_1$ to the standard basis $\mathcal{B}_c$ for $\mathbb{R}^2$ is given by

$$P_{\mathcal{B}_c \mathcal{B}_1} = ([b_1]_{\mathcal{B}_c} [b_2]_{\mathcal{B}_c}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that $\mathcal{B}_2 = \left\{ \overbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}^{[c_1]_{\mathcal{B}_c}}, \overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{[c_2]_{\mathcal{B}_c}} \right\}$. The change-of-basis matrix from $\mathcal{B}_2$ to the standard basis $\mathcal{B}_c$ for $\mathbb{R}^2$ is given by

$$P_{\mathcal{B}_c\mathcal{B}_2} = ([c_1]_{\mathcal{B}_c}[c_2]_{\mathcal{B}_c}) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

(b) The change-of-basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$ is given by

$$P_{\mathcal{B}_2\mathcal{B}_1} = P_{\mathcal{B}_2\mathcal{B}_c} P_{\mathcal{B}_c\mathcal{B}_1} = \left(P_{\mathcal{B}_c\mathcal{B}_2}\right)^{-1} P_{\mathcal{B}_c\mathcal{B}_1}.$$

Thus, we have to compute the inverse matrix of $P_{\mathcal{B}_c\mathcal{B}_2} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ that appeared in the first question.

$$\det P_{\mathcal{B}_c\mathcal{B}_2} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2.$$

Then,

$$\left(P_{\mathcal{B}_c\mathcal{B}_2}\right)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix},$$

because $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Thus,

$$P_{\mathcal{B}_2\mathcal{B}_1} = \left(P_{\mathcal{B}_c\mathcal{B}_2}\right)^{-1} P_{\mathcal{B}_c\mathcal{B}_1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ 1 & -1/2 \end{pmatrix}.$$

(c)

$$[x]_{\mathcal{B}_2} = P_{\mathcal{B}_2\mathcal{B}_1}[x]_{\mathcal{B}_1} = \begin{pmatrix} 0 & 1/2 \\ 1 & -1/2 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

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4. Let $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

   (a) Find the characteristic polynomial of A and the eigenvalues of A. Give the algebraic multiplicity of each eigenvalue. **[0.75]**

(b) Having in mind that the characteristic polynomial of A is $\lambda(\lambda-2)^3$, can the matrix A be diagonalized? Why? Give the geometric multiplicity of each eigenvalue. If A can be diagonalized, give the matrices P and D. **[1.75]**

Solution: (a) To compute the characteristic polynomial of the matrix A we have to compute the following determinant, we will do it as cofactor expansion across the first row and then across the second row:

$$\begin{aligned}
\det(A-\lambda\mathrm{Id}) &= \begin{vmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} \\
&= (2-\lambda)^2\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)^2((1-\lambda)^2-1) \\
&= (2-\lambda)^2(1-2\lambda+\lambda^2-1) = (2-\lambda)^2\lambda(-2+\lambda) = \lambda(\lambda-2)^3.
\end{aligned}$$

The eigenvalues are 0 and 2. The algebraic multiplicities are $n_0 = 1$ and $n_2 = 3$.

(b) We first have to compute the eigenvectors of each eigenvalue to decide if the matrix is diagonalizable or not. For $\lambda = 0$,

$$A - 0I = A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \underset{R_2-2R_4\to R_4}{\sim} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The vectors in the null space of the matrix $A - 0I = A$ verify that $\left\{\begin{array}{c} a = 0 \\ b = -d \\ c = 0 \\ d = d \end{array}\right.$.

Thus $V(0) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right\}$. The geometric multiplicity of $\lambda = 0$ is $1 = m_0$.

For $\lambda = 2$, we have

$$A - 2I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \underset{R_2+R_4\to R_4}{\sim} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The vectors int he null space of $A-2I$ verify that $\begin{cases} a=a \\ b=d \\ c=c \\ d=d \end{cases}$.

Thus $V(2) = \text{Gen}\left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right\}$. The geometric multiplicity of $\lambda = 2$ is $3 = m_2$.

As the algebraic and geometric multiplicities are the same for both eigenvalues, and they add 4, the matrix A is diagonalizable.

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad P = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

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5. Answer the following questions and **always** reason your answers.

   (a) Is the set $W = \{(x,y,z) : 2x+y-z+1=0\}$ a subspace of $\mathbb{R}^3$? Why? **[0.5]**

   Solution: No, it is not a subspace because $(0,0,0)$ is not in W. The point $(0,0,0)$ does not fulfill the equation defining W: $2\cdot 0 + 0 - 0 + 1 = 1 \neq 0$.

   (b) Find the matrix associated with the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (x, -y+z)$. **[0.5]**

   Solution: As T goes from $\mathbb{R}^3$ to $\mathbb{R}^2$, its associated matrix has size 2×3:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

   (c) If the rank of a 3×7 matrix A is 2, what is the dimension of the null space of A? Why? **[0.5]**

   Solution: By Rank Theorem, we know that

$$\text{rank}\, A + \dim \text{Nul}\, A = \text{number of columns of A}.$$

   In this case we have

$$2 + \dim \text{Nul}\, A = 7, \quad \dim \text{Nul}\, A = 7 - 2 = 5.$$

(d) Let A be a 5×5 matrix with two different eigenvalues. If one eigenspace has dimension 3 and the other one has dimension 2, is A diagonalizable? Why? **[0.5]**

Solution: The matrix A is diagonalizable because there exists a basis of egeinvectors for $\mathbb{R}^5$, $3 + 2 = 5$.

(e) Is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 1 & -1 \\ 6 & -4 \end{pmatrix}$? If so, find the eigenvalue. **[0.5]**

Solution: Let us check if the vector $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ satisfies $Av = \lambda v$ for any λ and $A = \begin{pmatrix} 1 & -1 \\ 6 & -4 \end{pmatrix}$:

$$Av = \begin{pmatrix} 1 & -1 \\ 6 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \stackrel{?}{=} \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From $\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we have $\lambda = 0$ and $\lambda = 2$. Thus the system is inconsistent because there does not exist a value λ such that it satisfies both equations in the system. Hence, the vector v is not an eigenvector of the matrix A.