Digital Communications

Telecommunications Engineering

Chapter 2

Pulse amplitude (linear) modulations

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Pulse amplitude (linear) modulations

- Linear modulation in a N-dimensional signal space

$$s(t) = \sum_n \sum_{j=0}^{N-1} A_j[n] \cdot \phi_j(t - nT)$$

  - Information is linearly conveyed
    - ⋆ In the amplitude of the set of functions $\{\phi_j(t)\}_{j=0}^{N-1}$
  - Encoder: $\boldsymbol{A}[n]$
    - ⋆ Constellation in a space of dimension N
    - ⋆ Designed considering energy (E_s) and performance (P_e, BER)
      - E_s: mean energy per symbol ($E_s = E[|A[n]|^2]$)
      - P_e: probability of symbol error
      - BER: bit error rate
  - Modulator: $\{\phi_j(t)\}_{j=0}^{N-1}$
    - ⋆ Designed considering channel characteristics
    - ⋆ Ideally: the only distortion appearing after the transmission is additive noise (white and Gaussian)

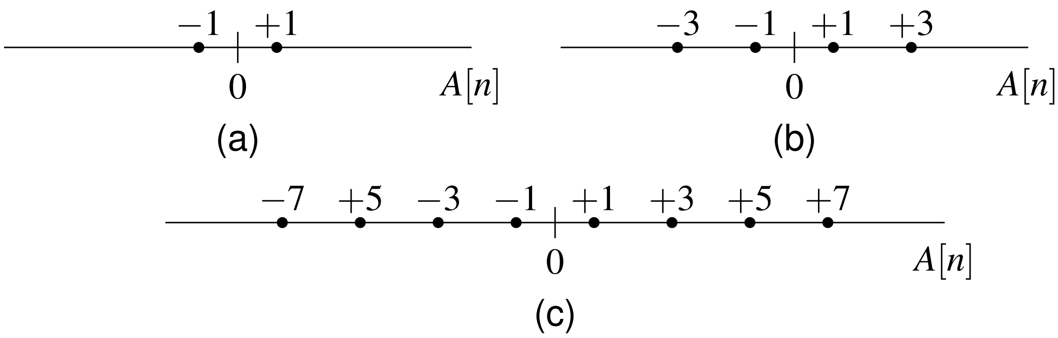
Baseband PAM modulation

- One-dimensional modulation: $N = 1$

$$s(t) = \sum_n A[n] \cdot g(t - nT)$$

PAM (*Pulse Amplitude Modulation*)
ASK (*Amplitude Shift Keying*)

- Constellations - Normalized: $A[n] \in \{\pm 1, \pm 3, \cdots, \pm(M-1)\}$
Examples: 2-PAM (a), 4-PAM (b), 8-PAM (c)



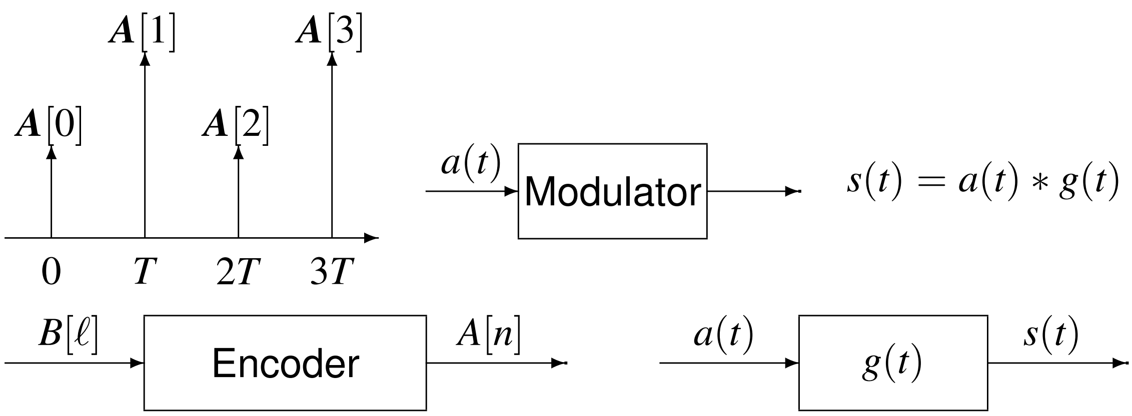
PAM modulation as a filtering process

- Signal of symbols: impulses with amplitudes $A[n]$

$$a(t) = \sum_n A[n] \cdot \delta(t - nT)$$

- Generation of PAM signal

$$s(t) = a(t) * g(t)$$



Selection of $g(t)$ waveforms

- Waveform $g(t)$ tipically receives two names:
  - Transmitter filter
  - Shaping pulse (although it is not necessarily a pulse)
- Selection to be able of identify sequence $A[n]$ from $s(t)$
  - Pulses with duration limited to symbol period T
    - ★ No overlapping between waveforms delayed nT seconds

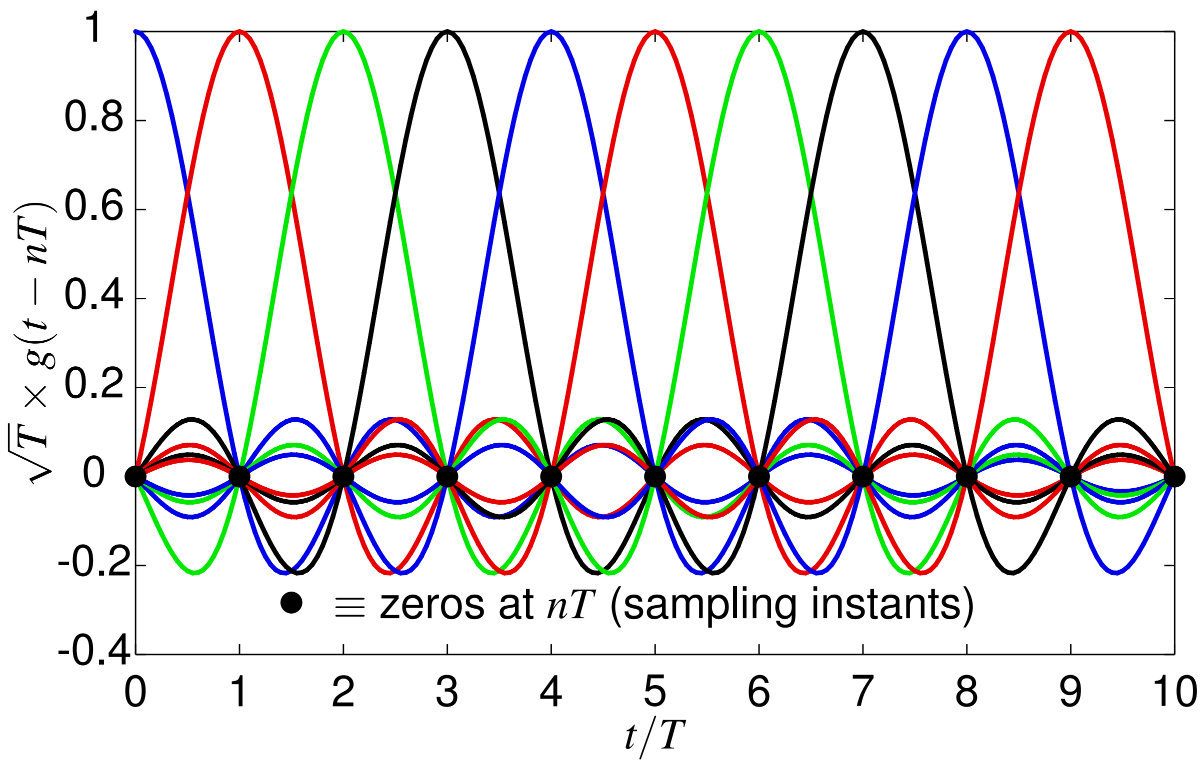
$$g_a(t) = \frac{1}{\sqrt{T}} \cdot \Pi\left(\frac{t}{T}\right)$$

    - ★ Symbol $A[n]$ determines signal amplitude in its associated symbol interval
  - Pulses with higher length
    - ★ Overlapping: non-destructive interference at some point each T seconds

$$g_b(t) = \frac{1}{\sqrt{T}} \cdot \text{sinc}\left(\frac{t}{T}\right)$$

    - ★ Symbol $A[n]$ determines signal amplitude at the nondestructive point associated to its symbol interval

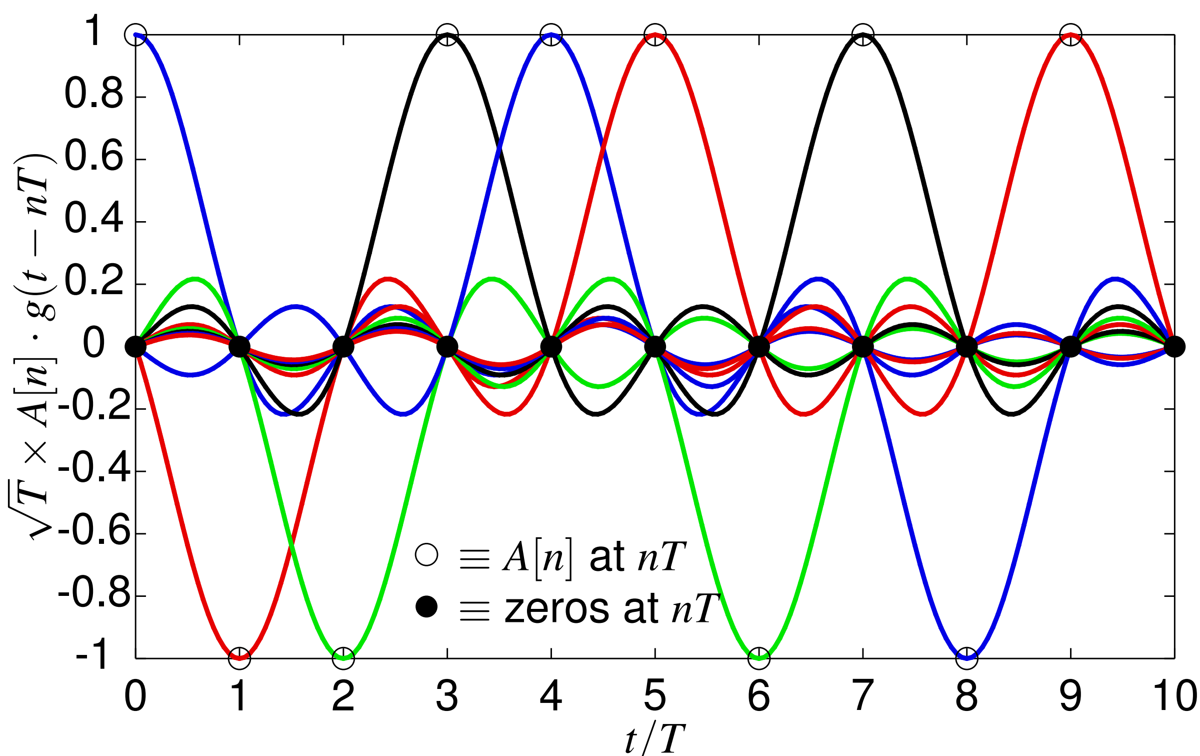
Sinc pulses - Pulses shifted T seconds



Sinc pulses - Contribution of each symbol

- Sequence:

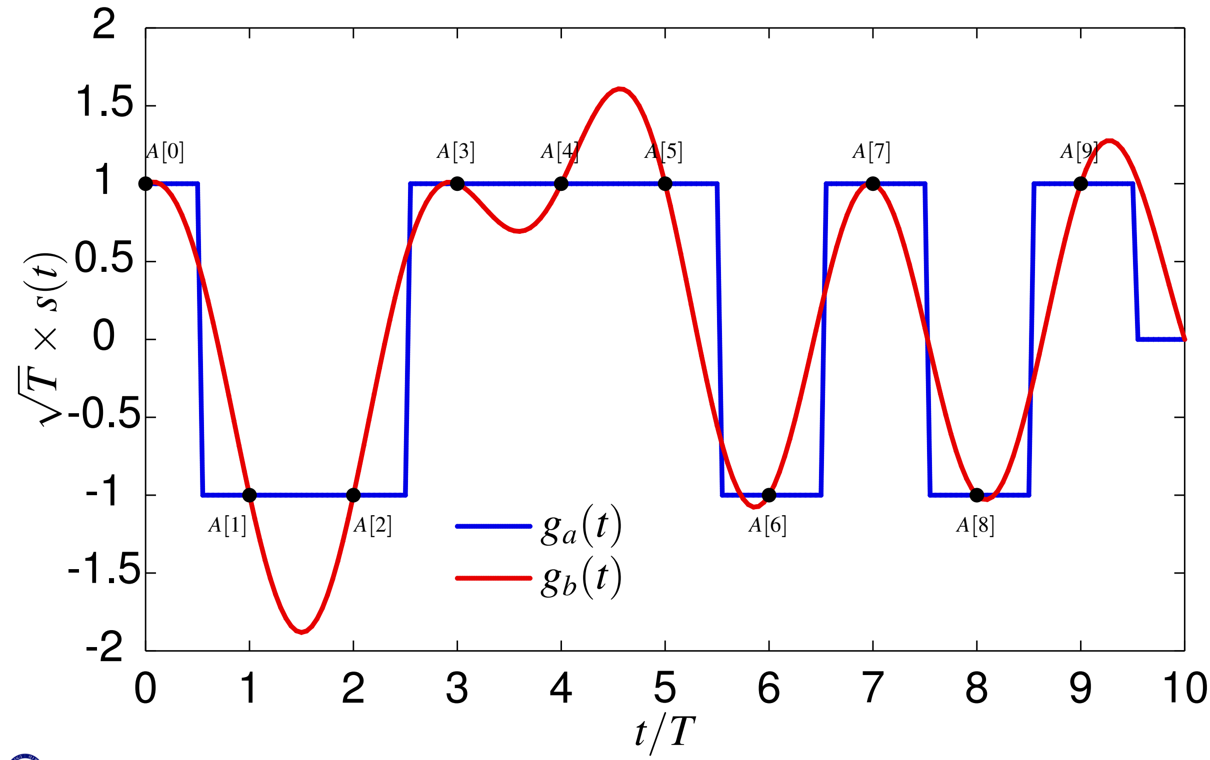
n	0	1	2	3	4	5	6	7	8	9
$A[n]$	$+1$	-1	-1	$+1$	$+1$	$+1$	-1	$+1$	-1	$+1$



Sinc pulses - Waveform

- Sequence:

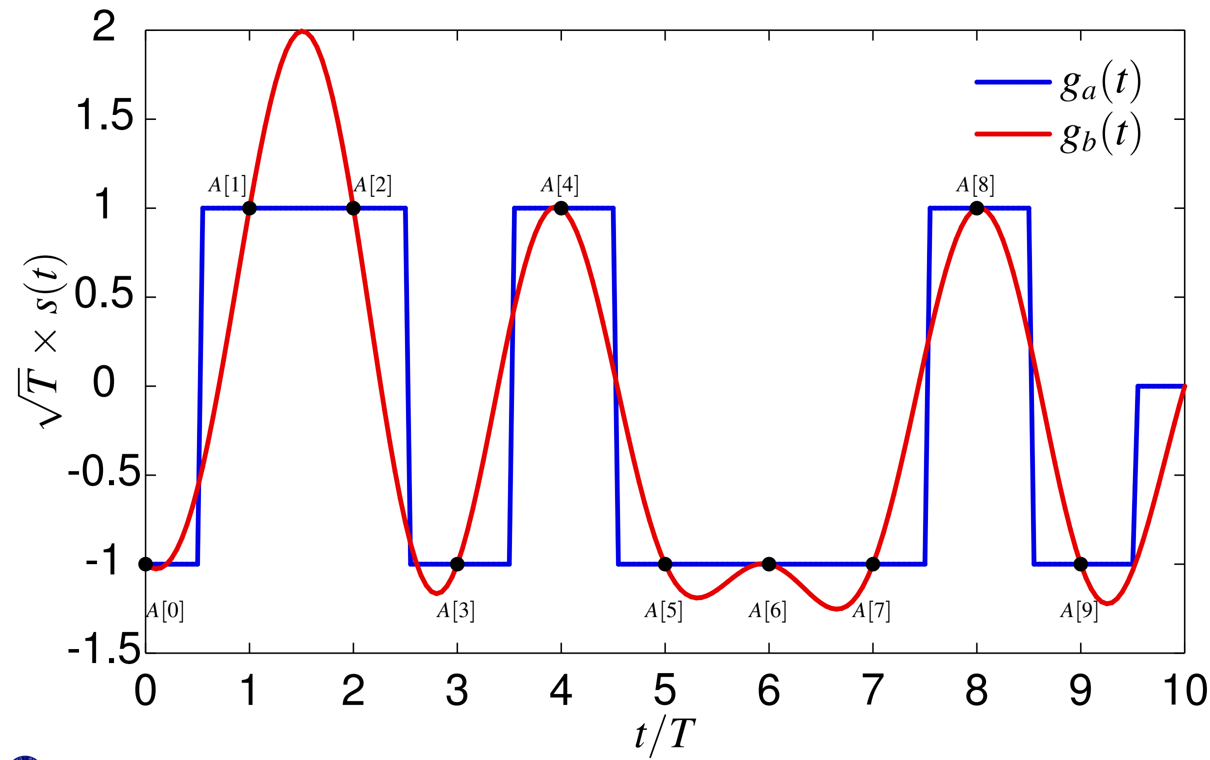
n	0	1	2	3	4	5	6	7	8	9
$A[n]$	$+1$	-1	-1	$+1$	$+1$	$+1$	-1	$+1$	-1	$+1$



Waveforms - Another example

- Sequence:

n	0	1	2	3	4	5	6	7	8	9
$A[n]$	-1	$+1$	$+1$	-1	$+1$	-1	-1	-1	$+1$	-1



Spectrum of a baseband PAM

- PAM baseband signal

$$s(t) = \sum_n A[n] \cdot g(t - nT)$$

- Let $\{A[n]\}_{n=-\infty}^{\infty}$ be a sequence of random variables (stationary random process):
  - $E[A[n]] = m$
  - $E[|A[n]|^2] = E_s$
  - $E[A[k] \cdot A^*[j]] = R_A[k-j] = R_A[j-k]$
  - Power spectral density function of $A[n]$ is

$$S_A(e^{j\omega}) = \sum_{n=-\infty}^{\infty} R_A[n] \cdot e^{-j\omega n}$$

- Let $g(t)$ be any deterministic function with Fourier transform $G(jw)$

Review: Wiener-Khinchin theorem

- Power spectral density

$$S_X(j\omega) \stackrel{def}{=} E\left[\lim_{T\to\infty} \frac{|X_T(j\omega)|^2}{T}\right] = \lim_{T\to\infty} \frac{E[|X_T(j\omega)|^2]}{T}.$$

- Wiener-Khinchin theorem
  If for any finite value τ ans any interval $\mathcal{A}$, of length $|\tau|$, the autocorrelationof random process fulfills

$$\left|\int_{\mathcal{A}} R_X(t+\tau, t)dt\right| < \infty,$$

  power spectral density of $X(t)$ is given by the Fourier transform of

$$< R_X(t+\tau, t) > \stackrel{def}{=} \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t+\tau, t) \cdot dt.$$

Corollary of Wiener-Khinchin theorem

- Corollary 1: If $X(t)$ is an stationary process and $\tau R_X(\tau) < \infty$ for all $\tau < \infty$, then
$$S_X(j\omega) = TF[R_X(\tau)].$$
- Corollary 2: If $X(t)$ is cyclostationary and
$$\left| \int_0^{T_o} R_X(t+\tau, t) dt \right| < \infty,$$
then
$$S_X(j\omega) = TF[\widetilde{R}_X(\tau)],$$
where
$$\widetilde{R}_X(\tau) = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} R_X(t+\tau, t) \cdot dt,$$
and T_o is the period of the cyclostationary process.

