Mathematical Methods II
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Solution: Quiz 3 (Units 5 and 6)

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Instructions

- You have 90 minutes to answer the quiz.
- Marks per question are given in bold. **Total marks: 10 points**
- The mark of this quiz is part of the 40% corresponding with the continuous evaluation of this course.
- Write your name in all the sheets.
- It is not allowed to use lecture notes, scientific calculators or cellphones during the exam.
- **The marks will be available on December 21, 2011**.

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1. Let $y = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$, $v_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ be vectors in $\mathbb{R}^3$. Let $W = \text{span}\{v_1, v_2\}$.

   (a) Are the vectors v_1 and v_2 orthogonal? Justify your answer. **[0.5]**

   (b) Find vectors u_1 and u_2 that are in the direction of v_1 and v_2, respectively, and have norm 1. Rationalize vectors u_1 and u_2 if required. **[0.75]**

   (c) Find the orthogonal projection of y onto $W = \text{span}\{v_1, v_2\}$. **[1]**

   (d) Compute the distance from y to the vector space W. **[0.75]**

Solution:

(a) To check if two vectors are orthogonal we have to compute the inner product of v_1 and v_2:

$$v_1 \cdot v_2 = v_1^T v_2 = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0 + 1 - 1 = 0.$$

As the inner product of v_1 and v_2 is zero, the vectors v_1 and v_2 are orthogonal.

(b) In this exercise we have to normalize the vectors v_1 and v_2. We first need to compute the norm of each of them:

$$\|v_1\| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}, \quad \|v_2\| = \sqrt{0^2 + 1^2 + (-1)^2} = \sqrt{2}.$$

Then,

$$\begin{aligned} u_1 &= \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{6}} v_1 = \frac{\sqrt{6}}{6} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \sqrt{6} \begin{pmatrix} 1/3 \\ 1/6 \\ 1/6 \end{pmatrix}, \\ u_2 &= \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}} v_2 = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \end{pmatrix}. \end{aligned}$$

(c) As the spanning set of the subspacel W are orthogonal, the orthogonal projection of y onto W is computed as follows:

$$\text{proj}_W y = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{8}{6} v_1 + \frac{4}{2} v_2 = \begin{pmatrix} 8/3 \\ 10/3 \\ -2/3 \end{pmatrix}.$$

(d) To compute the distance between y and W we have to find the norm of the following vector

$$y - \text{proj}_W y = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 8/3 \\ 10/3 \\ -2/3 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix}.$$

Hence the distance is

$$\|y - \text{proj}_W y\| = \left\| \begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix} \right\| = \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{12}{9}} = \frac{2}{3}\sqrt{3}.$$

2. A professor has observed that there exists some relationship between the distance that a student lives from the campus and the number of times he/she did not attend the lecture along the course. The professor collected the following data for four different students in the class:

Distance in km.	2	3	2	1
Number of absences	3	6	5	1

(a) Describe the model that produces a least-squares fit of the above data by a function of the form $y = \beta_0 + \beta_1 x$ where x is the distance and y is the number of absences. (Give the matrices of the system associated with the data and the straight-line). **[0.5]**

(b) Find β_0 and β_1 so that $y = \beta_0 + \beta_1 x$ is the least-squares line that best fits the above given data points. **[1]**

(c) According to the least-squares line you found in (b), how many absences would a student have if he lives 5 km away from the campus? **[0.25]**

Solution:

(a) The matrices of the system of linear equation that will be solved by least-squares are the following one:

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 6 \\ 5 \\ 1 \end{pmatrix}.$$

The system to be solved is $A \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = b$.

