Digital Communications
Telecommunications Engineering

Chapter 3

Angle modulations

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Phase modulations

- Phase shift keying (PSK) modulation
  - Linear modulation
  - Constellations: Constant modulus - Information in the phase

$$A[n] = \sqrt{E_s} \cdot e^{j\phi[n]}$$

$$x(t) = \sqrt{2E_s} \cdot \mathcal{R}e\left\{\sum_n g(t-nT) \cdot e^{j(\omega_c t + \phi[n])}\right\}$$
$$= \sqrt{2E_s} \cdot \sum_n g(t-nT) \cdot \cos(\omega_c t + \phi[n])$$

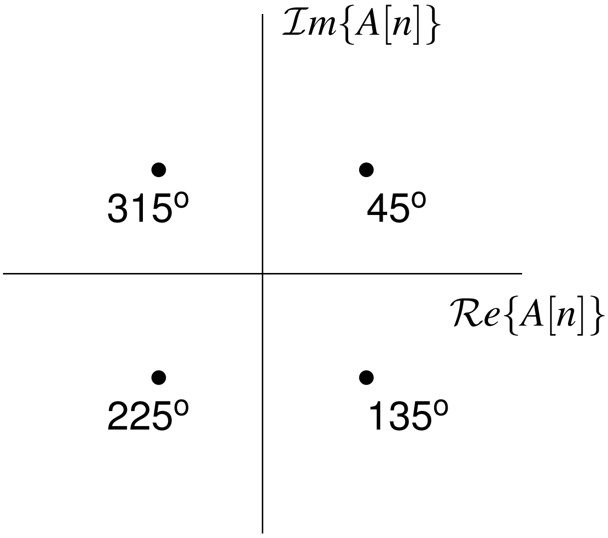
- Constant envelope modulation can be achieved by using

$$g(t) = \frac{1}{\sqrt{T}} \cdot w_T(t), \qquad w_T(t) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{other case} \end{cases}$$

- Drawback: high bandwith (phase shifts at $t = nT$)

$$S_s(j\omega) = E_s \cdot \text{sinc}^2\left(\frac{\omega T}{2\pi}\right)$$

QPSK Modulation - PSK for $M = 4$ - Constellation



- $\phi[n] = 45^o$: $A[n] = +1 + j$
- $\phi[n] = 135^o$: $A[n] = +1 - j$
- $\phi[n] = 225^o$: $A[n] = -1 - j$
- $\phi[n] = 315^o$: $A[n] = -1 + j$

Phase shifts in a QPSK modulation

- PSK signal

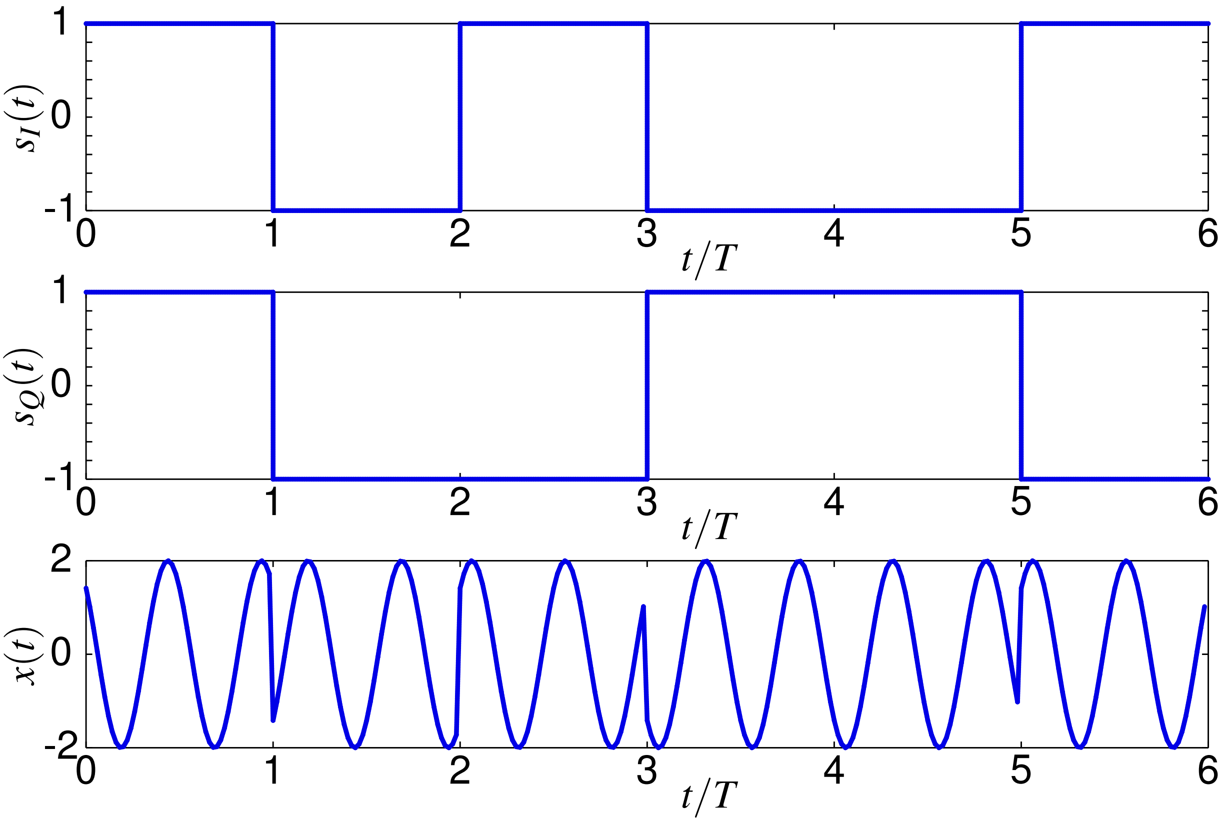
$$x(t) = \sqrt{2} \cdot s_I(t) \cdot \cos(\omega_c t) - \sqrt{2} \cdot s_Q(t) \cdot \sin(\omega_c t)$$
$$= \sqrt{2E_s} \cdot \sum_n g(t - nT) \cdot \cos(\omega_c t + \phi[n])$$