Mean and autocorrelation of a baseband PAM

- Definition of random process for PAM signal
$$X(t) = \sum_{n=-\infty}^{\infty} A[n] g(t - nT)$$
- Mean of random process $X(t)$
$$m_X(t) = E\left[\sum_n A[n] g(t-nT)\right] = \sum_n E[A[n]] g(t-nT) = m \sum_n g(t-nT)$$
- Autocorrelation function of random process $X(t)$
$$\begin{aligned} R_X(t, t+\tau) &= E[X(t) X^*(t+\tau)] \\ &= E\left[\left(\sum_k A[k] g(t-kT)\right)\left(\sum_j A^*[j] g^*(t+\tau-jT)\right)\right] \\ &= \sum_k \sum_j E[A[k] A^*[j]] g(t-kT) g^*(t+\tau-jT) \\ &= \sum_k \sum_j R_A[k-j] g(t-kT) g^*(t+\tau-jT) \end{aligned}$$

Cyclostationarity

- Mean is a periodical function of t (period T)

$$m_X(t+T) = m\sum_n g(t+T-nT) = m\sum_n g(t-(n-1)T)$$
$$= m\sum_j g(t-jT) = m_X(t)$$

- Autocorrelation is a periodical function of t (period T)

$$R_X(t+T, t+\tau+T) =$$
$$= \sum_k\sum_j R_A[k-j]g(t+T-kT)g^*(t+T+\tau-jT)$$
$$= \sum_k\sum_j R_A[k-j]g(t-(k-1)T)g^*(t-(j-1)T+\tau)$$
$$= \sum_m\sum_n R_A[m+1-(n+1)]g(t-mT)g^*(t-nT+\tau)$$
$$= \sum_m\sum_n R_A[m-n]g(t-mT)g^*(t-nT+\tau) = R_X(t,t+\tau)$$

Time average of autocorrelation function

$$\tilde{R}_X(\tau) = \frac{1}{T}\int_0^T R_X(t,t+\tau)dt$$
$$= \frac{1}{T}\int_0^T \sum_k\sum_j R_A[k-j]g(t-kT)g^*(t+\tau-jT)dt$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_A[-m]\int_0^T g(t-kT)g^*(t+\tau-(k+m)T)dt$$
$$= \frac{1}{T}\sum_{m=-\infty}^{\infty} R_A[m]\sum_{k=-\infty}^{\infty}\int_{-kT}^{-(k-1)T} g(u)g^*(u+\tau-mT)du$$
$$= \frac{1}{T}\sum_{m=-\infty}^{\infty} R_A[m]\int_{-\infty}^{\infty} g(u)g^*(u+\tau-mT)du$$
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} R_A[n]r_g(nT-\tau),$$

$$r_g(t) = g(t)*g^*(-t)$$

Power spectral density (PSD)

$$\begin{aligned}\tilde{R}_X(\tau) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} R_A[n]\cdot r_g(nT-\tau)\\ &= \frac{1}{T}r_g(\tau) * \sum_{n=-\infty}^{\infty} R_A[n]\cdot\delta(\tau-nT)\\ &= \frac{1}{T}\cdot g(\tau) * g^*(-\tau) * \sum_{n=-\infty}^{\infty} R_A[n]\cdot\delta(\tau-nT)\end{aligned}$$

$$\begin{aligned}S_X(j\omega) &= \mathcal{FT}\left\{\tilde{R}_X(\tau)\right\}\\ &= \frac{1}{T}\cdot G(j\omega)\cdot G^*(j\omega)\cdot\sum_{n=-\infty}^{\infty} R_A[n]\cdot e^{-j\omega nT}\\ &= \frac{1}{T}\cdot|G(j\omega)|^2\cdot S_A(e^{j\omega T})\end{aligned}$$

Power spectral density (II)

$$S_s(j\omega) = \frac{1}{T}\cdot S_A(e^{j\omega T})\cdot|G(j\omega)|^2$$

- Three contributions:
  - A constant factor scale: $\frac{1}{T} = R_s$ bauds
  - A deterministic component given by $g(t)$: $|G(j\omega)|^2$
  - A statistical component given by $A[n]$: $S_A(e^{j\omega})$
- For white sequences $A[n]$ (the most typical case)

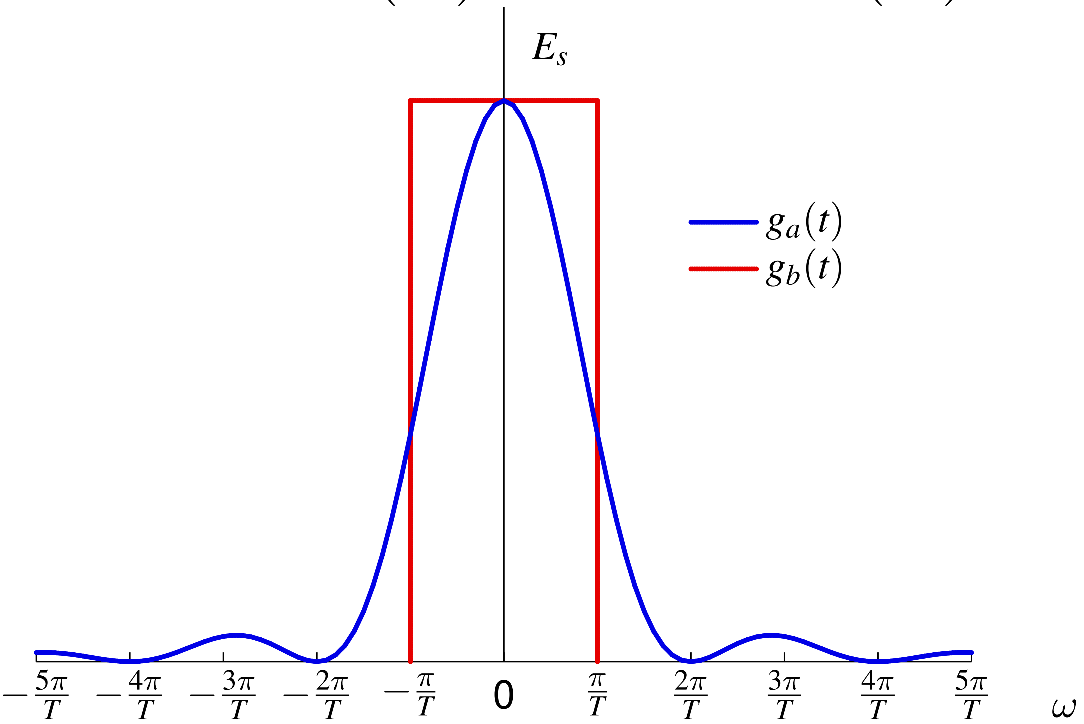
$$R_A[n] = E_s\cdot\delta[n],\qquad S_A(e^{j\omega}) = E_s = E\left\{|A[n]|^2\right\}$$

$$S_s(j\omega) = \frac{E_s}{T}\cdot|G(j\omega)|^2$$

  - $g(t)$: Shaping pulse (determines the shape of spectrum)

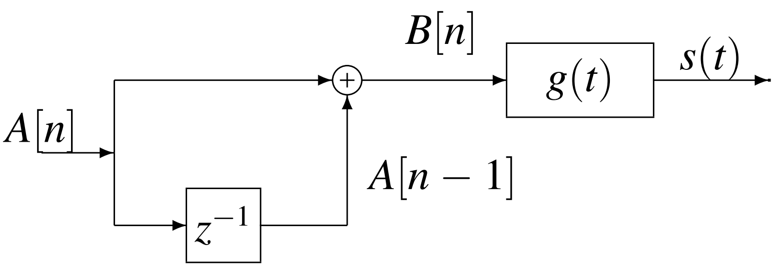
Spectrum for white sequences - PSDs

$$G_a(j\omega) = \sqrt{T} \cdot \text{sinc}\left(\frac{\omega T}{2\pi}\right), \quad G_b(j\omega) = \sqrt{T} \cdot \Pi\left(\frac{\omega T}{2\pi}\right)$$



PSD for coloured data sequence

- PSD shape can be modified by introducing correlation in the sequence



- White sequence $A[n]$: 2-PAM ($A[n] \in \{\pm 1\}$)
  - Mean energy per symbol: $E_s = E\left[|A[n]|^2\right] = 1$
- Coloured sequence

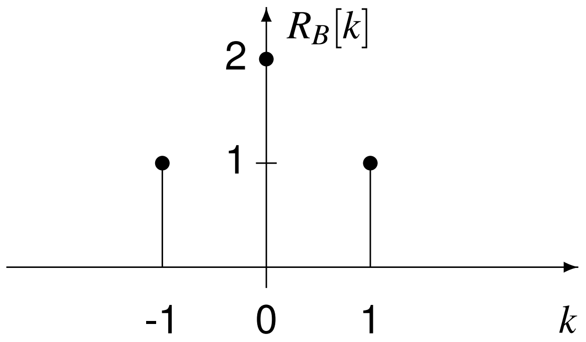
$$B[n] = A[n] + A[n-1]$$

$$s(t) = \sum_{n=-\infty}^{\infty} B[n] \cdot g(t - nT)$$

Autocorrelation function of $B[n]$

- Autocorrelation of $A[n]$: $R_A[k] = E_s \cdot \delta[k] = \delta[k]$
- Autocorrelation function of $B[n]$

$$\begin{aligned} R_B[k] =& E\left[B[n]B^*[n+k]\right] \\ =& E\left[(A[n]+A[n-1])\cdot(A[n+k]+A[n+k-1])\right] \\ =& E\left[A[n]A[n+k]\right]+E\left[A[n]A[n+k-1]\right] \\ &+E\left[A[n-1]A[n+k]\right]+E\left[A[n-1]A[n+k-1]\right] \\ =& R_A[k]+R_A[k-1]+R_A[k+1]+R_A[k] \\ =& 2R_A[k]+R_A[k-1]+R_A[k+1] \end{aligned}$$



Power spectral density

- PSD for sequence $B[n]$

$$\begin{aligned} S_B\left(e^{j\omega}\right) =& \mathcal{FT}\left\{R_B[k]\right\} = \sum_k R_B[k]\cdot e^{-j\omega k} \\ =& 2\cdot e^{j\omega\cdot 0}+e^{j\omega\cdot 1}+e^{-j\omega\cdot 1} \\ =& 2\cdot[1+\cos(\omega)] \end{aligned}$$

- PSD for baseband PAM signal $s(t)$

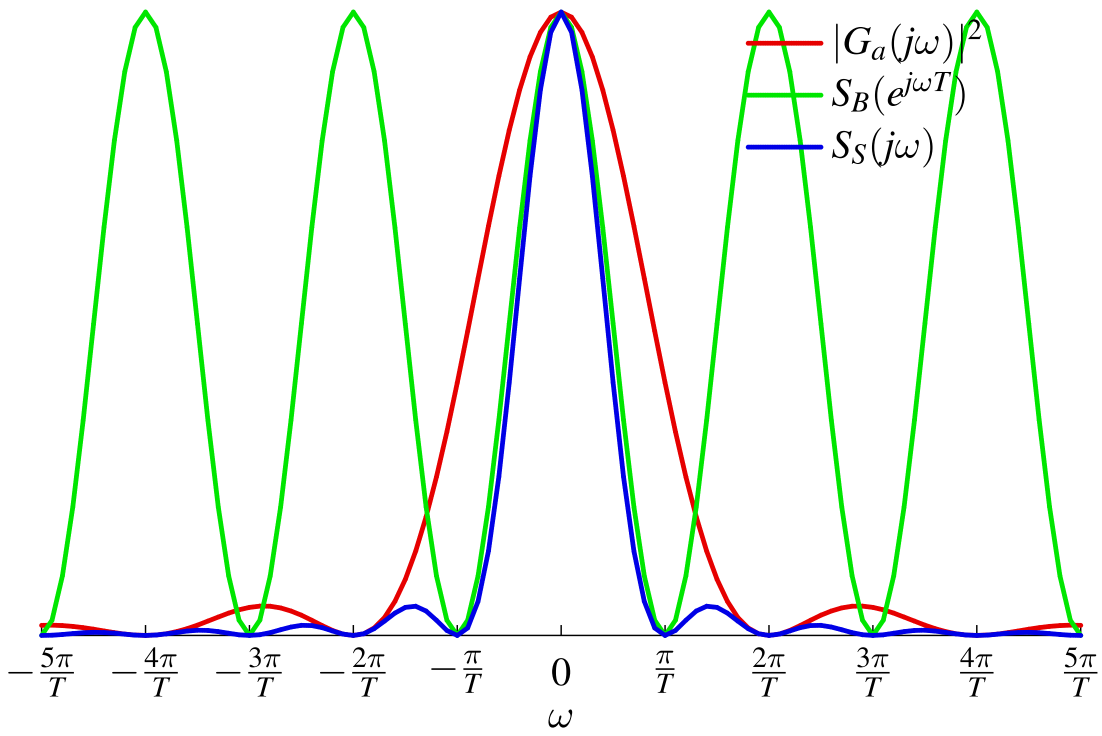
  This system transmits data sequence $B[n]$

$$S_S(j\omega) = \frac{1}{T}\cdot S_B(e^{j\omega T})\cdot|G(j\omega)|^2$$

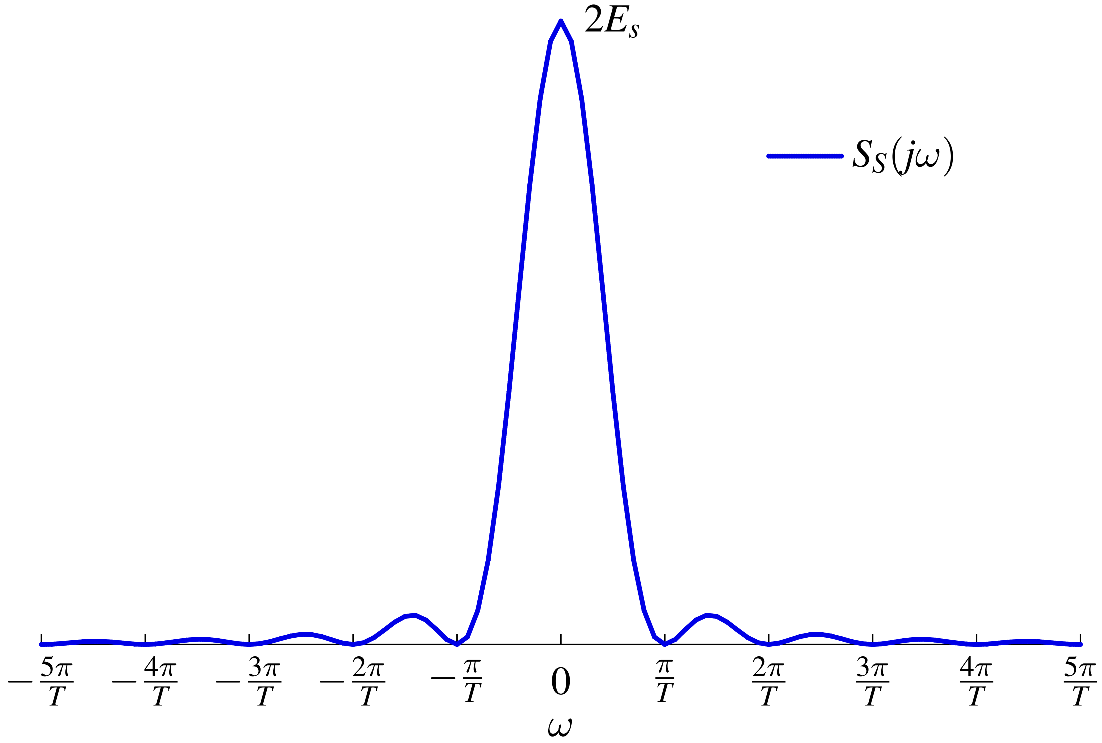
Evaluating the previously obtained expression for $S_B(e^{j\omega})$ in ωT we have

$$S_S(j\omega) = \frac{2}{T}\left[1+\cos(\omega T)\right]\cdot|G(j\omega)|^2$$

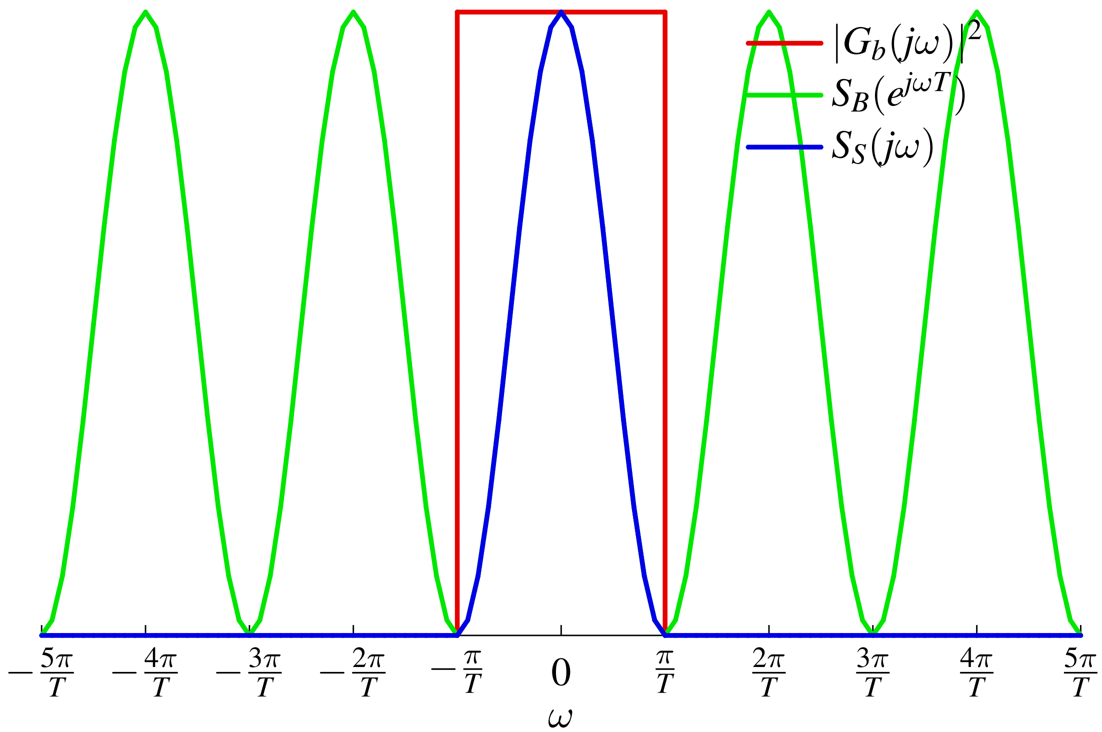
Power spectral density with $g_a(t)$



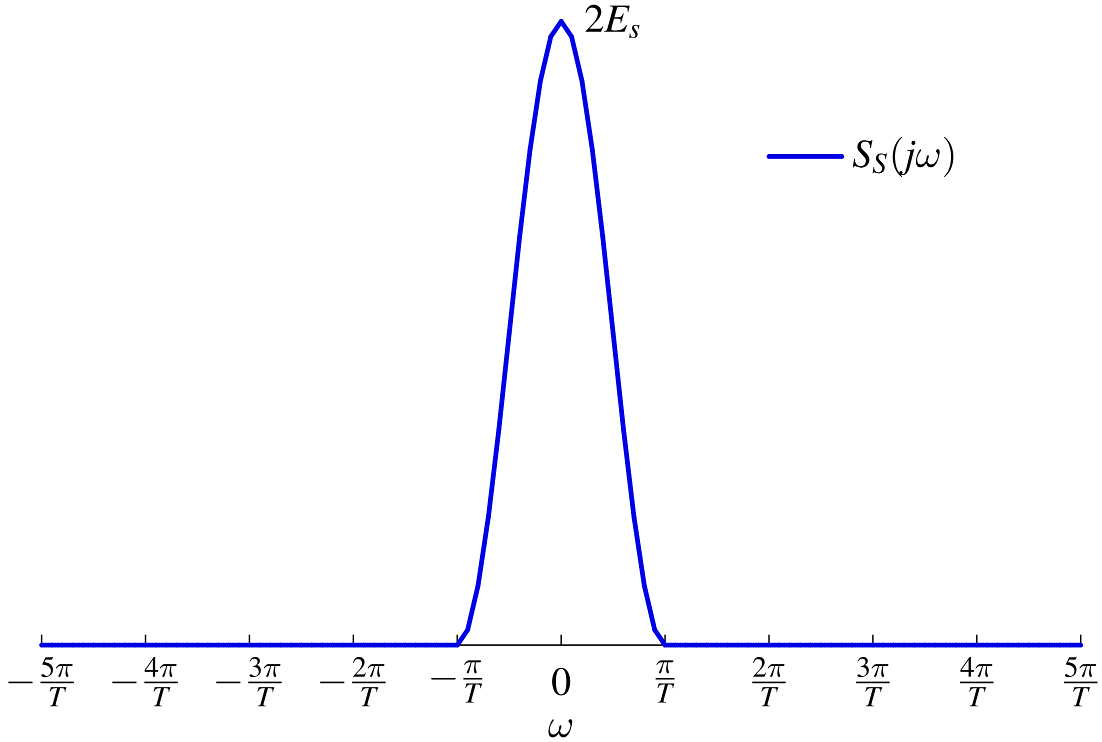
Power spectral density with $g_a(t)$



Power spectral density with $g_b(t)$



Power spectral density with $g_b(t)$



Power of a baseband PAM modulation

- Power can be obtained from $S_s(j\omega)$

$$P_S = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_s(j\omega)\; d\omega$$

