Bachelor in Statistics and Business
Universidad Carlos III de Madrid

Mathematical Methods II
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Name ____________________ Student number ____________________

Quiz 3 (Units 5 and 6)

December 12, 2011

Instructions

- You have 90 minutes to answer the quiz.
- Marks per question are given in bold. **Total marks: 10 points**
- The mark of this quiz is part of the 40% corresponding with the continuous evaluation of this course.
- Write your name in all the sheets.
- It is not allowed to use lecture notes, scientific calculators or cellphones during the exam.
- **The marks will be available on December 21, 2011**.

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1. Let $y = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$, $v_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ be vectors in $\mathbb{R}^3$. Let $W = \text{span}\{v_1, v_2\}$.

   (a) Are the vectors v_1 and v_2 orthogonal? Justify your answer. **[0.5]**

   (b) Find vectors u_1 and u_2 that are in the direction of v_1 and v_2, respectively, and have norm 1. Rationalize vectors u_1 and u_2 if required. **[0.75]**

   (c) Find the orthogonal projection of y onto $W = \text{span}\{v_1, v_2\}$. **[1]**

   (d) Compute the distance from y to the vector space W. **[0.75]**

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2. A professor has observed that there exists some relationship between the distance that a student lives from the campus and the number of times he/she did not attend the lecture along the course. The professor collected the following data for four different students in the class:

Distance in km.	2	3	2	1
Number of absences	3	6	5	1

(a) Describe the model that produces a least-squares fit of the above data by a function of the form $y = \beta_0 + \beta_1 x$ where x is the distance and y is the number of absences. (Give the matrices of the system associated with the data and the straight-line). **[0.5]**

(b) Find β_0 and β_1 so that $y = \beta_0 + \beta_1 x$ is the least-squares line that best fits the above given data points. **[1]**

(c) According to the least-squares line you found in (b), how many absences would a student have if he lives 5 km away from the campus? **[0.25]**

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3. Let $A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix}$.

(a) Find the singular values of the matrix A. **[0.75]**

(b) Find the matrix norm of the matrix A. **[0.5]**

(c) Find a singular value decomposition of the matrix A. **[1.5]**

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4. Answer the following questions and **always** justify your answers.

(a) Let $u = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $v = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$. Compute $\dfrac{u \cdot v}{v \cdot v}$ and $\dfrac{v \cdot u}{u \cdot u}$. **[0.5]**

(b) Let $u = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $v = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$. Use the Gram-Schmidt process to produce an **orthogonal** basis for the subspace spanned by u and v. **[0.5]**

(c) If A is a symmetric matrix of size $n \times n$ and B is a matrix of size $n \times m$, prove that $B^T AB$ is a symmetric matrix. Indicate at each step the matrix operation properties you use. **[0.5]**

(d) Is the matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & -2 \\ 0 & 1 & 5 \end{pmatrix}$ orthogonal? Justify your answer. **[0.5]**

(e) Plot the image of the unit sphere under the linear transformation given by the matrix $C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$. **[0.5]**