(b) To solve the least-squares problem, we have to solve the normal equations, that is, $A^T A\widehat{x} = A^T b$. Let us find the matrices appearing in that system:

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 8 & 18 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 35 \end{pmatrix}$$

Let us consider the augmented matrix and compute its reduced echelon form:

$$\begin{pmatrix} 4 & 8 & 15 \\ 8 & 18 & 35 \end{pmatrix} \underset{-2R_1+R_2 \to R_2}{\sim} \begin{pmatrix} 4 & 8 & 15 \\ 0 & 2 & 5 \end{pmatrix} \underset{-4R_2+R_1 \to R_1}{\sim} \begin{pmatrix} 4 & 0 & -5 \\ 0 & 2 & 5 \end{pmatrix}$$

$$\underset{\substack{R_1/4 \to R_1 \\ R_2/2 \to R_2}}{\sim} \begin{pmatrix} 1 & 0 & -5/4 \\ 0 & 1 & 5/2 \end{pmatrix}$$

Hence, $\widehat{x} = \begin{pmatrix} -5/4 \\ 5/2 \end{pmatrix}$. The best straight line approximation of the data given is $y = -\frac{5}{4} + \frac{5}{2}x$.

(c) To know how many absences a student who lives 5 km away from campus will have we just have to substitute in the straight line equation x by 5,

$$y = -\frac{5}{4} + \frac{5}{2}5 = \frac{45}{4} = 11,25.$$

Thus, the student would have 11,25 absences.

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3. Let $A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix}$.

(a) Find the singular values of the matrix A. **[0.75]**

(b) Find the matrix norm of the matrix A. **[0.5]**

(c) Find a singular value decomposition of the matrix A. **[1.5]**

Solution:

(a) To find the singular values we have to compute the eigenvalues of the matrix A^TA:

$$A^TA = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 1 \\ 1 & 11 \end{pmatrix}.$$

The characteristic polynomial of A^TA is

$$\begin{vmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{vmatrix} = (11-\lambda)^2 - 1 = \lambda^2 - 22\lambda + 120.$$

The roots of such polynomial are:

$$\lambda = \frac{22 \pm \sqrt{22^2 - 4 \cdot 120}}{2} = \frac{22 \pm 2}{2} = \begin{cases} 12 \\ 10 \end{cases}$$

Hence the singular values of $A^T A$ are $\sqrt{12} = 2\sqrt{3}$ and $\sqrt{10}$.

(b) The Euclidean norm of the matrix A is the largest singular value of A. In this case

$$\|A\|_2 = \sqrt{12} = 2\sqrt{3}.$$

(c) To find a singular value decomposition of A we first compute the eigenvectors of $A^T A$. For $\lambda = 12$ we have to compute the null space of

$$A^T A - \lambda \mathrm{I}_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \underset{R_1 + R_2 \to R_2}{\sim} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

$$\text{Then, } \begin{cases} -a + b = 0 \\ b = b \end{cases} \Rightarrow \begin{cases} a = b \\ b = b \end{cases}.$$

Hence, $V(12) = \mathrm{Gen}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \mathrm{span}\{w_1\}$.

For $\lambda = 10$ we have to find the null space of the following matrix

$$A^T A - \lambda \mathrm{I}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \underset{-R_1 + R_2 \to R_2}{\sim} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

$$\text{Then, } \begin{cases} a + b = 0 \\ b = b \end{cases} \Rightarrow \begin{cases} a = -b \\ b = b \end{cases}.$$

Hence $V(10) = \mathrm{span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} = \mathrm{span}\{w_2\}$.

We have to normalize the vectors to have an orthogonal matrix V for the singular value decomposition of A.

$$v_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Thus, $V = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$.

To find the other orthogonal matrix U we first obtain vectors u from the singular values as follows:

$$u_1 = \frac{1}{\sigma_1}Av_1 = \frac{1}{\sqrt{12}}\begin{pmatrix}3 & -1\\ 1 & 3\\ 1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2\sqrt{6}}\begin{pmatrix}2\\4\\2\end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix},$$

$$u_2 = \frac{1}{\sigma_2}Av_2 = \frac{1}{\sqrt{10}}\begin{pmatrix}3 & -1\\ 1 & 3\\ 1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} = \frac{1}{2\sqrt{5}}\begin{pmatrix}-4\\2\\0\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\\0\end{pmatrix}.$$