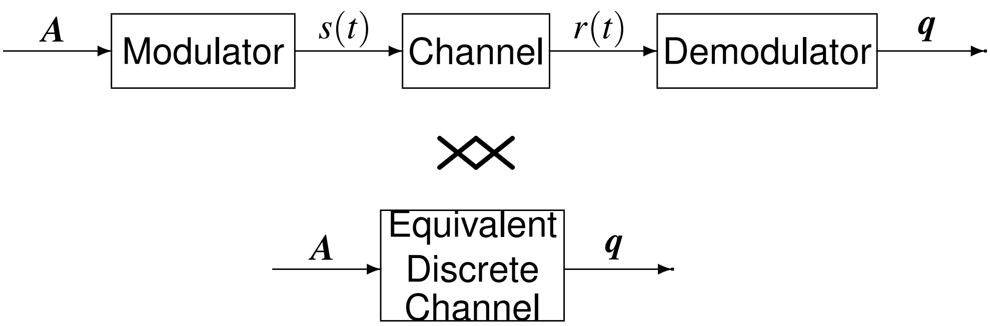
- For white symbol sequences $A[n]$

$$P_S = \frac{E_s}{T}\cdot \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2\; d\omega}_{\mathcal{E}\{g(t)\}}$$

  - If $g(t)$ is normalized, by applying Parseval's relationship

$$P_S = \frac{E_s}{T} = E_s \times R_s \text{ Watts}$$

Equivalent discrete channel

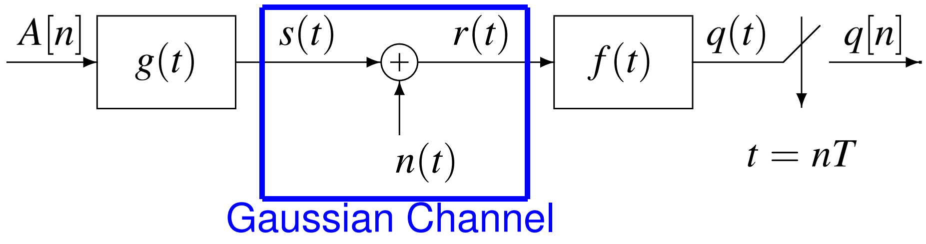


- Provides the discrete time expression for observations at the output of the demodulator $\boldsymbol{q}[n]$ as a function of the transmitted sequence $\boldsymbol{A}[n]$
  - In ideal systems: $\boldsymbol{q}[n] = \boldsymbol{A}[n] + \boldsymbol{n}[n]$
    Gaussian distributions for observations (conditioned to $\boldsymbol{A}[n] = \boldsymbol{a}_i$)

$$f_{\boldsymbol{q}[n]|\boldsymbol{A}[n]}(\boldsymbol{q}|\boldsymbol{a}_i) = \frac{1}{(\pi N_o)^{N/2}} e^{-\frac{||\boldsymbol{q}-\boldsymbol{a}_i||^2}{N_0}}$$

- Expressions will now be obtained for two channel models
  - Gaussian channel
  - Linear channel

Transmission of PAM signals over Gaussian channels



- Gaussian channel model
  - Distortion during transmission is limited to noise addition

$$r(t) = s(t) + n(t)$$

$n(t)$: stationary random process, white, Gaussian, zero mean, $S_n(j\omega) = N_0/2$

- Receiver filter $f(t)$
  - Typical set up: matched filter

$$f(t) = g^*(-t) = g(-t), \text{ because } g(t) \text{ is real}$$

- Signal at the input of the sampler

$$q(t) = s(t) * f(t) + n(t) * f(t)$$

Equivalent discrete channel for Gaussian channels

- Signal before sampling

$$q(t) = \underbrace{\left(\overbrace{\sum_k A[k] \cdot g(t-kT)}^{s(t)}\right) * f(t)}_{\text{Noiseless output } o(t)} + \underbrace{n(t) * f(t)}_{\text{Filtered noise } z(t)}$$

$$o(t) = \sum_k A[k] \cdot \left(g(t-kT) * f(t)\right) = \sum_k A[k] \cdot p(t-kT)$$

- $p(t) = g(t) * f(t)$: joint transmitter-receiver response
  - This joint response determines the noiseless output at the receiver
- Observation at demodulator output

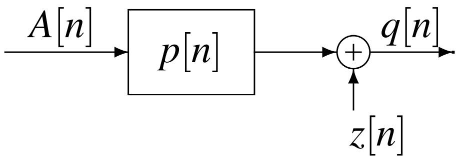
$$q[n] = q(t)|_{t=nT} = \sum_k A[k] \cdot p\left((n-k)T\right) + z(nT)$$

Equivalent discrete channel for Gaussian channels (II)

- Definition of equivalent discrete channel $p[n]$

$$p[n] = p(t)\big|_{t=nT} = o[n] + z[n]$$

$$\text{Noiseless output } o[n] = \sum_k A[k] \cdot p[n-k] = A[n] * p[n]$$



- Ideal: $p[n] = \delta[n] \rightarrow q[n] = A[n] + z[n]$
- Real: Intersymbol interference (ISI)

$$q[n] = A[n] \cdot p[0] + \sum_{\substack{k \\ k \neq n}} A[k] \cdot p[n-k] + z[n]$$

$$\text{ISI } = \sum_{\substack{k \\ k \neq n}} A[k] \cdot p[n-k]$$

Effect of ISI - Extended constellation

- ISI produces an extended constellation at the receiver side

Values of noiseless discrete output $o[n] = A[n] * p[n]$

- Example: 2-PAM modulation

Channel A

$p[n] = \delta[n] + \frac{1}{4}\delta[n-1]$

$o[n] = A[n] + \frac{1}{4}A[n-1]$

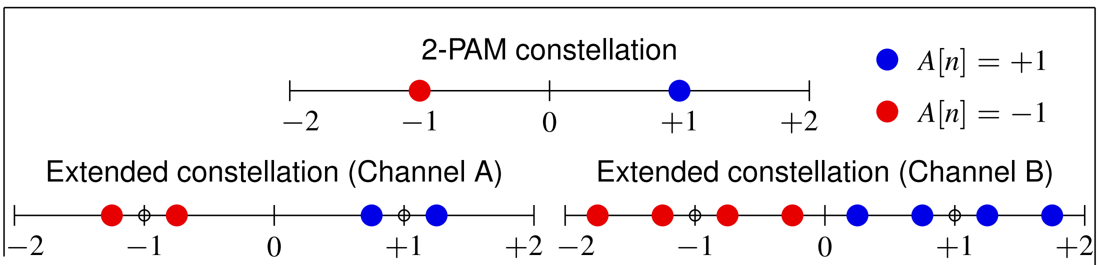
$A[n]$	$A[n-1]$	$o[n]$
+1	+1	$+\frac{5}{4}$
+1	−1	$+\frac{3}{4}$
−1	+1	$-\frac{3}{4}$
−1	−1	$-\frac{5}{4}$

Channel B

$p[n] = \delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]$

$o[n] = A[n] + \frac{1}{2}A[n-1] + \frac{1}{4}A[n-2]$

$A[n]$	$A[n-1]$	$A[n-2]$	$o[n]$
+1	+1	+1	$+\frac{7}{4}$
+1	+1	−1	$+\frac{5}{4}$
+1	−1	+1	$+\frac{3}{4}$
+1	−1	−1	$+\frac{1}{4}$
−1	+1	+1	$-\frac{1}{4}$
−1	+1	−1	$-\frac{3}{4}$
−1	−1	+1	$-\frac{5}{4}$
−1	−1	−1	$-\frac{7}{4}$



Joint transmitter receiver response $p(t)$

- Response $p(t)$ determines the ISI behavior
  - Noiseless output depends on the value of $p[n]$, which is obtained by sampling at symbol rate the joint transmitter-receiver response $p(t)$
- Usual receiver: matched filter $f(t) = g^*(-t)$

$$p(t) = g(t) * g^*(-t) \equiv r_g(t)$$

$r_g(t)$: continuous autocorrelation of $g(t)$ (or temporal ambiguity function of $g(t)$)

Some properties of the continuous autocorrelation function

- Definition for deterministic finite energy function $x(t)$

$$r_x(t) = x(t) * x^*(-t)$$

Informally: measure of similarity between a function and itself with a delay t

- Expresión in the frequency domain

$$R_x(j\omega) = \mathcal{FT}\{r_x(t)\} = \mathcal{FT}\{x(t)\} \times \mathcal{FT}\{x^*(-t)\}$$
$$= X(j\omega) \cdot X^*(j\omega) = |X(j\omega)|^2$$

- Maximum value is at $t = 0$: $|r_x(0)| \geq |r_x(t)|$
- Energy of the signal

$$\text{Parseval: } \mathcal{E}\{x(t)\} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

Using the continuous autocorrelation function (temporal ambiguity func.)

$$\mathcal{E}\{x(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(j\omega) \, d\omega \rightarrow \boxed{\mathcal{E}\{x(t)\} = r_x(0)}$$

Nyquist criterion for zero ISI

- Conditions for avoiding ISI expressed in the time domain

$$p[n] = p(t)\Big|_{t=nT} = \delta[n]$$

- Equivalent condition in the frequency domain

$$P\left(e^{j\omega}\right) = 1$$

- Equivalent continuous-time expressions

$$p(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT) = \delta(t)$$

$$P(j\omega) * \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(j\omega - j\frac{2\pi}{T}k\right) = 1$$

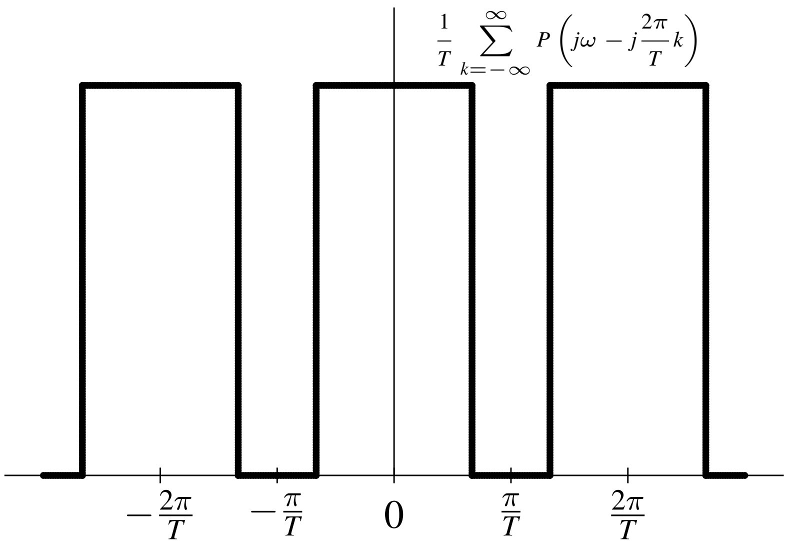
$$\frac{1}{T} \sum_{k=-\infty}^{\infty} P\left(j\omega - j\frac{2\pi}{T}k\right) = 1$$

  - Replicas of $P(j\omega)$ displaced multiples of $\frac{2\pi}{T}$ sum a constant

Application: band-limited pulses

- Example using a badwidth $W < \frac{\pi}{T}$ rad/s (or $B < \frac{1}{2T} = \frac{R_s}{2}$ Hz)
  - Simplest choice for $P(j\omega)$: squared pulse

$$P(j\omega) = \Pi\left(\frac{\omega}{2W}\right) = \begin{cases} 1 & |\omega| < W = 2\pi B \\ 0 & |\omega| > W = 2\pi B \end{cases}$$



  - Impossible to fulfill Nyquist criterion for $W < \frac{\pi}{T}$ rad/s

Application: band-limited pulses (II)

- Nyquist ISI criterion for squared pulses: only pulses with

$$W = n \cdot \frac{\pi}{T} = n \cdot \pi \cdot R_s \text{ rad/s} \quad \left(B = n \cdot \frac{R_s}{2} \text{ Hz} \right)$$

- Which in the time domain means that

$$p(t) = \text{sinc}\left(n \frac{t}{T} \right)$$

- Relationship bandwidth / transmission rate: optimal $p(t)$
  - Minimum bandwidth to transmit without ISI at rate $R_s = \frac{1}{T}$ bauds

$$W_{min} = \frac{\pi}{T} = \pi \cdot R_s \text{ rad/s} \quad \left(B_{min} = \frac{R_s}{2} \text{ Hz} \right)$$

  - Maximun rate without ISI through a bandwidth W rad/s (B Hz)

$$R_s\big|_{max} = \frac{W}{\pi} = 2 \cdot B \text{ bauds (symbols/s)}$$

  - Optimal pulses

$$p(t) = \text{sinc}\left(\frac{t}{T} \right), \; P(j\omega) = T \cdot \Pi\left(\frac{\omega T}{2\pi} \right)$$

Raised cosine pulses

- Family of bandlimited pulses : parameter $\alpha \in [0, 1]$: roll-off factor
  - Particular case $\alpha = 0$ is a sinc function
- Expression for pulses in time domain

$$h_{RC}^{\alpha,T}(t) = \left(\frac{\sin(\pi t/T)}{\pi t/T} \right) \left(\frac{\cos(\alpha \pi t/T)}{1 - (2\alpha t/T)^2} \right)$$

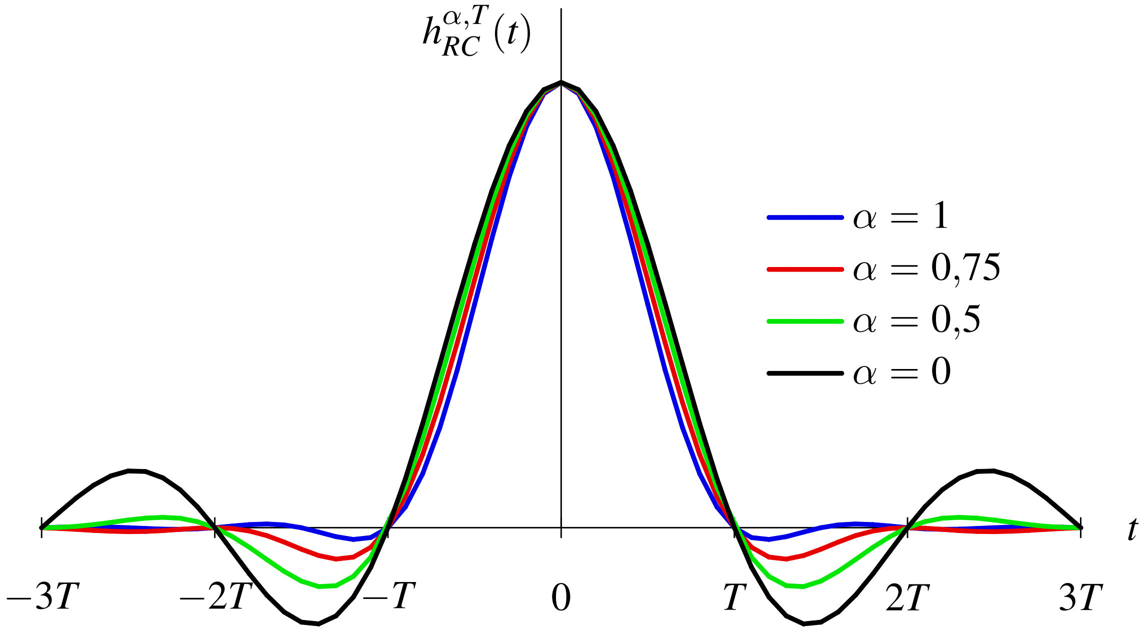
- Fourier transform

$$H_{RC}^{\alpha,T}(j\omega) = \begin{cases} T & 0 \leq |\omega| < (1-\alpha)\frac{\pi}{T} \\ \frac{T}{2}\left[1 - \sin\left(\frac{T}{2\alpha}\left(|\omega| - \frac{\pi}{T} \right) \right) \right] & (1-\alpha)\frac{\pi}{T} \leq |\omega| \leq (1+\alpha)\frac{\pi}{T} \\ 0 & |\omega| > (1+\alpha)\frac{\pi}{T} \end{cases}$$

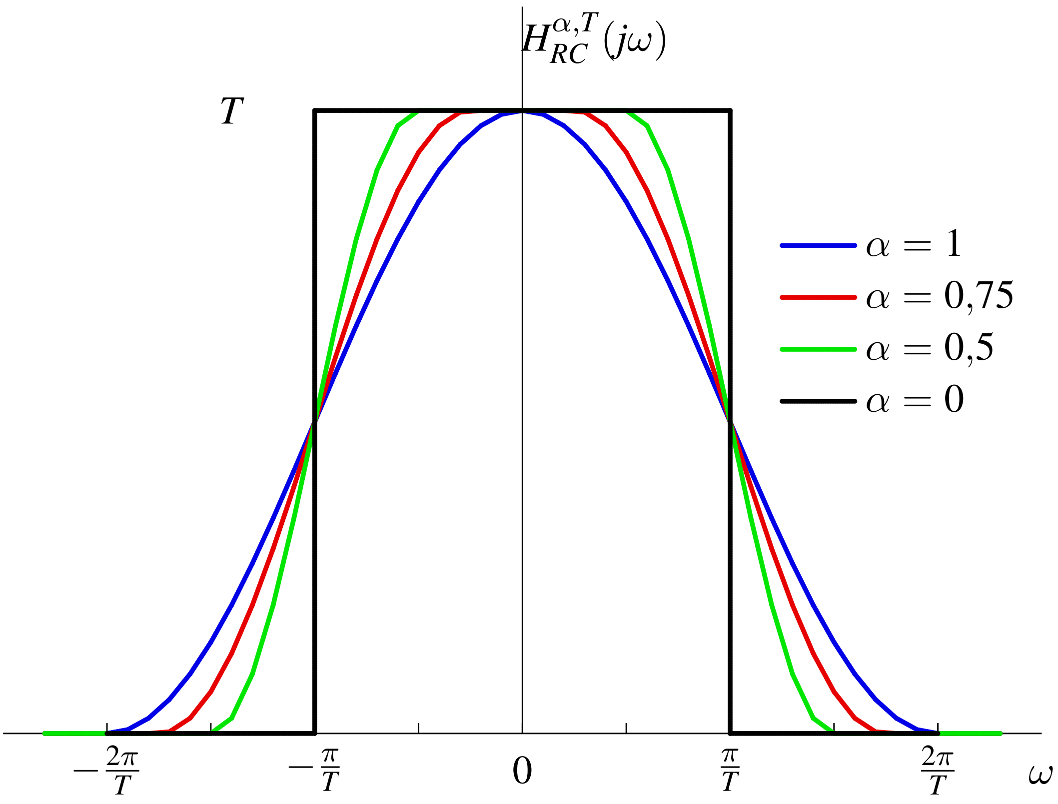
- Bandwidth for a transmission rate depends on the roll-off factor

$$W = (1+\alpha) \cdot \frac{\pi}{T} \text{ rad/s}, \quad B = (1+\alpha) \cdot \frac{R_s}{2} \text{ Hz}$$

Raised cosine pulses: $h_{RC}^{\alpha,T}(t)$



Raised cosine pulses: $H_{RC}^{\alpha,T}(j\omega)$



Root-raised cosine pulses

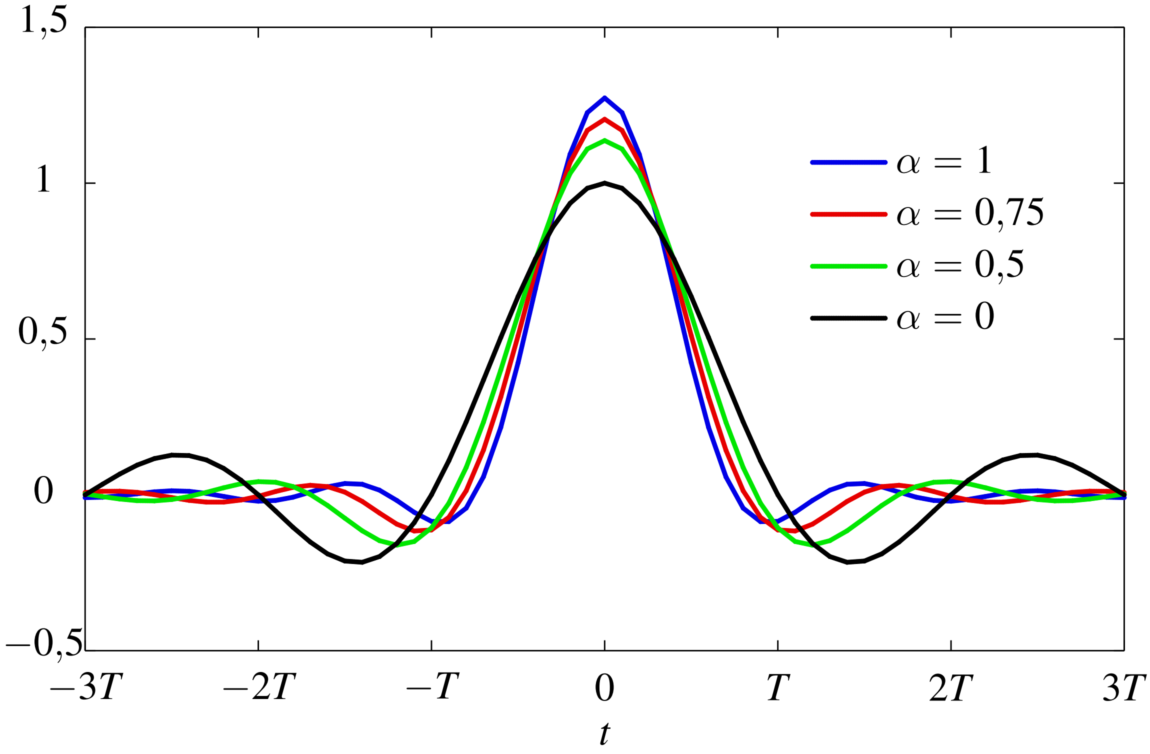
- Filter whose convolution is a raised cosine

$$h_{RRC}^{\alpha,T}(t) * h_{RRC}^{\alpha,T}(t) = h_{RC}^{\alpha,T}(t),\ H_{RRC}^{\alpha,T}(j\omega) \cdot H_{RRC}^{\alpha,T}(j\omega) = H_{RC}^{\alpha,T}(j\omega)$$

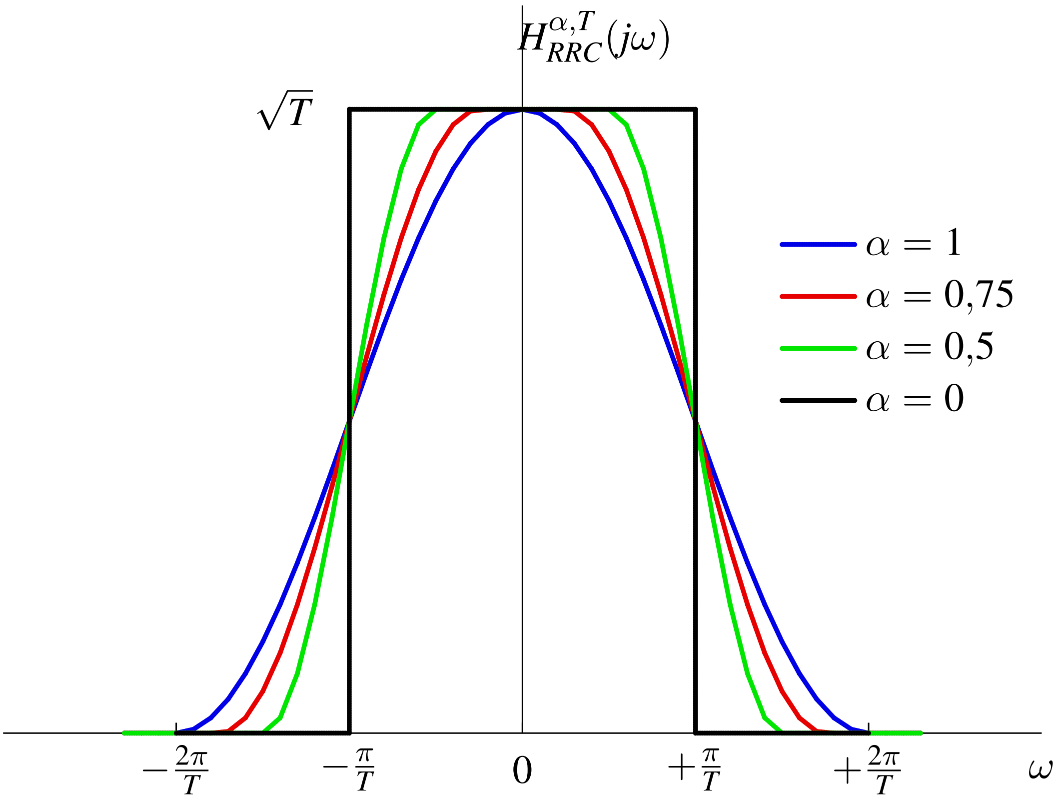
- General procedure to obtain transmission filter $h_{RRC}(t)$
  1. Design in frequency domain from $H_{RC}^{\alpha,T}(j\omega)$
  2. Divide in two contributions: $H_{RRC}^{\alpha,T}(j\omega) = \sqrt{H_{RC}^{\alpha,T}(j\omega)}$
  3. $h_{RRC}^{\alpha,T}(t) = TF^{-1}\left\{H_{RRC}^{\alpha,T}(j\omega)\right\}$
- Root-raised cosine pulses

$$h_{RRC}^{\alpha,T}(t) = \frac{\sin\left((1-\alpha)\frac{\pi t}{T}\right) + \frac{4\alpha t}{T}\cdot\cos\left((1-\alpha)\frac{\pi t}{T}\right)}{\frac{\pi t}{T}\cdot\left[1-\left(\frac{4\alpha t}{T}\right)^2\right]}$$

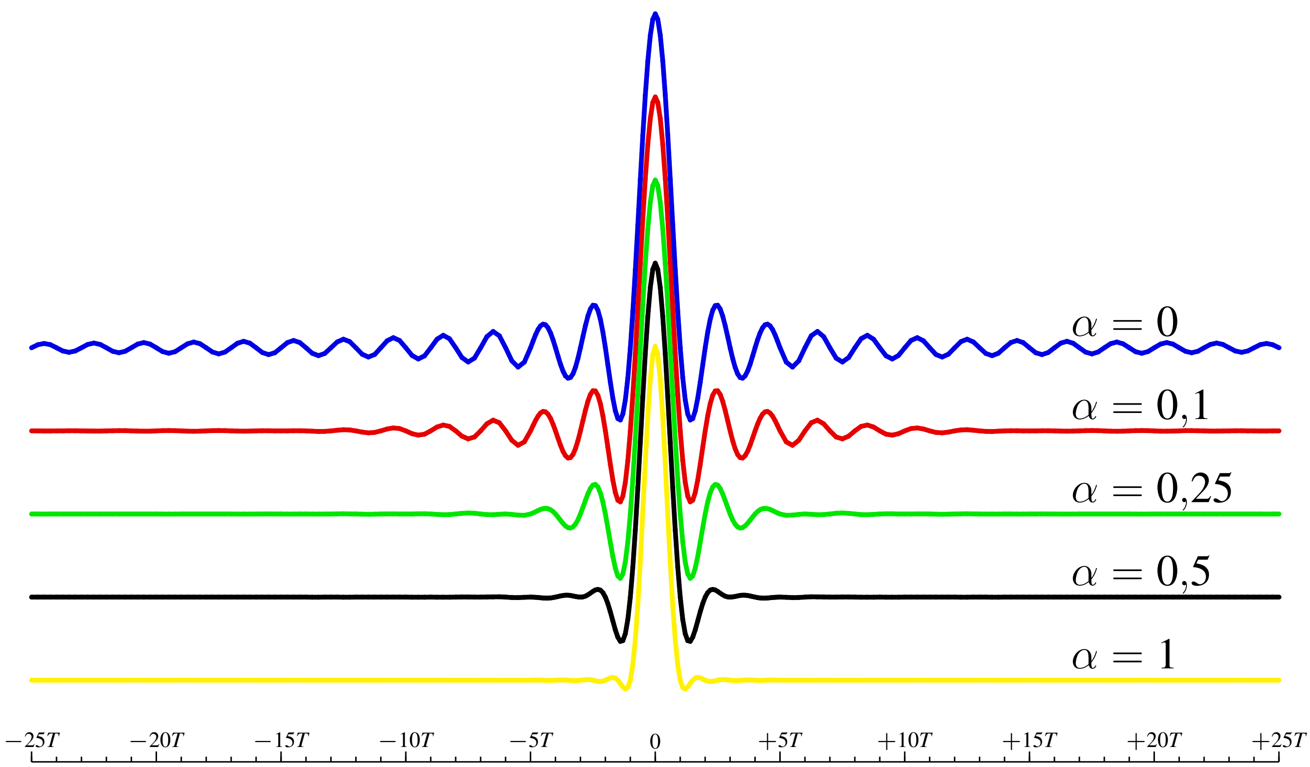
Root-raised cosine pulses: $h_{RRC}^{\alpha,T}(t)$



Root-raised cosine pulses: $H_{RRC}^{\alpha,T}(j\omega)$

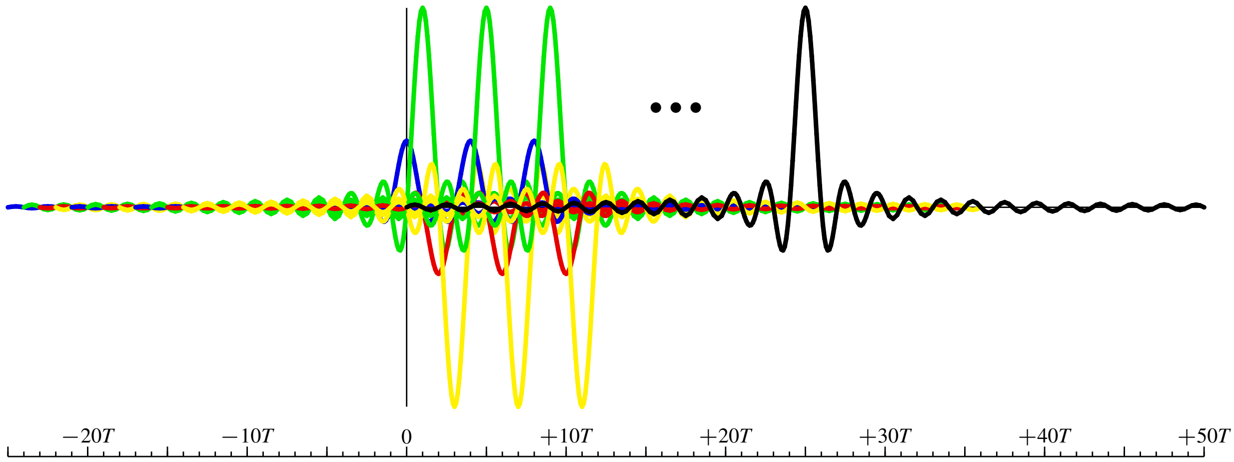


Raised cosines - side lobe attenuation



Raised cosines - implementation delay

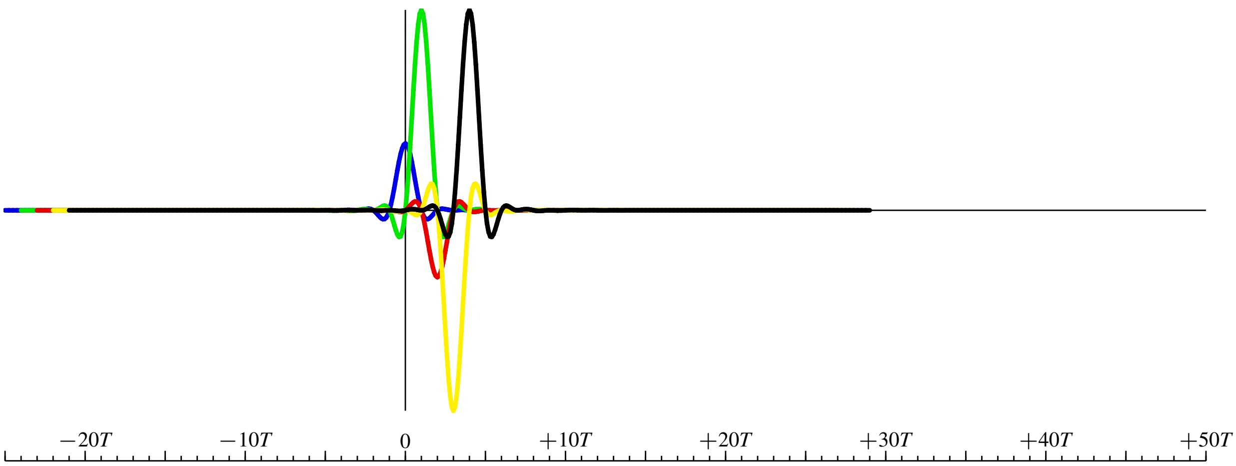
- A raised cosine has a number of "*relevant*" side lobes that is decreasing with roll-off factor
  - Non-relevant lobes could be truncated to make easier the implementation
- For implementententing the modulated waveform, a delay is necessary
  - Delay is related with the number of relevant side lobes that have to be cosidered before truncation
  - Delay is lower for higher values of α (higher bandwidth requirement)
- Example: generation of a 4-PAM waveform with $\alpha = 0$
  - In the example, 25 side lobes are considered relevant (and therefore 25 side lobes are depicted)
  - A delay of $25 \times T$ seconds is necessary to compute the addition
  - Black signal is the last one with relevant contribution at $t = 0$ (related with $A[25]$)



Raised cosines - implementation delay (II)

- Lower delays can be achieved by using higher roll-off factors
  - The price to be paid is a higher required bandwidth
- Example: generation of a 4-PAM waveform with $\alpha = 0{,}5$
  - In the example, 4 side lobes are considered relevant
  - A delay of $4 \times T$ seconds is necessary to compute the addition
  - Black signal is the last one with relevant contribution at $t = 0$ (related with $A[4]$)
  - Delay is decreased from $25 \times T$ to $4 \times T$ in this example (more than 6 times lower)
  - Required bandwidth is 50 % higher

  NOTE: the number of "*relevant*" lobes depends on required accuracy, this is just a simple example (numbers can not be taken as a precise reference)



Review: spectrum of continuous/discrete time signals

- Continuous signal $x(t)$ and discretized $x[n]$ sampled at T seg.

$$x[n] = x(t)\big|_{t=nT} = x(nT)$$

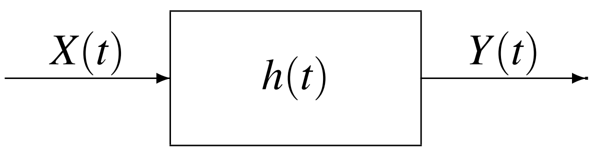
- Usual notation
  - $X(j\omega)$: spectrum (Fourier transform) of $x(t)$
  - $X\left(e^{j\omega}\right)$: spectrum of $x[n]$
- Relationship between both spectral responses
  - To obtain discrete from continuous

$$X\left(e^{j\omega}\right) = \frac{1}{T} \cdot \sum_k X\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)$$

  - To obtain continuous from discrete

$$X(j\omega) = T \cdot X\left(e^{j\omega T}\right), \ |\omega| \leq \pi$$

Review: random processes and linear systems



Theorem: $X(t)$ is stationary, mean m_X and autocorrelation function $R_X(\tau)$. The process is the input of a time-invariant linear system with impulse response $h(t)$. In this case, *input and output processes, $X(t)$ and $Y(t)$, are jointly stationary*, being

$$m_Y = m_X \int_{-\infty}^{\infty} h(t) \cdot dt$$

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

$$R_{XY}(\tau) = R_X(\tau) * h(-\tau)$$

Moreover, it can be seen that

$$R_Y(\tau) = R_{XY}(\tau) * h(\tau)$$

Review: expressions in the frequency domain

- Mean for output process

$$m_Y = m_X \cdot H(0)$$

- Power spectral density of the output process

$$S_Y(j\omega) = S_X(j\omega) \cdot |H(j\omega)|^2$$

- Crossed power spectral densities

$$S_{XY}(j\omega) \stackrel{def}{=} TF[R_{XY}(\tau)]$$

$$S_{XY}(j\omega) = S_X(j\omega)H^*(j\omega)$$

$$S_{YX}(j\omega) = S^*_{XY}(j\omega) = S_X(j\omega)H(j\omega)$$

Properties of the noise at the receiver

- Noise $n(t)$ is filtered by receiver filter $f(t)$ (producing filtered noise $z(t)$) and then sampled ($z[n]$)
- Analysis in the frequency domain
  - PSD of filtered noise $z(t)$

$$S_z(j\omega) = S_n(j\omega) \cdot |F^*(j\omega)|^2 = \frac{N_0}{2} \cdot |F(j\omega)|^2$$

    - Coloured noise (non-flat PSD)
  - PSD of sampled noise $z[n]$

$$S_z(e^{j\omega}) = \frac{N_0}{2} \cdot \frac{1}{T} \sum_k \underbrace{\left| F\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)\right|^2}_{R_f\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)}$$

    - Sampled noise can be white !!!!

Condition: $\frac{1}{T} \sum_k R_f\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right) = \text{constant}$

Conditions for sampled noise $z[n]$ being white

- Sampled noise $z[n]$ is white if

$$\frac{1}{T}\sum_k R_f\left(j\frac{\omega}{T}-j\frac{2\pi k}{T}\right)=C \text{ is equivalent to } R_f(e^{j\omega})=C$$

  - Equivalent condition in the time domain

$$r_f[n]=r_f(t)\Big|_{t=nT}=C\cdot\delta[n], \text{ which implies } C=r_f(0)$$

- Equivalent statement for $z[n]$ being white
  - $z[n]$ is white if the continuous autocorrelation function of receiver filter fulfills Nyquist condition for zero ISI
- REMARK
  - Condition for $z[n]$ being white only depends on the shape of receiver filter $f(t)$!!!

Consequences of Nyquist criterion for Gaussian channels

- A matched filter is assumed at the receiver

$$f(t)=g(-t) \text{ since } g(t) \text{ is a real function}$$

- Condition to avoid ISI
  - Joint response $p(t)=g(t)*f(t)$ fulfills Nyquist criterion
    - ⋆ Using matched filters $p(t)=r_g(t)$
- Condition for $z[n]$ being white
  - Continuous autocorrelation of the receiver filter, $r_f(t)$, fulfills Nyquist criterion
    - ⋆ Using matched filters $r_f(t)=r_g(t)$
- Conclusion: both conditions are equivalent
  - Transmitting through a Gaussian channel using matched filters, if ISI is avoided, sampled noise $z[n]$ is white

Signal to noise relationship

- If Nyquist ISI criterion is satisfied, the received observation is

$$q[n] = A[n] + z[n]$$

- In this case, signal to noise ratio is

$$\left(\frac{S}{N}\right)_q = \frac{E\{|A[n]|^2\}}{\sigma_z^2} = \frac{E_s}{\sigma_s^2}$$

- σ_z^2 is the varianze of noise sequence $z[n]$

$$\sigma_z^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_z(e^{j\omega})\, d\omega$$

  - If Nyquist ISI criterion is fulfilled
    - ⋆ For a normalized receiver filter $\sigma_z^2 = \frac{N_0}{2}$
    - ⋆ For a non-normalized receiver filter

$$\sigma_z^2 = \frac{N_0}{2} \times \mathcal{E}\{f(t)\} = \frac{N_0}{2} \times r_f(0)$$

Evaluation of Probability of Symbol Error (P_e)

- Definition

$$P_e = P(\hat{A}[n] \neq A[n])$$

- Evaluation - Averaging of probability of symbol error for each symbol in the constellation

$$P_e = \sum_{i=0}^{M-1} p_A(a_i) \cdot P_{e|a_i}$$

- Calculation of contitional probabilities of symbol error (conditional probabilities of error)

$$P_{e|\boldsymbol{a}_i} = \int_{q \notin I_i} f_{\boldsymbol{q}|\boldsymbol{A}}(q|a_i)\, dq$$

  Conditional distribution of observations conditioned to transmission of the symbol $\boldsymbol{a}_i$ is integrated out of its decision region I_i

Calculation of Bit Error Rate (BER)

- Conditional BER for each symbol $\boldsymbol{a}_i$ are averaged

$$BER = \sum_{i=0}^{M-1} p_A(a_i) \cdot BER_{a_i}$$

- Calculation of conditional BER for $\boldsymbol{a}_i$

$$BER_{a_i} = \sum_{\substack{j=0 \\ j \neq i}}^{M-1} P_{e|a_i \to a_j} \cdot \frac{m_{e|a_i \to a_j}}{m}$$

  - $P_{e|a_i \to a_j}$: probability of deciding $\hat{A} = \boldsymbol{a}_j$ when $\boldsymbol{A} = \boldsymbol{a}_i$ was transmitted

$$P_{e|a_i \to a_j} = \int_{\boldsymbol{q}_0 \in I_j} f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}_0|\boldsymbol{a}_i)\, d\boldsymbol{q}_0$$

  - $m_{e|a_i \to a_j}$: number of bit errors associated to that decision
  - m: number of bits per symbol in the constellation

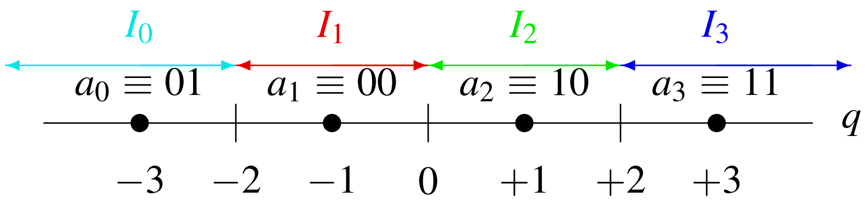
Example - 1-D M-ary constellation

- Example:
  - $M = 4$, equiprobable symbols $p_A(\boldsymbol{a}_i) = \frac{1}{4}$
  - Constellation: $\boldsymbol{a}_0 = -3,\ \boldsymbol{a}_1 = -1,\ \boldsymbol{a}_2 = +1,\ \boldsymbol{a}_3 = +3$
  - Decision regions: thresholds $q_{u1} = -2,\ q_{u2} = 0,\ q_{u3} = +2$

$$I_0 = (-\infty, -2],\ I_1 = (-2, 0],\ I_2 = (0, +2],\ I_3 = (+2, +\infty)$$

  - Binary assignment

$$\boldsymbol{a}_0 \equiv 01,\ \boldsymbol{a}_1 \equiv 00,\ \boldsymbol{a}_2 \equiv 10,\ \boldsymbol{a}_3 \equiv 11$$