We need to extend u_1 and u_2 to an orthonormal basis of $\mathbb{R}^3$. We can follow three different methods:

i. Find a vector $w = \begin{pmatrix}a\\b\\c\end{pmatrix}$ in $\mathbb{R}^3$ orthogonal to u_1 and u_2. We obtain a system of two equations with three unknowns by imposing the two conditions of orthogonality:

$$\left.\begin{aligned} w\cdot u_1 &= \frac{1}{\sqrt{6}}(a+2b+c)=0\\ w\cdot u_2 &= \frac{1}{\sqrt{5}}(-2a+b)=0\end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} a+2b+c=0\\ -2a+b=0\end{aligned}\right.$$

The augmented matrix of the system is:

$$\begin{pmatrix}1 & 2 & 1 & 0\\ -2 & 1 & 0 & 0\end{pmatrix} \underset{2R_1+R_2\to R_2}{\sim} \begin{pmatrix}1 & 2 & 1 & 0\\ 0 & 5 & 2 & 0\end{pmatrix} \underset{R_1-(2/5)R_2\to R_1}{\sim} \begin{pmatrix}1 & 0 & 1/5 & 0\\ 0 & 5 & 2 & 0\end{pmatrix}$$

Thus $w = \begin{pmatrix}-c/5\\-2c/5\\c\end{pmatrix}$. If we take $c=5$, we have $w = \begin{pmatrix}-1\\-2\\5\end{pmatrix}$. The norm of this vector is $\|w\| = \sqrt{(-1)^2+(-2)^2+5^2} = \sqrt{30}$.
If we normalize w, we obtain u_3:

$$u_3 = \frac{q_3}{\|q_3\|} = \frac{q_3}{\sqrt{\frac{1}{36}+\frac{1}{9}+\frac{25}{36}}}q_3 = \frac{6}{6\sqrt{30}}\begin{pmatrix}-1\\-2\\5\end{pmatrix} = \frac{1}{\sqrt{30}}\begin{pmatrix}-1\\-2\\5\end{pmatrix}.$$

ii. Another option is to compute the eigenvectors of AA^T for the eigenvalues have not been used yet:

$$AA^T = \begin{pmatrix}3 & -1\\ 1 & 3\\ 1 & 1\end{pmatrix}\begin{pmatrix}3 & 1 & 1\\ -1 & 3 & 1\end{pmatrix} = \begin{pmatrix}10 & 0 & 2\\ 0 & 10 & 4\\ 2 & 4 & 2\end{pmatrix}.$$

The characteristic polynomial of AA^T is

$$\begin{vmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{vmatrix} = (10-\lambda)^2(2-\lambda)-4(10-\lambda)-16(10-\lambda) = (10-\lambda)\lambda(\lambda-12).$$

The eigenvalues of AA^T are 12, 10, 0. Compute the eigenvector associated with the eigenvalue 0:

$$AA^T - 0\mathrm{I}_2 = AA^T = \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix} \underset{\substack{\text{Divide all the} \\ \text{rows by 2}}}{\sim} \begin{pmatrix} 5 & 0 & 1 \\ 0 & 5 & 2 \\ 1 & 2 & 1 \end{pmatrix} \underset{R_1 - 5R_3 \to R_3}{\sim}$$

$$\begin{pmatrix} 5 & 0 & 1 \\ 0 & 5 & 2 \\ 0 & -10 & -4 \end{pmatrix} \underset{2R_2 + R_3 \to R_3}{\sim} \begin{pmatrix} 5 & 0 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\text{Then, } \begin{cases} 5a + c = 0 \\ 5b + 2c = 0 \\ c = c \end{cases} \Rightarrow \begin{cases} a = -\dfrac{1}{5}c \\ b = -\dfrac{2}{5}c \\ c = c \end{cases}.$$

$$\text{Hence } V(0) = \text{span}\left\{ \begin{pmatrix} -1/5 \\ 2/5 \\ 1 \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \right\} = \text{span}\{w_3\}.$$

Normalize the vector w_3 to obtain the last column of U:

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{1}{\sqrt{30}} q_3 = \begin{pmatrix} -1/\sqrt{30} \\ -2/\sqrt{30} \\ 5/\sqrt{30} \end{pmatrix}.$$

iii. The last method is to use the Gramd-Schmidt process. But we first need a basis for $\mathbb{R}^3$ that contains u_1 and u_2. For example, $\left\{ u_1, u_2, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a set of linear independent vectors. From this set we can obtain an orthonormal set by taking $q_1 = u_1$, $q_2 = u_2$ and

$$q_3 = e_3 - \text{proj}_{\{u_1,u_2\}} e_3 = e_3 - \frac{e_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{e_3 \cdot u_2}{u_2 \cdot u_2} u_2 = e_3 - \frac{1}{\sqrt{6}} u_1 - 0 u_2 = \begin{pmatrix} -1/6 \\ -1/3 \\ 5/6 \end{pmatrix}.$$

Normalize q_3 to obtain u_3:

$$u_3 = \frac{q_3}{\|q_3\|} = \frac{q_3}{\sqrt{\dfrac{1}{36} + \dfrac{1}{9} + \dfrac{25}{36}}} q_3 = \frac{6}{6\sqrt{30}} \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{30}} \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}.$$

With the three methods we obtain the same singular valued decomposition of A:

$$A = U\Sigma V^T = \begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{5} & -1/\sqrt{30} \\ 2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \end{pmatrix} \begin{pmatrix} 2\sqrt{3} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

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4. Answer the following questions and **always** justify your answers.

(a) Let $u = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $v = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$. Compute $\frac{u \cdot v}{v \cdot v}$ and $\frac{v \cdot u}{u \cdot u}$. **[0.5]**

Solution:

$$\frac{u \cdot v}{v \cdot v} = \frac{-4+12}{16+36} = \frac{8}{52} = \frac{2}{13},$$
$$\frac{v \cdot u}{u \cdot u} = \frac{8}{1+4} = \frac{8}{5}.$$

(b) Let $u = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $v = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$. Use the Gram-Schmidt process to produce an **orthogonal** basis for the subspace spanned by u and v. **[0.5]**

Solution: Take $w_1 = u$ and

$$w_2 = v - \text{proj}_{w_1} v = v - \frac{v \cdot w_1}{w_1 \cdot w_1} w_1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} - \frac{8}{5} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 28/5 \\ 14/5 \end{pmatrix}.$$

(c) If A is a symmetric matrix of size $n \times n$ and B is a matrix of size $n \times m$, prove that $B^T AB$ is a symmetric matrix. Indicate at each step the matrix operation properties you use. **[0.5]**

Solution: The matrix $B^T AB$ is symmetric if $(B^T AB)^T = B^T AB$. Having in mind the sizes of A and B, it is straightforward that the matrix $B^T AB$ is square. Hence, it makes sense to study the symmetry of the matrix. Note that

$$(B^T AB)^T = B^T A^T (B^T)^T = B^T AB,$$

because $(AB)^T = B^T A^T$, as A is symmetric $A^T = A$ and $(B^T)^T = B$. Hence $B^T AB$ is a symmetric matrix.

(d) Is the matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & -2 \\ 0 & 1 & 5 \end{pmatrix}$ orthogonal? Justify your answer. **[0.5]**

Solution: The matrix A is orthogonal if all the columns are orthonormal. Let $a_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $a_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $a_3 = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$, we have to compute the following inner products:

$$a_1 \cdot a_2 = 2 - 2 + 0 = 0, \quad a_1 \cdot a_3 = -2 + 2 + 0, \quad a_2 \cdot a_3 = -1 - 4 + 5 = 0,$$
$$a_1 \cdot a_1 = 5 \neq 1.$$

(e) Plot the image of the unit sphere under the linear transformation given by the matrix $C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$. **[0.5]**

Solution: Compute the image by the matrix C of the standard basis for $\mathbb{R}^3$,

$$C\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$
$$C\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$
$$C\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, the image of the unit sphere under that transformation is an ellipse in $\mathbb{R}^2$ with center at $(0, 0)$ and semiaxis of length 2 and 1. The semiaxis are on the axis OX and OY.