Example - 1-D M-ary constellation (II)

- Probability of error

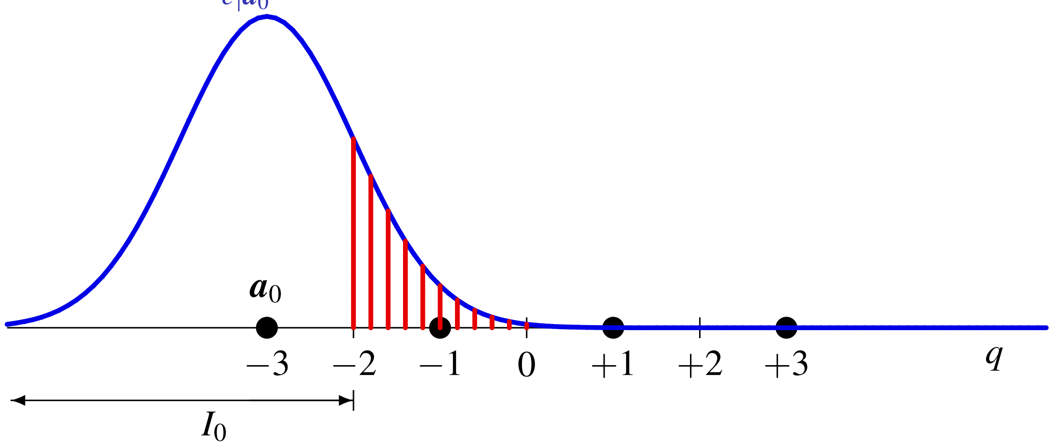
$$P_e = \frac{1}{4}\sum_{i=0}^{M-1} P_{e|\boldsymbol{a}_i} = \frac{3}{2} Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

- Bit error rate (BER)

$$BER = \frac{3}{4} Q\left(\frac{1}{\sqrt{N_o/2}}\right) + \frac{1}{2} Q\left(\frac{3}{\sqrt{N_o/2}}\right) - \frac{1}{4} Q\left(\frac{5}{\sqrt{N_o/2}}\right)$$

- Analytic developments are detailed in Annex

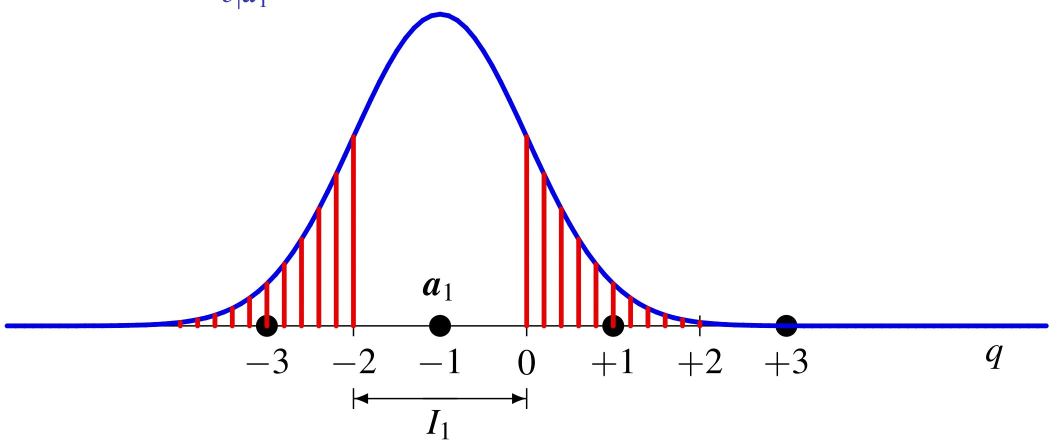
Calculation of $P_{e|\boldsymbol{a}_0}$



- Distribution $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_0)$
  - Gaussian with mean $\boldsymbol{a}_0 = -3$ and variance $N_0/2$
- Conditional probability of error
  - Integration of $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_0)$ out of I_0

$$P_{e|\boldsymbol{a}_0} = \int_{q \notin I_0} f_{\boldsymbol{q}|\boldsymbol{A}}(q|a_0)\, dq = Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

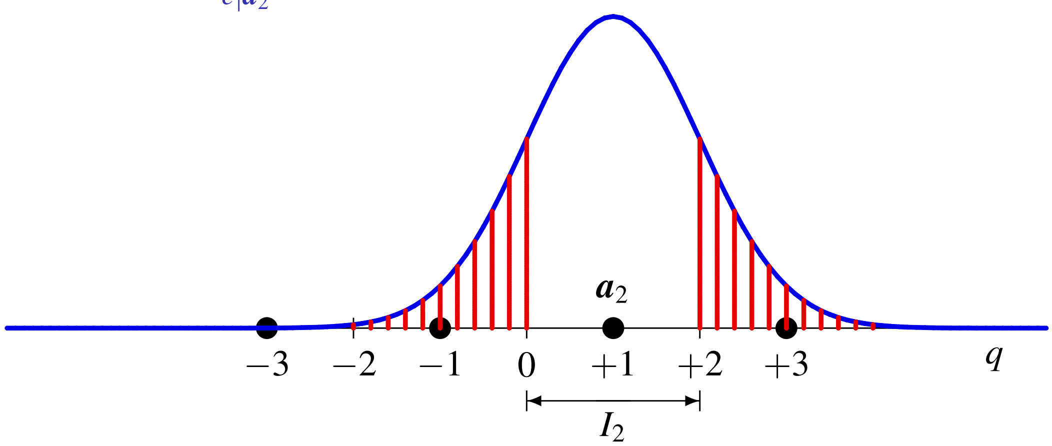
Calculation of $P_{e|\boldsymbol{a}_1}$



- Distribution $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_1)$
  - Gaussian with mean $\boldsymbol{a}_1 = -1$ and variance $N_0/2$
- Conditional probability of error
  - Integration of $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_1)$ out of I_1

$$P_{e|\boldsymbol{a}_1} = \int_{q \notin I_1} f_{\boldsymbol{q}|\boldsymbol{A}}(q|a_1)\, dq = 2Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

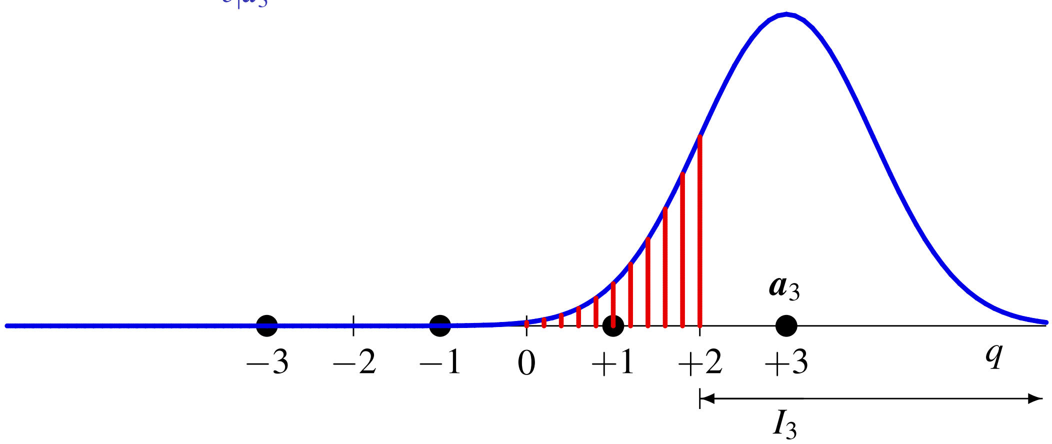
Calculation of $P_{e|\boldsymbol{a}_2}$



- Distribution $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_2)$
  - Gaussian with mean $\boldsymbol{a}_2 = +1$ and variance $N_0/2$
- Probability of error
  - Integration of $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_2)$ out of I_2

$$P_{e|\boldsymbol{a}_2} = \int_{q \notin I_2} f_{\boldsymbol{q}|\boldsymbol{A}}(q|a_2)\, dq = 2Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

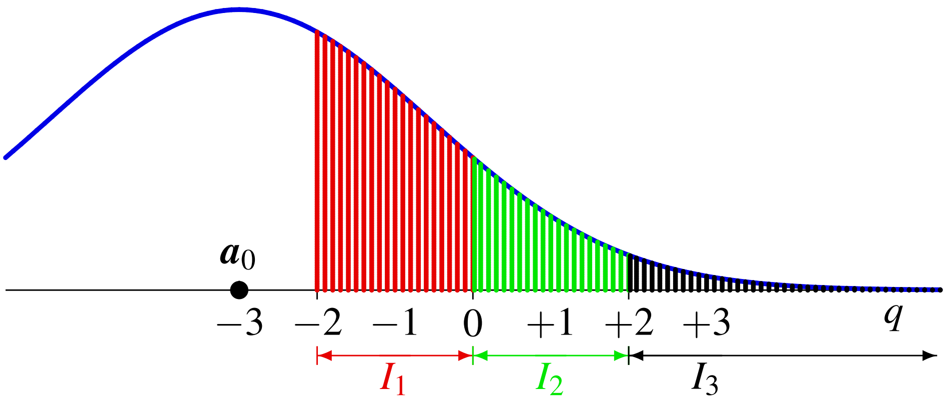
Calculation of $P_{e|\boldsymbol{a}_3}$



- Distribution $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_3)$
  - Gaussian with mean $\boldsymbol{a}_3 = -3$ and variance $N_0/2$
- Probability of error
  - Integration of $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_3)$ out of I_3

$$P_{e|\boldsymbol{a}_3} = \int_{q \notin I_3} f_{\boldsymbol{q}|\boldsymbol{A}}(q|a_3)\, dq = Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

Calculation of $BER_{\boldsymbol{a}_0}$

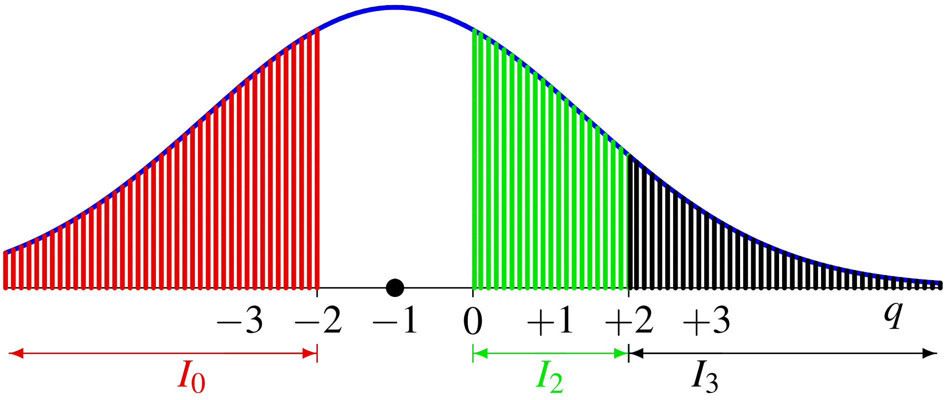


- Binary assignment: $\boldsymbol{a}_0 \equiv 01,\ \boldsymbol{a}_1 \equiv 00,\ \boldsymbol{a}_2 \equiv 10,\ \boldsymbol{a}_3 \equiv 11$
- Distribution $f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_0)$: Gaussian with mean $\boldsymbol{a}_0$ and variance $N_0/2$

$$BER_{\boldsymbol{a}_0} = \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right) - Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_0\to\boldsymbol{a}_1}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\boldsymbol{a}_0\to\boldsymbol{a}_1}}{m}}$$

$$+ \underbrace{\left[Q\left(\frac{3}{\sqrt{N_0/2}}\right) - Q\left(\frac{5}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_0\to\boldsymbol{a}_2}} \times \underbrace{\frac{2}{2}}_{\frac{m_{e|\boldsymbol{a}_0\to\boldsymbol{a}_2}}{m}} + \underbrace{\left[Q\left(\frac{5}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_0\to\boldsymbol{a}_3}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\boldsymbol{a}_0\to\boldsymbol{a}_3}}{m}}$$

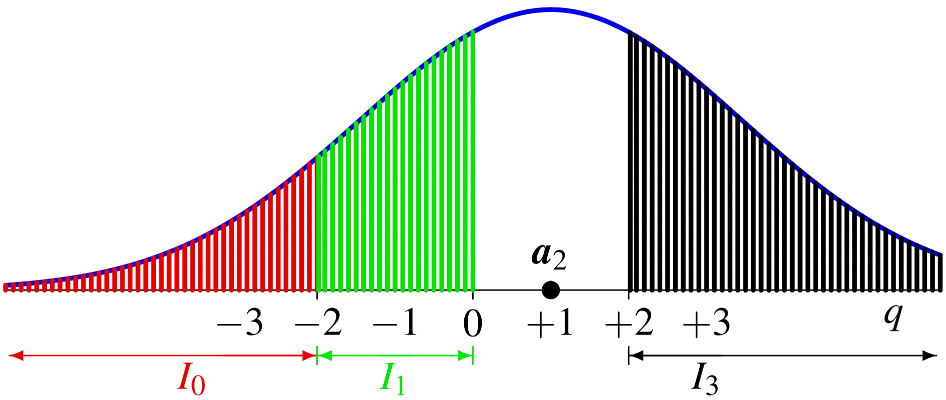
Cálculation of BER_{a_1}



- Binary assignment: $\boldsymbol{a}_0 \equiv 01,\ \boldsymbol{a}_1 \equiv 00,\ \boldsymbol{a}_2 \equiv 10,\ \boldsymbol{a}_3 \equiv 11$
- Distribution $f_{\boldsymbol{q}|A}(\boldsymbol{q}|\boldsymbol{a}_1)$: Gaussian with mean $\boldsymbol{a}_1$ and variance $N_0/2$

$$BER_{\boldsymbol{a}_1} = \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_1\to\boldsymbol{a}_0}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\boldsymbol{a}_1\to\boldsymbol{a}_0}}{m}} + \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right) - Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_1\to\boldsymbol{a}_2}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\boldsymbol{a}_1\to\boldsymbol{a}_2}}{m}}$$
$$+ \underbrace{\left[Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_1\to\boldsymbol{a}_3}} \times \underbrace{\frac{2}{2}}_{\frac{m_{e|\boldsymbol{a}_1\to\boldsymbol{a}_3}}{m}}$$

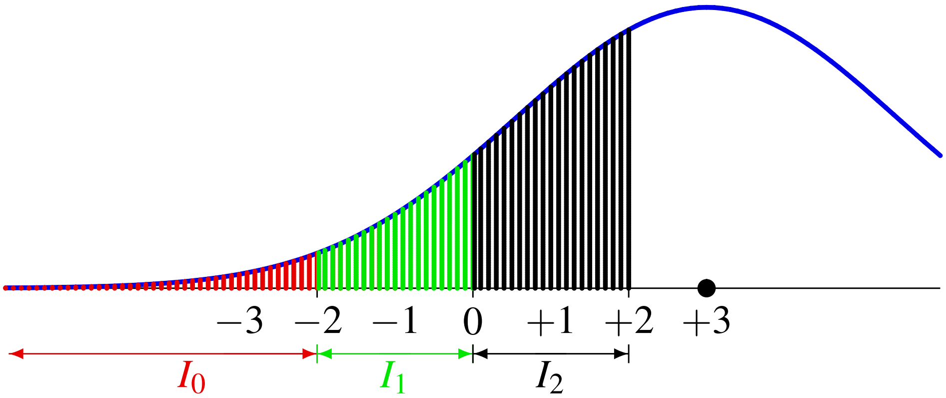
Calculation of BER_{a_2}



- Binary assignment: $\boldsymbol{a}_0 \equiv 01,\ \boldsymbol{a}_1 \equiv 00,\ \boldsymbol{a}_2 \equiv 10,\ \boldsymbol{a}_3 \equiv 11$
- Distribution $f_{\boldsymbol{q}|A}(\boldsymbol{q}|\boldsymbol{a}_2)$: Gaussian with mean $\boldsymbol{a}_2$ and variance $N_0/2$

$$BER_{\boldsymbol{a}_2} = \underbrace{\left[Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_2\to\boldsymbol{a}_0}} \times \underbrace{\frac{2}{2}}_{\frac{m_{e|\boldsymbol{a}_2\to\boldsymbol{a}_0}}{m}} + \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right) - Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_2\to\boldsymbol{a}_1}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\boldsymbol{a}_2\to\boldsymbol{a}_1}}{m}}$$
$$+ \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_2\to\boldsymbol{a}_3}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\boldsymbol{a}_2\to\boldsymbol{a}_3}}{m}}$$

Calculation of BER_{a_3}

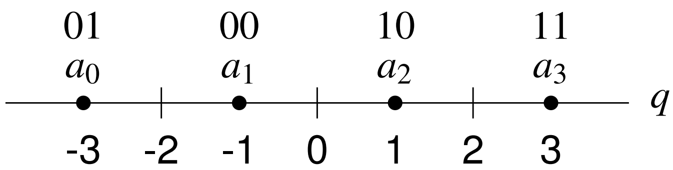


- Binary assignment: $\boldsymbol{a}_0 \equiv 01,\ \boldsymbol{a}_1 \equiv 00,\ \boldsymbol{a}_2 \equiv 10,\ \boldsymbol{a}_3 \equiv 11$
- Distribution $f_{\boldsymbol{q}|A}(\boldsymbol{q}|\boldsymbol{a}_3)$: Gaussian with mean $\boldsymbol{a}_3$ and variance $N_0/2$

$$BER_{\boldsymbol{a}_3} = \underbrace{\left[Q\left(\frac{5}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_3\to\boldsymbol{a}_0}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\boldsymbol{a}_3\to\boldsymbol{a}_0}}{m}} + \underbrace{\left[Q\left(\frac{3}{\sqrt{N_0/2}}\right) - Q\left(\frac{5}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_3\to\boldsymbol{a}_1}} \times \underbrace{\frac{2}{2}}_{\frac{m_{e|\boldsymbol{a}_3\to\boldsymbol{a}_1}}{m}}$$
$$+ \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right) - Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\boldsymbol{a}_3\to\boldsymbol{a}_2}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\boldsymbol{a}_3\to\boldsymbol{a}_2}}{m}}$$

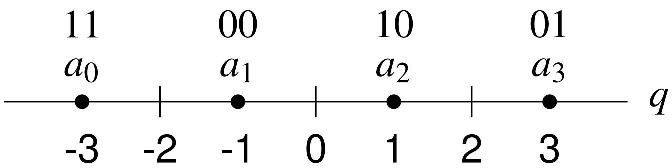
Modification of the binary assignment

- Final result for previous binary assignment



$$BER = \frac{3}{4}Q\left(\frac{1}{\sqrt{N_o/2}}\right) + \frac{1}{2}Q\left(\frac{3}{\sqrt{N_o/2}}\right) - \frac{1}{4}Q\left(\frac{5}{\sqrt{N_o/2}}\right)$$

- If binary assignment is modified

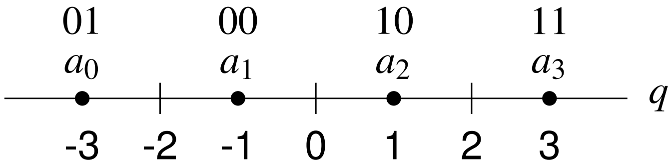


  - Terms $P_{e|a_i\to a_j}$ do not vary
  - Terms $m_{e|a_i\to a_j}$ do vary $\Rightarrow$ BER is modified !!!

$$BER = \frac{5}{4}Q\left(\frac{1}{\sqrt{N_o/2}}\right) - \frac{1}{4}Q\left(\frac{3}{\sqrt{N_o/2}}\right)$$

Gray Coding

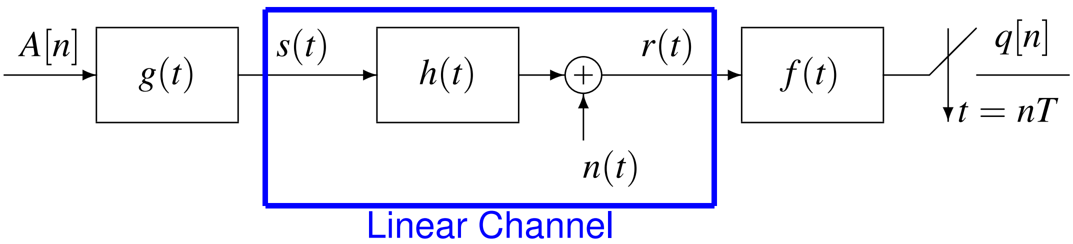
- Blocks of m bits assigned to symbols at minimum distance differ in only a single bit



  - This assignment minimizes BER for a given constellation
- Terms $P_{e|\boldsymbol{a}_i \to \boldsymbol{a}_j}$ depend on the constellation
  - Values depend on distance between $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$
  - Highest values for symbols at minimum distance
- Terms $\frac{m_{e|\boldsymbol{a}_i \to \boldsymbol{a}_j}}{m}$ depend on bit assignment
  - These terms weight the contribution of $P_{e|\boldsymbol{a}_i \to \boldsymbol{a}_j}$
    - ★ Gray coding: minimizes impact of highest values of $P_{e|\boldsymbol{a}_i \to \boldsymbol{a}_j}$
    - ★ For high values of signal to noise ratio (SNR), in most cases, a symbol error produces a single erroneous bit

$$BER \approx \frac{1}{m} \cdot P_e$$

Transmission of PAM through linear channels



- Linear channel model
  - PAM signal $s(t)$ suffers a linear distortion during transmission
  - Gaussian noise is also added

$$r(t) = s(t) * h(t) + n(t)$$

$h(t)$: linear system impulse response modeling linear distortion

$n(t)$: stationary random process, white, Gaussian, zero mean, $S_n(j\omega) = N_0/2$

- Receiver filter $f(t)$
  - Typical set up: matched filter $f(t) = g^*(-t) = g(-t)$
- Signal at the input of the sampler

$$q(t) = r(t) * f(t) = s(t) * h(t) * f(t) + n(t) * f(t)$$

Equivalent discrete channel for linear channels

- Signal before sampling

$$\begin{aligned} q(t) &= \left(\sum_k A[k]\cdot g(t-kT)\right) * h(t) * f(t) + n(t) * f(t) \\ &= \sum_k A[k]\cdot \Big(g(t-kT) * h(t) * f(t)\Big) + n(t) * f(t) \\ &= \sum_k A[k]\cdot p(t-kT) + z(t) \end{aligned}$$

- $p(t) = g(t) * h(t) * f(t)$: joint transmitter-channel-receiver response
  - For a matched filter at the receiver

$$p(t) = g(t) * h(t) * g^*(-t) = r_g(t) * h(t)$$

$r_g(t)$: continuous autocorrelation of $g(t)$ (or temporal ambiguity function of $g(t)$)

- Observation at demodulator output

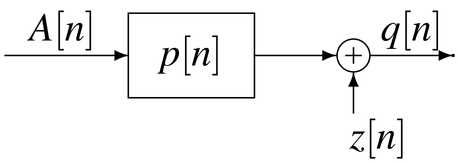
$$q[n] = q(t)\big|_{t=nT} = \sum_k A[k]\cdot p\left((n-k)T\right) + z(nT)$$

Equivalent discrete channel for linear channels (II)

- Definition of equivalent discrete channel $p[n]$

$$p[n] = p(t)\big|_{t=nT}$$

$$q[n] = \sum_k A[k]\cdot p[n-k] + z[n] = A[n] * p[n] + z[n]$$



- Same basic model as for Gaussian channels but with a new definition por joint response $p(t)$
  - Now definition includes the effect of $h(t)$

$$p(t) = g(t) * h(t) * f(t),\ \ P(j\omega) = G(j\omega)\cdot H(j\omega)\cdot F(j\omega)$$

  - Using matched filters: $f(t) = g(-t)$, $F(j\omega) = G^*(j\omega)$

$$p(t) = r_g(t) * h(t),\ \ P(j\omega) = |G(j\omega)|^2\cdot H(j\omega)$$

Avoidance of ISI

- Nyquist ISI criterion must be fulfilled for $p[n]$ (or $P(j\omega)$)
  - Definition of $p(t)$ includes now the effect of linear channel $h(t)$
- Design of $p(t)|P(j\omega)$ to fulfill Nyquist at symbol period T
- Design usign matched filters in the receiver
  Response of transmitter filter in the frequency domain
  - $P(j\omega) = H(j\omega) \cdot |G(j\omega)|^2$
  - Therefore

$$G(j\omega) = \begin{cases} \sqrt{\frac{P(j\omega)}{H(j\omega)}}, & \text{if } H(j\omega) \neq 0 \\ 0, & \text{in other case} \end{cases}$$

  If the receiver filter is matched to the transmitter filter, this choice for the transmitter filter eliminates ISI
  - $P(j\omega)$ is a design option
    - ⋆ Tipically, a raised-cosine response is selected

$$P(j\omega) = H_{RC}^{\alpha,T}(j\omega)$$

Avoidance of ISI - Problems

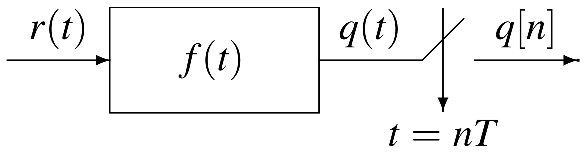
- Channel response, $H(j\omega)$, must be known
  - It can be difficult to know it
  - Channel can be time variant
- Discrete noise sequence, $z[n]$, is not white

$$S_z\left(e^{j\omega}\right) = \frac{N_0}{2T} \sum_k \left| \frac{P\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)}{H\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)} \right|$$

  - Memoryless symbol by symbol detector is not optimal
  - All sequence $q[n]$ has to be used to estimate the symbol at a given discrete instant n_0, $A[n_0]$
  - Noise can be amplified
    - ⋆ Channels with deep attenuation at some frequencies in the band

Using a generic receiver filter

- Generic receiver, not necesarily a matched filter



- Definition of joint response $p(t)$

$$p(t) = g(t) * h(t) * f(t), \; P(j\omega) = G(j\omega) \cdot H(j\omega) \cdot F(j\omega)$$

- Equivalent discrete channel at symbol rate $p[n]$

$$p[n] = p(nT) = (g(t) * h(t) * f(t))\,\big|_{t=nT}$$

- Filtered noise

$$z(t) = n(t) * f(t), \; z[n] = z(nT)$$

  - Power spectral density for discrete noise $z[n]$

$$S_z\left(e^{j\omega}\right) = \frac{N_0}{2} \times \frac{1}{T}\sum_k \left|F\left(j\frac{\omega}{T} - j\frac{2\pi}{T}k\right)\right|^2$$

Criteria to design $f(t)$

- Filter matched to the joint transmitter-channel response

$$f(t) = g_h(-t), \text{ with } g_h(t) = g(t) * h(t)$$

  - Maximizes the signal to noise ratio
  - Does not provides zero ISI and noise $z[n]$ is not white

- Minimum mean squared error criterion: to maximize

$$\frac{E\left\{(A[n]p[0])^2\right\}}{E\left\{\left(\sum_{\substack{k\\k\neq n}} A[k]p[n-k] + z[n]\right)^2\right\}}$$

- Simultaneously avoidance of ISI and white noise
  - Selection of $P(j\omega)$ fulfilling Nyquist
  - Selection of $F(j\omega)$ with $R_f(j\omega) = |F(j\omega)|^2$ fulfilling Nyquist

$$G(j\omega) = \frac{P(j\omega)}{H(j\omega) \cdot F(j\omega)}$$

    - ★ Usually presents serious implementation problems

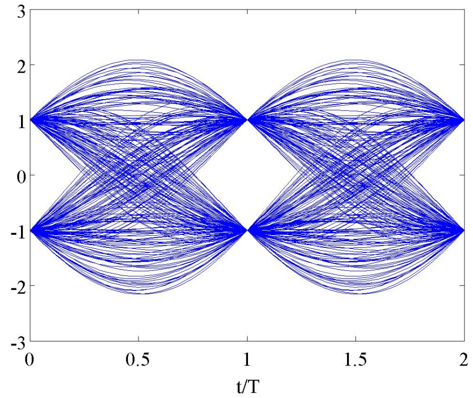
Typical set up for linear channels

- Receiver uses a matched filter $f(t) = g(-t)$ with $r_f(t) = r_g(t)$ fulfilling Nyquist
  - This ensures discrete filtered noise $z[n]$ is white
- Joint response $p(t)$ then does not fulfills Nyquist
  - ISI is present in the system
    - ⋆ Receivers can be specifically designed to deal with ISI (as it will be seen in chapter 4)

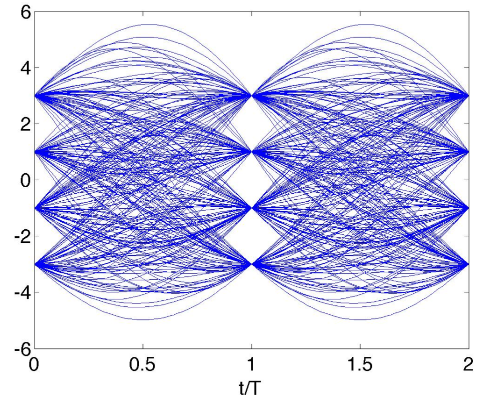
Eye diagram

- Visualization tool for a digital communication system
  - Superposition of waveform pieces around a sampling point
  - Duration of each piece: $2T$
- Main features
  - In the middle and in both sides (horizontaly), there are sampling instants
    - ⋆ Traces should have to go through values of the constellation
  - Diversity of transition between sampling instants depend on the shape of transmitter and receiver filters
- It allows to detect several problems:
  - Problems/sensitivity to synchronism
  - Level of noise
  - Presence (and level) of ISI

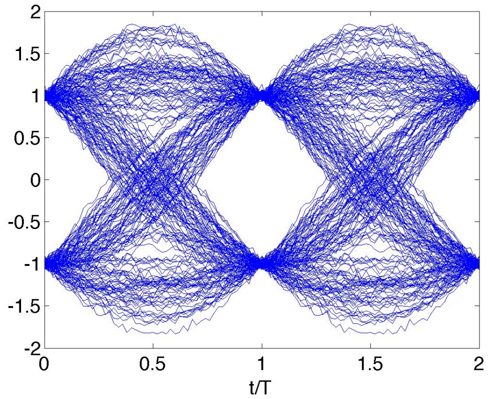
Eye diagram - Examples



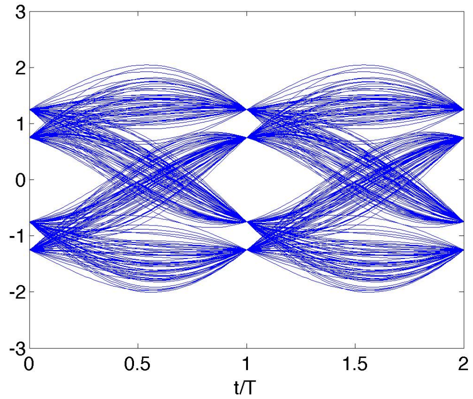
2-PAM $\alpha = 0$



4-PAM $\alpha = 0$

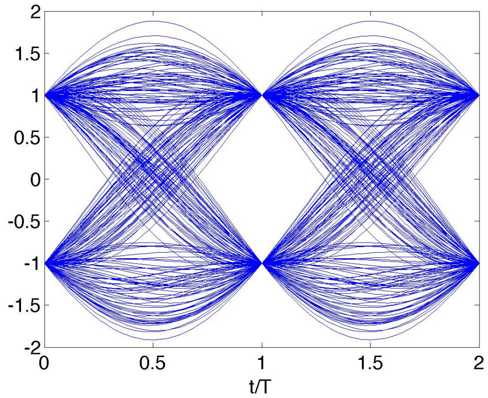


2-PAM $\alpha = 0{,}25$ Noisy

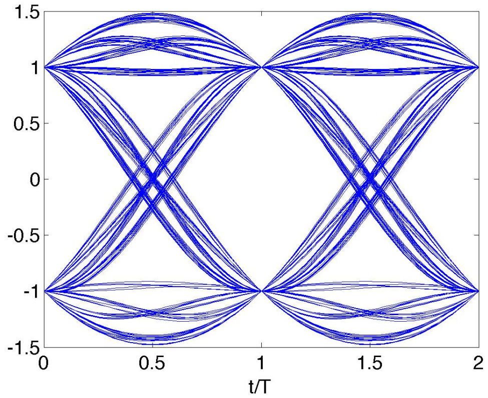


2-PAM $\alpha = 0$, ISI

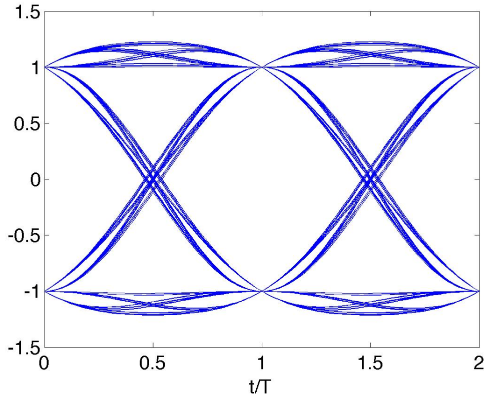
Eye diagram - Examples (II)



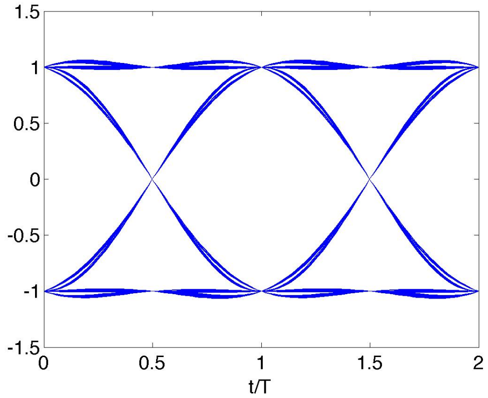
2-PAM $\alpha = 0{,}25$



2-PAM $\alpha = 0{,}5$



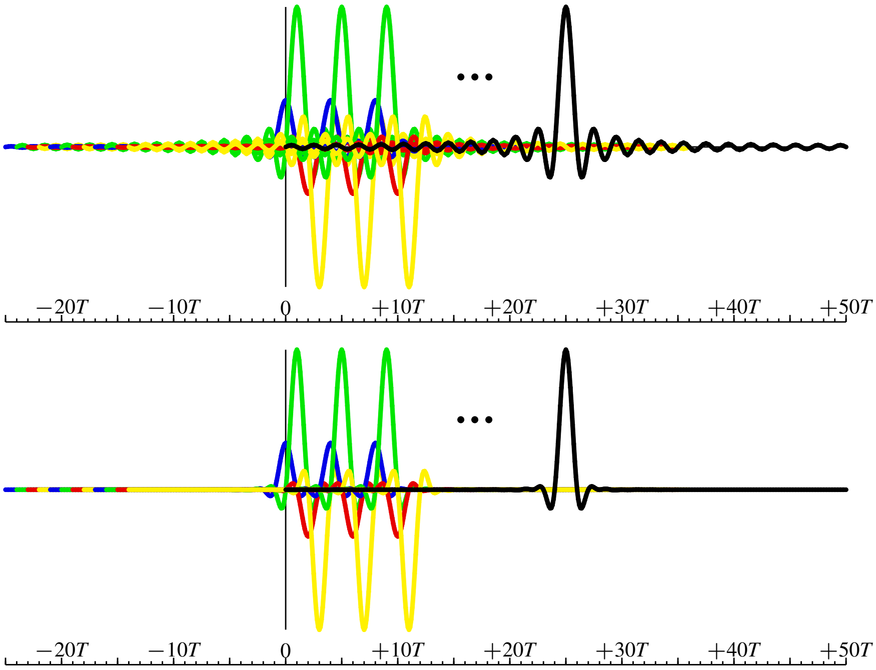
2-PAM $\alpha = 0{,}75$



2-PAM $\alpha = 1$

Variability of signal transitions related with α value

- Pictures illustrate the different variability of signals using $\alpha = 0$ (above) and $\alpha = 0{,}5$ (below)



Band pass PAM - Generation by AM modulación

- A baseband PAM is initially generated

$$s(t) = \sum_n A[n] \cdot g(t - nT)$$

- Then, this baseband PAM signal is modulated with an amplitude modulation. Several options are available
  - Conventional AM (double sided band with carrier)
  - Double sided band PAM (DSB-PAM)
  - Single sided band PAM (SSB-PAM)
    - Lower sided band
    - Upper sided band
  - Vestigial sided band PAM (VSB-PAM)
    - Lower sided band
    - Upper sided band

Drawbacks of using a AM modulation

- Conventional AM and double sided band PAM (DSB-PAM)
  - Spectral efficiency is reduced to the half (bandwidth is doubled)
- Single sided band PAM (SSB-PAM)
  - Ideal analog side band filters are required
    - Real filters introduce a distortion
- Vestigial sided band PAM (VSB-PAM)
  - Analog vestigial band filters are required
    - Strong constraints

Modulation by using quadrature carriers

- Two sequences of symbols (not necessarily independent) are simultaneously transmitted (rate $R_s = \frac{1}{T}$ in both cases)

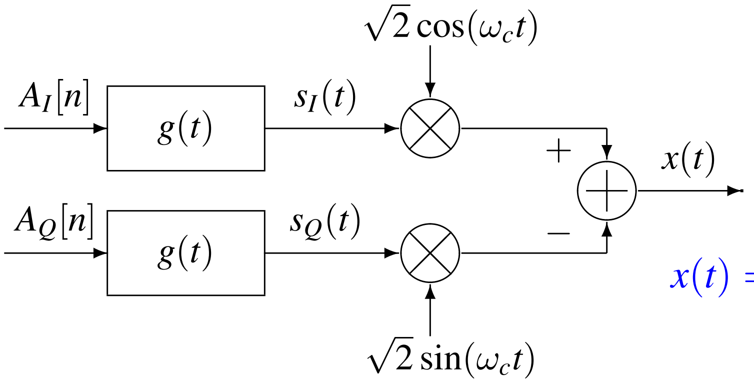
$$A_I[n] \text{ and } A_Q[n]$$

- Two baseband PAM signals are generated using $g(t)$

$$s_I(t) = \sum_n A_I[n] \cdot g(t - nT) \qquad s_Q(t) = \sum_n A_Q[n] \cdot g(t - nT)$$

$s_I(t)$: in-phase component, $s_Q(t)$: quadrature component

- Generation of the band pass signal, $x(t)$, from $s_I(t)$ and $s_Q(t)$



$$x(t) = \sqrt{2} \cdot s_I(t) \cdot \cos(\omega_c t) - \sqrt{2} \cdot s_Q(t) \cdot \sin(\omega_c t)$$

Complex notation for band pass PAM

- Complex sequence of symbols

$$A[n] = A_I[n] + jA_Q[n]$$

  - $A_I[n] = \mathcal{R}e\{A[n]\}, \quad A_Q[n] = \mathcal{I}m\{A[n]\}$

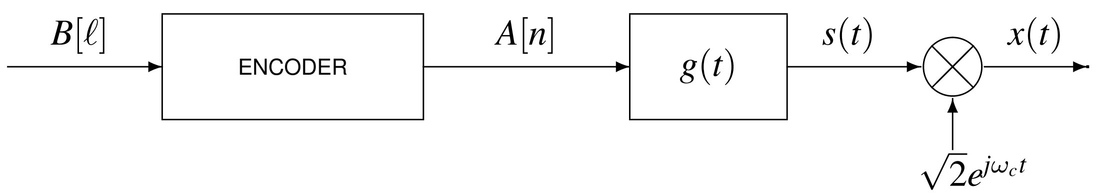
- Complex baseband signal, $s(t)$:

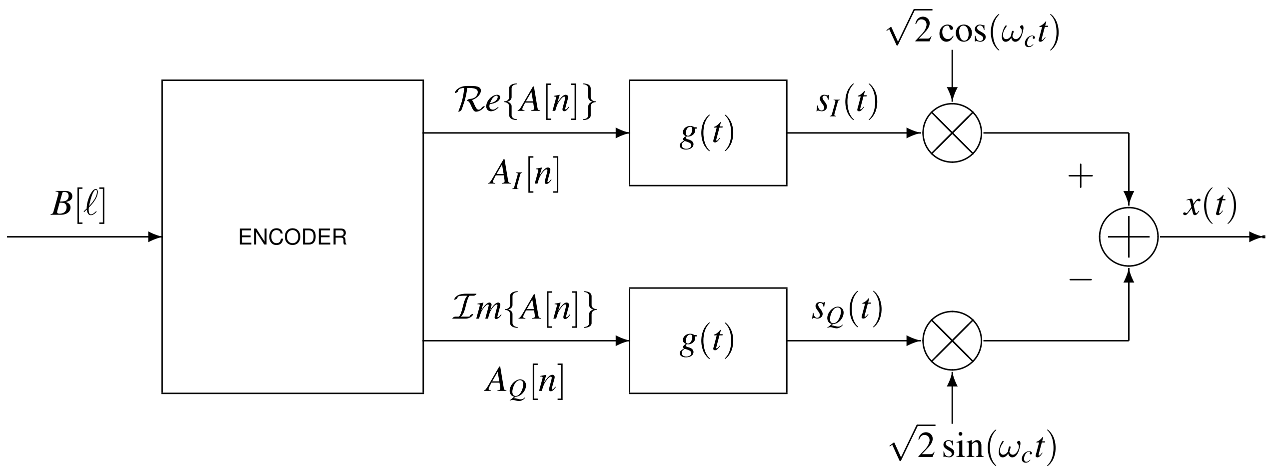
$$s(t) = s_I(t) + js_Q(t) = \sum_n A[n] \cdot g(t-nT)$$

- The band pass PAM signal can be written as follows

$$x(t) = \sqrt{2} \cdot \mathcal{R}e\left\{s(t) \cdot e^{j\omega_c t}\right\} = \sqrt{2} \cdot \mathcal{R}e\left\{\sum_n A[n] \cdot g(t-nT) \cdot e^{j\omega_c t}\right\}$$

Bandpass PAM modulator





Relationship with a 2D signal space

- Signal in a 2D signal space can be written as

$$x(t) = \sum_n A_0[n] \cdot \phi_0(t - nT) + \sum_n A_1[n] \cdot \phi_1(t - nT)$$

  - $\phi_0(t)$ and $\phi_1(t)$ are orthonormal signals

- In this case, this only happens if

$$\omega_c = \frac{2\pi}{T} \times k, \quad \text{with } k \in \mathbb{Z}$$

  In this case

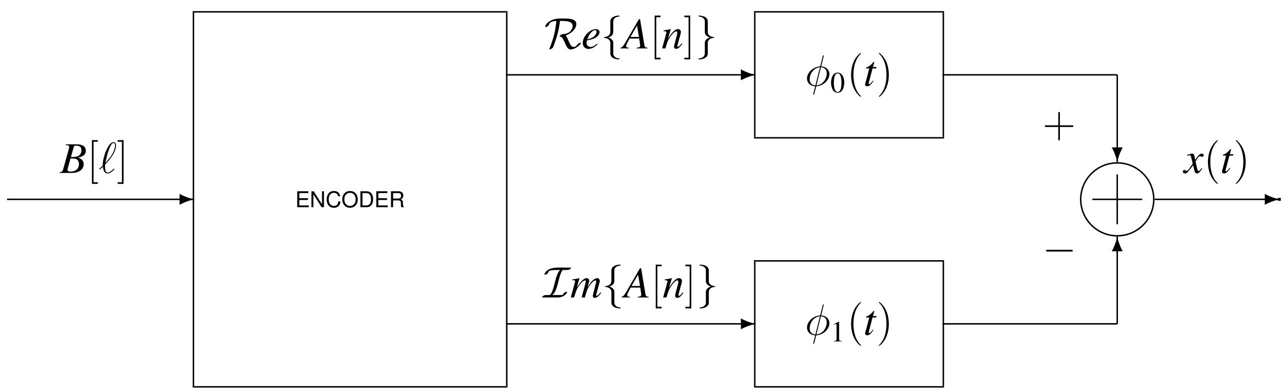
$$A_0[n] = A_I[n], \; A_Q[n] = A_1[n]$$

$$\phi_0(t) = g(t) \cdot \cos(\omega_c t), \qquad \phi_1(t) = -g(t) \cdot \sin(\omega_c t)$$

$$\phi_0(t - nT) = g(t - nT) \cdot \cos(\omega_c(t - nT)) = g(t - nT) \cdot \cos(\omega_c t)$$

$$\phi_1(t - nT) = -g(t - nT) \cdot \sin(\omega_c(t - nT)) = -g(t - nT) \cdot \sin(\omega_c t)$$

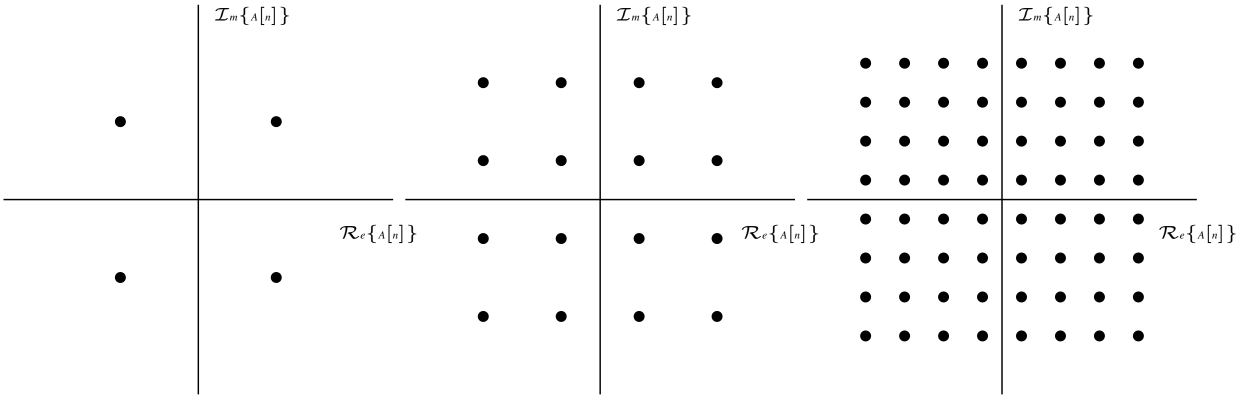
Modulator 2D signal space



Bandpass PAM constellations

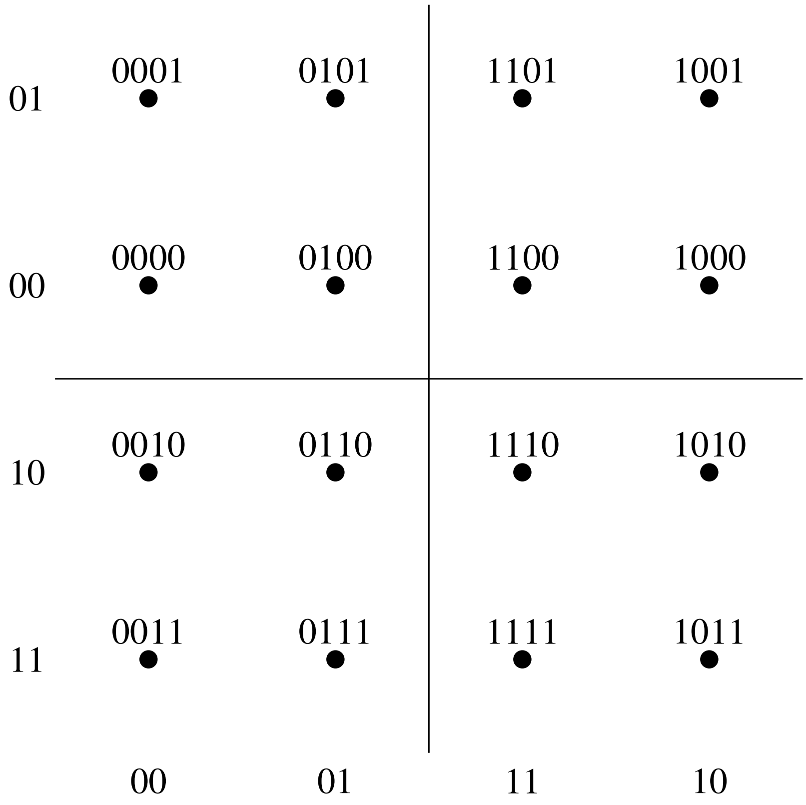
- 2D plotting of possible combinations for $A_I[n]$ and $A_Q[n]$
- Typical constellations
  - QAM (Quadrature Amplitude Modulation) constellations
    - ⋆ $M = 2^m$ symbols, with m even
    - ⋆ Symbols arranged in a full squared lattice ($2^{m/2} \times 2^{m/2}$ levels)
      - Both $A_I[n]$ and $A_Q[n]$ use baseband PAM constellations
      - Independent symbol mapping, bit assignment, and definition or decision regions are possible
  - Crossed QAM constellations
    - ⋆ $M = 2^m$ symbols, with m odd
    - ⋆ Symbols arranged in a non-full squared lattice
      - Independent symbol mapping, bit assignment, and definition of decision regions are not possible
  - PSK (Phase Shift Keying) constellations
    - ⋆ Symbols are drawn as points in a circle
      - Constant energy for all symbols

QAM constellations

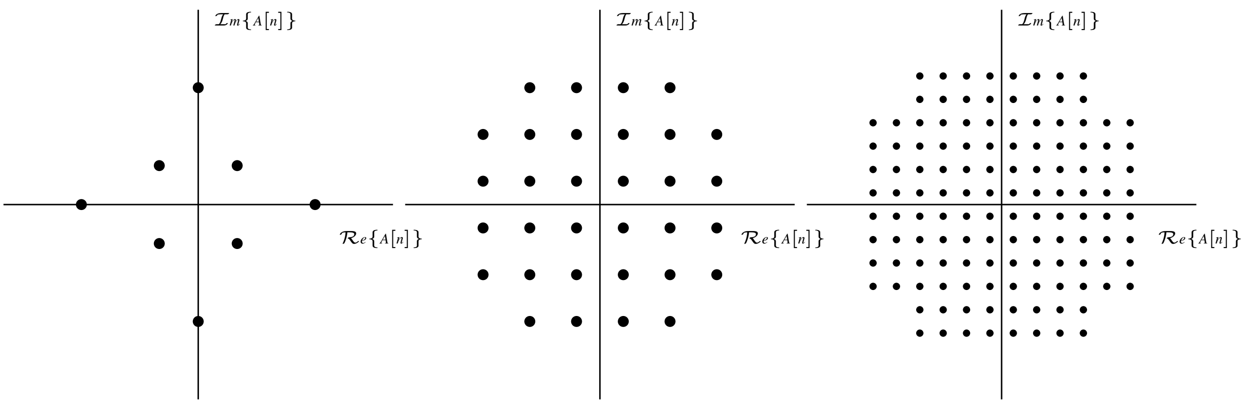


Constelaciones 4-QAM (QPSK), 16-QAM y 64-QAM

Gray coding for QAM



Crossed QAM constellations



Constellations: 8-QAM, 32-QAM y 128-QAM

Phase shift keying (PSK) modulation

- PSK constellation

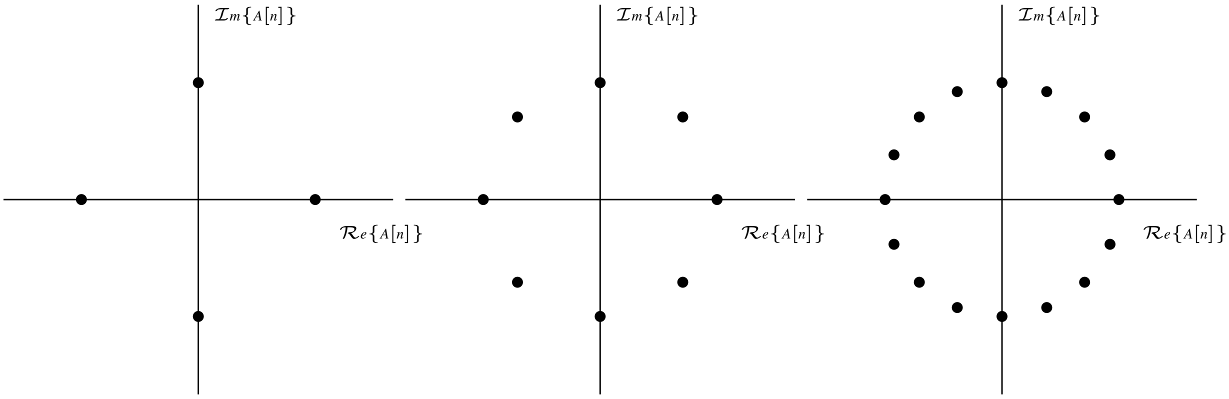
$$A[n] = \sqrt{E_s} \cdot e^{j\varphi[n]}$$

  - Constant modulus
  - Information is conveyed in the symbol phase
- Waveform for PSK modulations

$$x(t) = \sqrt{2E_s} \mathcal{R}e \left\{ \sum_n g(t-nT) \cdot e^{j(\omega_c t + \varphi[n])} \right\}$$
$$= \sqrt{2E_s} \sum_n g(t-nT) \cos(\omega_c t + \varphi[n])$$

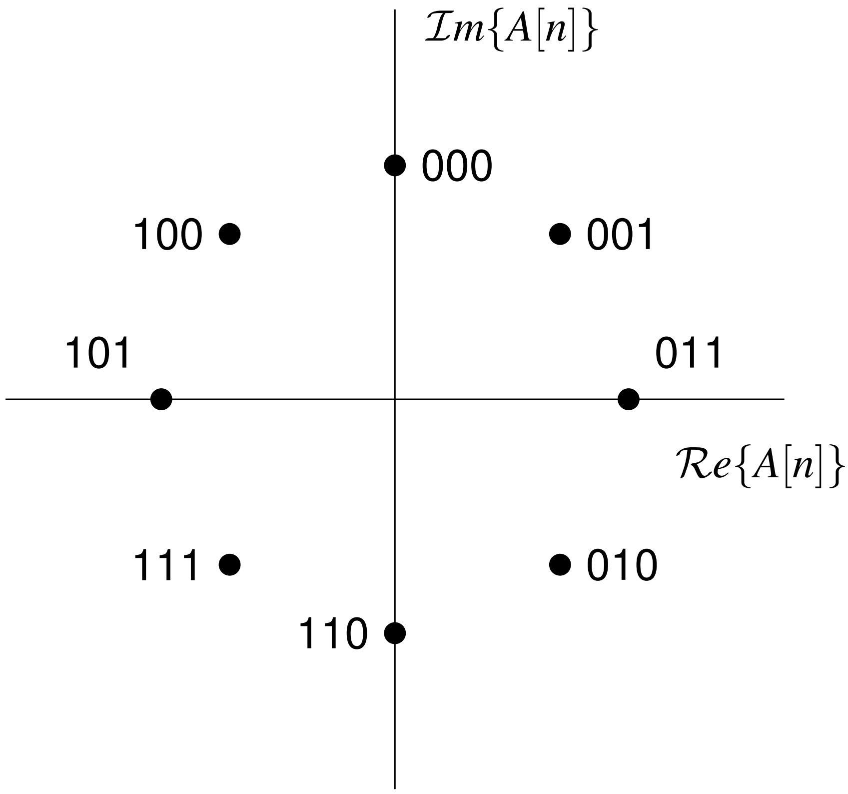
  - Phase shifts in transitions from symbol to symbol

PSK constellations

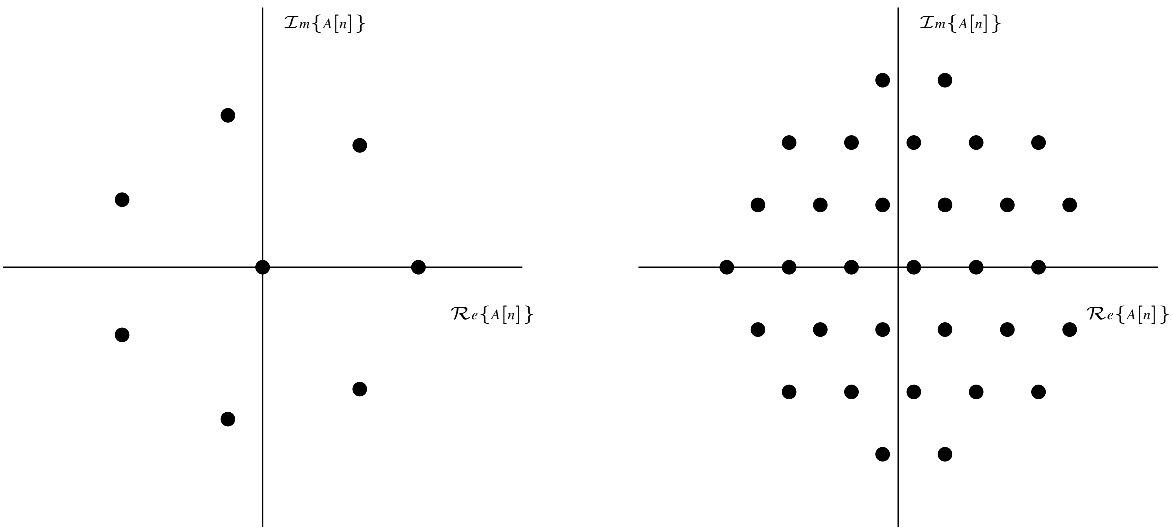


Constellations: 4-PSK (QPSK), 8-PSK y 16-PSK

Gray coding for PSK



Other constellations



Constellations 1-7-AM-PM y 32-hexagonal

Spectrum of a band pass PAM

- Condition for cyclostationarity of signal $x(t)$:

$$E\{A[k+m]\cdot A[k]\}=0,\ \text{for all } k,m,m\neq 0$$

  - Conditions for QAM constellations
    - Symbol sequences $A_I[n]$ and $A_Q[n]$ are mutually independent
    - Autocorrelation functions of $A_I[n]$ and $A_Q[n]$ are identical
  - Conditions for PSK constellations
    - Samples of $\varphi[n]$ are independent

- Under cyclostationarity the power spectral density function is

$$S_x(j\omega)=\frac{1}{2}\left[S_s(j\omega-j\omega_c)+S_s^*(-j\omega-j\omega_c)\right]$$

$$S_s(j\omega)=\frac{1}{T}\cdot S_A\left(e^{j\omega T}\right)\cdot|G(j\omega)|^2$$

Spectrum of a band pass PAM (II)

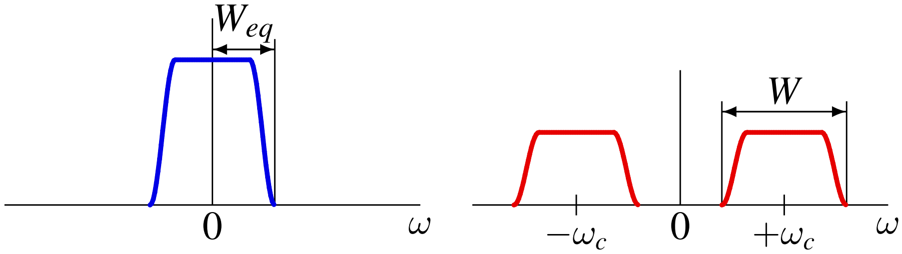
- For white sequences of symbols: $S_A\left(e^{j\omega}\right)=E_s$

$$S_s(j\omega)=\frac{E_s}{T}\cdot|G(j\omega)|^2$$

The shaping pulse is responsible of the shape of the spectrum

$$S_x(j\omega)=\frac{1}{2}\frac{E_s}{T}\left[|G(j\omega-j\omega_c)|^2+|G(j\omega+j\omega_c)|^2\right]$$

  - Example using pulses of raised cosine family



Bandpass bandwidth W is double of equivalent baseband bandwidth W_{eq}

Spectral efficiency is the same because now two sequences are transmitted

Transmitted power

- The mean transmitted power is

$$P_X = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(j\omega)\, d\omega$$

- If symbol sequence $A[n]$ is white

$$S_A\left(e^{j\omega}\right) = E_s$$

  - Power for a white symbol sequence

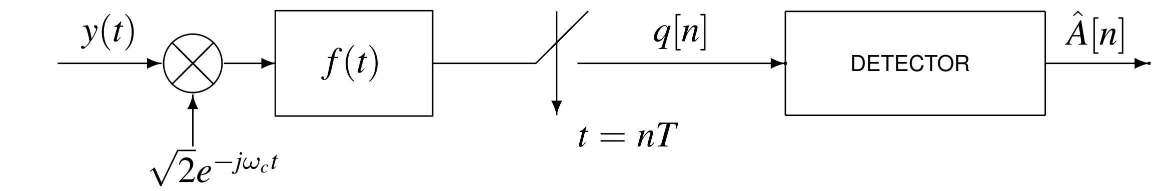
$$P_X = \frac{Es}{T} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 \; d\omega = \frac{E_s}{T} \cdot \mathcal{E}\{g(t)\}$$

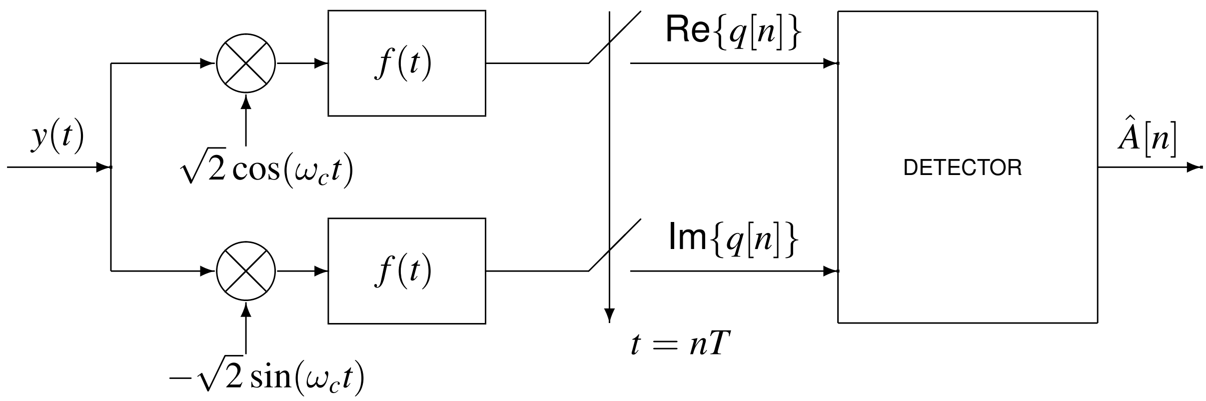
    - ★ For normalized pulses (with unitary energy)

$$P_X = \frac{E_s}{T} = E_s \times R_s \text{ Watts}$$

Demodulator for band pass PAM

- Demodulation and a baseband filter structure can be used
  - Complex notation and implementation by components can be seen in the following pictures





Equivalent alternative demodulator

- Signal at the input of the sampler (using complex notation)

$$q(t) = \left(y(t) \cdot e^{-j\omega_c t}\right) * \left(\sqrt{2} \cdot f(t)\right)$$

- Expression for the convolution

$$q(t) = \sqrt{2} \int_{-\infty}^{\infty} f(\tau) \cdot y(t-\tau) \cdot e^{j\omega_c \tau} \cdot e^{-j\omega_c t} \, d\tau$$

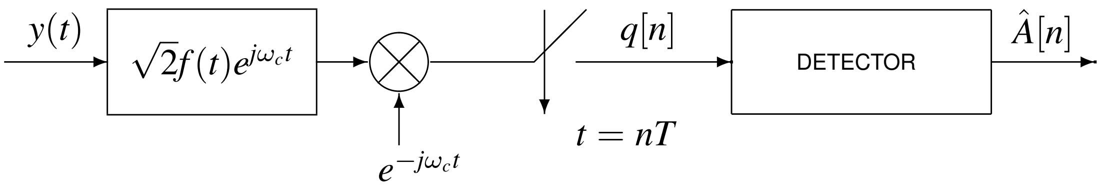
- Rearranging terms, an equivalent demodulation scheme is obtained

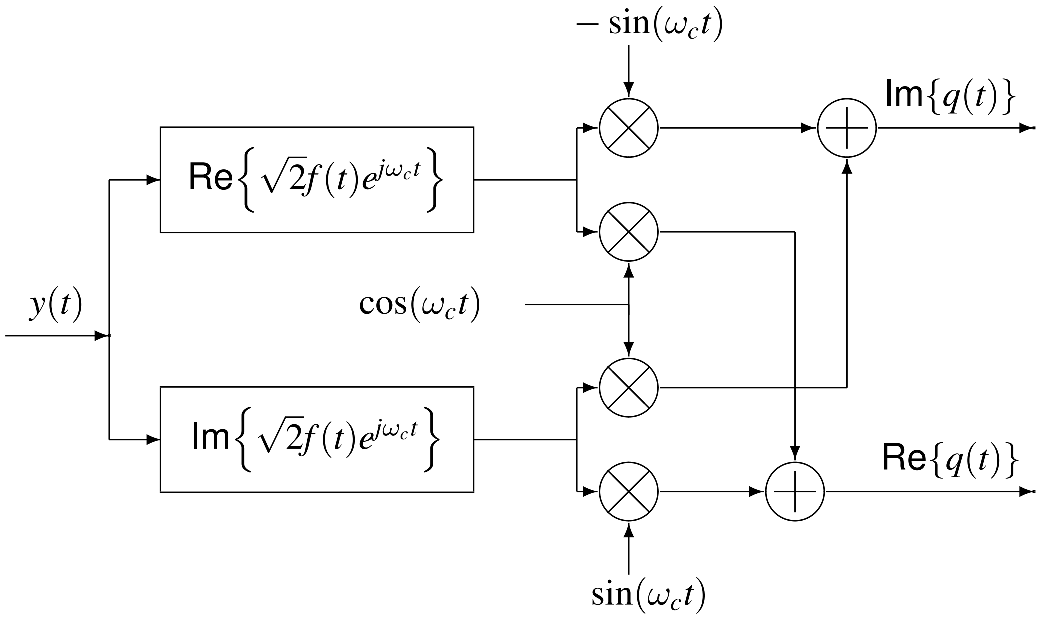
$$q(t) = e^{-j\omega_c t} \cdot \int_{-\infty}^{\infty} \sqrt{2} \cdot f(\tau) \cdot e^{j\omega_c \tau} \cdot y(t-\tau) \, d\tau$$

$$q(t) = e^{-j\omega_c t} \cdot \left(y(t) * \left(\sqrt{2} \cdot f(t) \cdot e^{j\omega_c t}\right)\right)$$

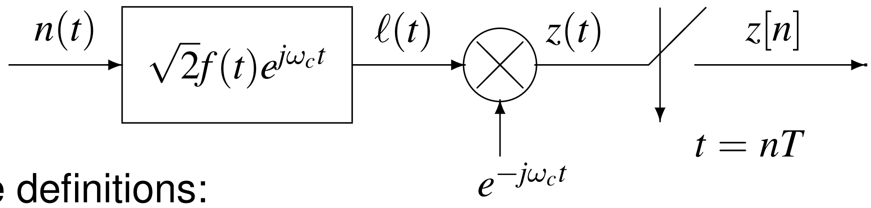
Bandpass filtering and then demodulation

Equivalent alternative demodulator (II)





Noise characteristics at the receiver



- Some definitions:

$$f_c(t) = \sqrt{2} \cdot f(t) \cdot e^{j\omega_c t}, \; F_c(j\omega) = \sqrt{2} \cdot F(j\omega - j\omega_c)$$

- Properties:
  1. $z(t)$ is strict sense stationary only if $\ell(t)$ es circularly symmetric
     NOTE: A complex process $X(t)$ is circularly symmetric if real and imaginary parts, $X_r(t)$ and $X_i(t)$, are jointly stationary, and their correlations satisfy

$$R_{X_r}(\tau) = R_{X_i}(\tau), \; R_{X_r,X_i}(\tau) = -R_{X_i,X_r}(\tau)$$

  2. $\ell(t)$ is circularly symmetric if ω_c is higher than bandwidth of filter $f_c(t)$ (narrow band system)

$$S_\ell(j\omega) = 2 \cdot S_n(j\omega) \cdot |F(j\omega - j\omega_c)|^2$$

Noise signal $z(t)$ at the receiver

- $z(t)$ is circularly symmetric and its power spectral density is

$$S_z(j\omega) = 2 \cdot S_n(j\omega + j\omega_c) \cdot |F(j\omega)|^2$$

  - In the process is symmetric, its real and imaginary parts, $z_I(t)$ and $z_Q(t)$, have the same variance and are independer for any time instant t
  - In general, $z_I(t_1)$ and $z_Q(t_2)$, for $t_1 \neq t_2$ are not independent
  - If spectrum is hermitic, $S_z(j\omega) = S_z^*(-j\omega)$, $z_I(t_1)$ and $z_Q(t_2)$, for $t_1 \neq t_2$ are also independent
    - ★ If $n(t)$ is white, this is fulfilled when $f(t)$ is real

Discrete noise sequence $z[n]$ at the receiver

- $z[n]$ is circularly symmetric

$$S_z\left(e^{j\omega}\right) = \frac{2}{T} \cdot \sum_k S_n\left(j\frac{\omega}{T} + j\frac{\omega_c}{T} - j\frac{2\pi k}{T}\right) \cdot \left|F\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)\right|^2$$

For white noise $n(t)$

$$S_n(j\omega) = \frac{N_0}{2}$$

Now

- $z_I[n]$ and $z_Q[n]$ are independent for any instant n
- $z_I[n_1]$ and $z_Q[n_2]$, for $n_1 \neq n_2$, are only independent if $S_z\left(e^{j\omega}\right)$ is a symmetric function
  - This happens for white noise if the ambiguity function of $f(t)$, $r_f(t) = f(t) * f^*(-t)$, satisfies the Nyquist ISI criterion at symbol period T

Variance and distribution of $z[n]$

- The variance of complex discrete noise is

$$\sigma_z^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_z\left(e^{j\omega}\right)\, d\omega$$

- In noise $n(t)$ is white, with $S_n(j\omega) = N_0/2$ W/Hz, and if $r_f(t)$ is normalized and satisfies the Nyquist ISI criterion

$$\sigma_z^2 = N_0$$

- If noise is circularly symmetric
  - Real and imaginary parts ($z_I[n]$ and $z_Q[n]$) are independent and both have variance $N_0/2$
  - Probability density function of noise level is

$$f_Z(z) = \frac{1}{\pi N_0} e^{-\frac{|z|^2}{N_0}}$$

NOTE: If receiver filter is not normalized, noise variance is multiplied by $\mathcal{E}\{f(t)\}$

Baseband equivalent discrete channel

- Definition of the complex equivalent baseband channel, $h_{eq}(t)$

$$h_{eq}(t) = e^{-j\omega_c t} \cdot h(t) \leftrightarrow H_{eq}(j\omega) = H(j\omega + j\omega_c)$$

The behavior of the channel around central frequency ω_c is shifted down to baseband

- Signal at the output of the matched filter

$$q(t) = \sum_n A[n] \cdot p(t - nT) + z(t)$$

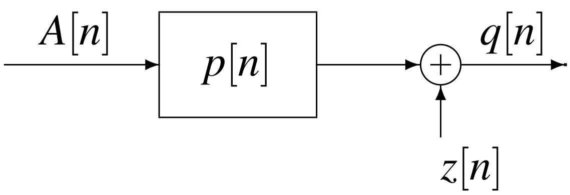
  - $p(t) = g(t) * h_{eq}(t) * f(t)$, $P(j\omega) = G(j\omega) \cdot H_{eq}(j\omega) \cdot F(j\omega)$

- Baseband equivalent discrete channel:

$$p[n] = p(t)\big|_{t=nT} = p(nT)$$

$$\begin{aligned} P\left(e^{j\omega}\right) &= \frac{1}{T}\sum_k P\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right) \\ &= \frac{1}{T}\sum_k G\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right) \cdot H_{eq}\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right) \cdot F\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right) \end{aligned}$$

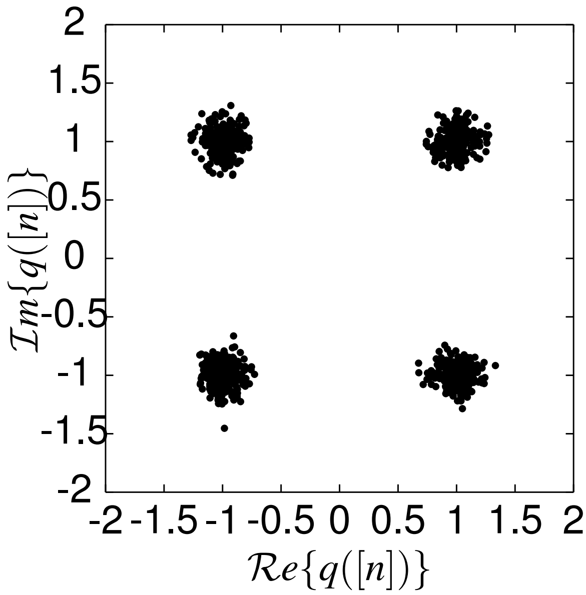
Equivalent discrete channels - baseband and band pass PAM



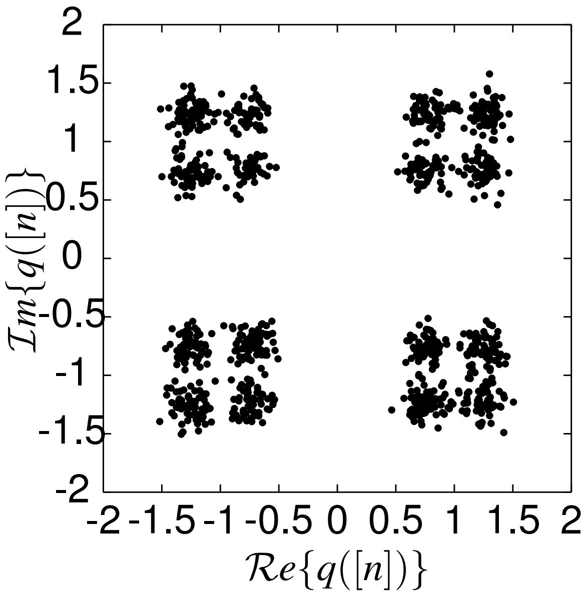
- Identification of baseband and band pass PAM
  - Symbols $A[n]$
  - Equivalent discrete channel $p[n]$
  - Discrete noise $z[n]$
    - ⋆ Are real in baseband PAM
    - ⋆ Are complex in band pass PAM

Scattering diagram

- Monitoring tool for band pass system
  - Plotting of $\mathcal{R}e(q[n])$ versus $\mathcal{I}m(q[n])$
  - Ideally: the transmitted constellation must be plotted
  - Allows to monitor noise level, ISI level, synchronism errors



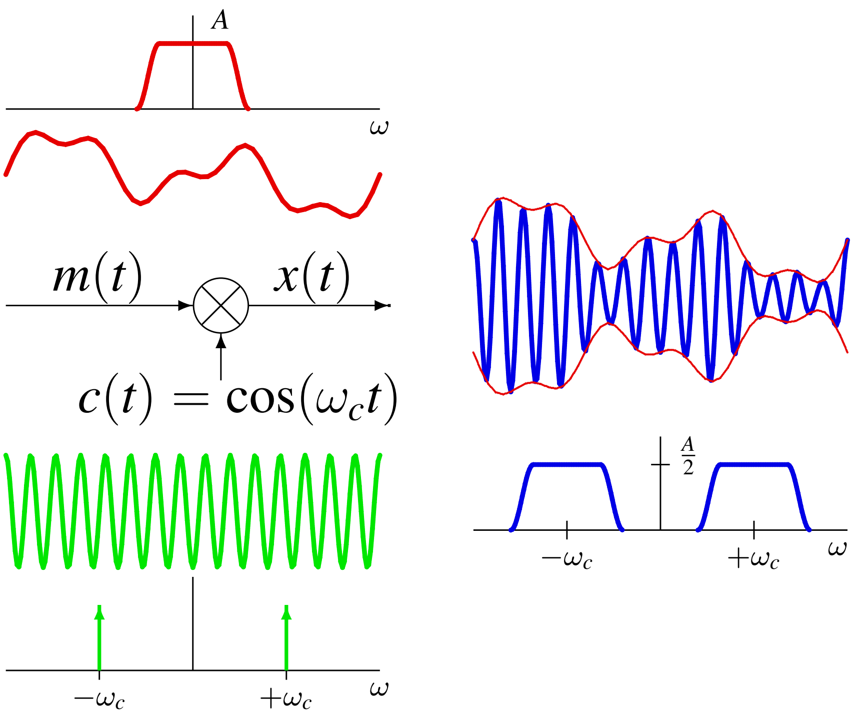
$p[n] = \delta[n]$



$p[n] = \delta[n] - 0{,}25\delta[n-1]$

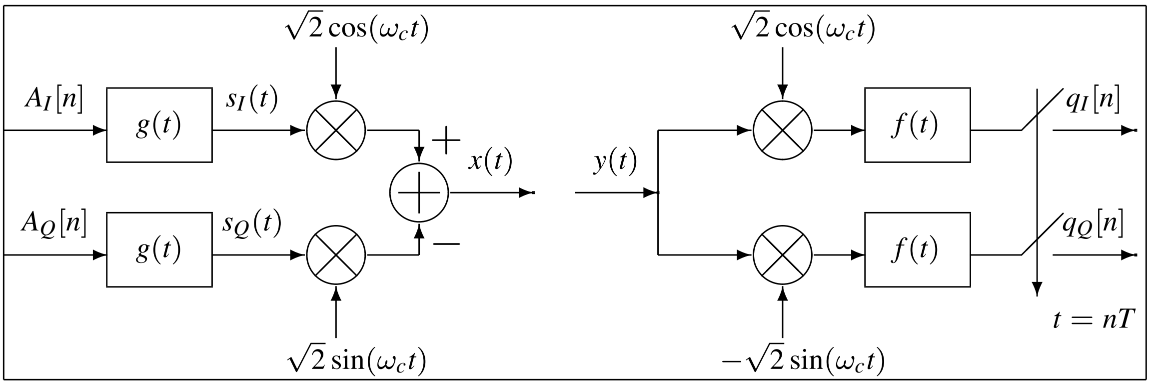
Reminder - AM modulations

- Product by a sinusoid of frequency ω_c shifts spectrum ω_c



Analytic analysis of modulation / demodulation

- Block diagram for transmitter and receiver



- Transmitter multiplies two baseband signals by two orthogonal carriers
- Receiver demodulates each component and then filters with $f(t)$
  - Receiver filter $f(t)$ has a baseband characteristic
  - Typical set-up: root-raised cosine filter

Analytic analysis of modulation / demodulation (II)

- Undistorted received signal (modulated signal) has the shape

$$y(t) = A \cdot \cos(\omega_c t) + B \cdot \sin(\omega_c t)$$

- At the receiver, signal processing is splitted in two components

$$y_A(t) = [A \cdot \cos(\omega_c t) + B \cdot \sin(\omega_c t)] \times \cos(\omega_c t)$$

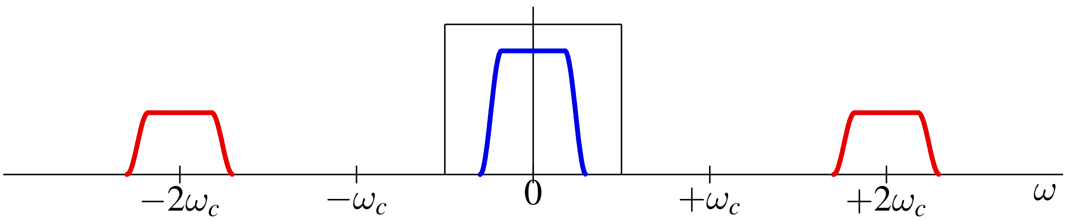
$$y_B(t) = [A \cdot \cos(\omega_c t) + B \cdot \sin(\omega_c t)] \times \sin(\omega_c t)$$

- Trigonometric identities and removing (filtering) of bandpass terms

$$X \cdot \cos(\omega_c t) \cdot \cos(\omega_c t) = \underbrace{\frac{X}{2}}_{Desired} + \underbrace{\frac{X}{2} \cdot \cos(2\omega_c t)}_{Bandpass\ at\ 2\omega_c}$$

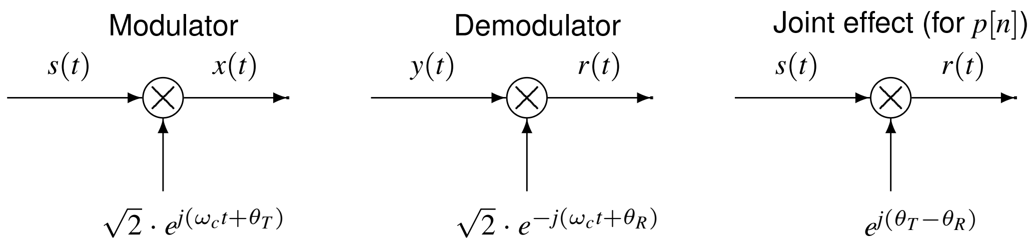
$$X \cdot \sin(\omega_c t) \cdot \cos(\omega_c t) = \underbrace{\frac{X}{2} \cdot \sin(2\omega_c t)}_{Bandpass\ at\ 2\omega_c}$$

$$X \cdot \sin(\omega_c t) \cdot \sin(\omega_c t) = \underbrace{\frac{X}{2}}_{Desired} - \underbrace{\frac{X}{2} \cdot \cos(2\omega_c t)}_{Bandpass\ at\ 2\omega_c}$$



Analytic analysis of modulation / demodulation (III)

- The product of two carriers allows to recover the transmitted baseband signals
  - Products $\cos(\omega_c t) \times \cos(\omega_c t)$ or $\sin(\omega_c t) \times \sin(\omega_c t)$ introduce a $\frac{1}{2}$ factor
    - ★ Factors $\sqrt{2}$ are introduced at transmiter and receiver to compensate it
  - Complex notation fails to represent this scaling
    - ★ This has to be taken into account



- Non-coherent receivers
  - Receiver whose demodulator has a phase that is different than phase at modulator
  - Produces a rotation in the received constellation

Binary transmission rate (R_b bits/s)

- Binary transmission rate is obtained as $R_s = m \times R_s$
  - Symbol rate (R_s bauds)
  - Number of bits per symbol in the constellation (m)

$$m = \log_2(M)$$

M: number of symbols of the constellation

- Limitation in the achievable binary rate
  - Limitation in R_s: available bandwidth (B Hz)

    Using filters of the raised cosine family

BASEBAND	BAND PASS
$R_{s\|max} = \frac{2B}{1+\alpha}$	$R_{s\|max} = \frac{B}{1+\alpha}$

  - Limitation on the number of symbols M (and therefore in m)
    - ★ Power limitation limits mean energy per symbol $E_s = E\left[|A[n]|^2\right]$

      - This limits the maximum modulus of the constellation
    - ★ Performance requirements limit the minimum distance between symbols

$$P_e \approx k \cdot Q\left(\frac{d_{min}}{2\sqrt{N_0/2}}\right)$$

    - ★ E_s and P_e determine a maximum constellation density

Constellation density - Example - QAM