with

$$s_I(t) = \sum_n \mathcal{Re}\{A[n]\} \cdot g(t - nT)$$
$$s_Q(t) = \sum_n \mathcal{Im}\{A[n]\} \cdot g(t - nT)$$

- Phase shifts
  - $\pm 90^o$: a change in $s_I(t)$ or in $s_Q(t)$
  - 180^o: simultaneous change in both $s_I(t)$ and $s_Q(t)$

Waveforms for QPSK modulation



Offset QPSK modulation (OQPSK)

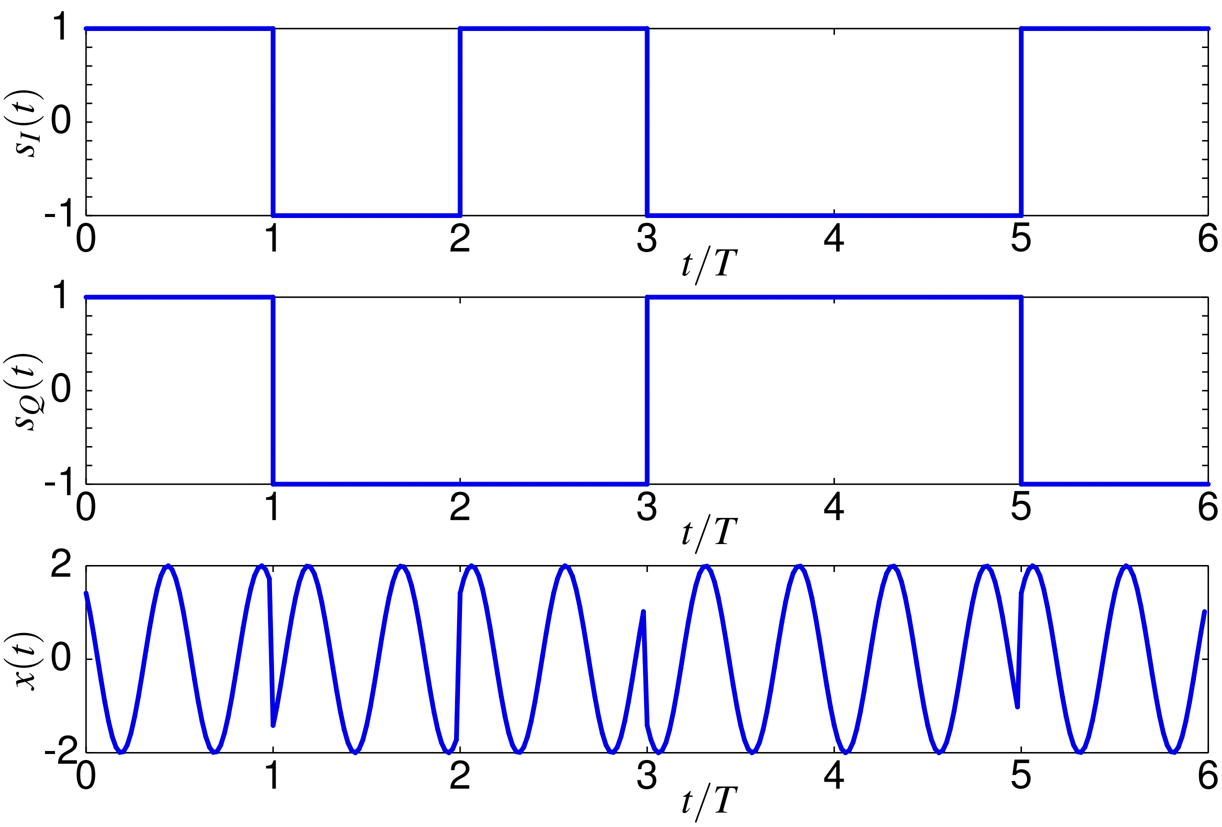
- Goal: to avoid 180^o phase shifts
  - Avoidance of simultaneous transitions of $s_I(t)$ and $s_Q(t)$
- OQPSK signal
  - Quadrature component is delayed $T/2$ seconds
  - Phase shifts are limited to $\pm 90^o$
  - Phase shifts happen often (can occur each $T/2$ seconds)

$$x(t) = \sqrt{2} \cdot s_I(t) \cdot \cos(\omega_c t) - \sqrt{2} \cdot s_Q(t) \cdot \sin(\omega_c t)$$

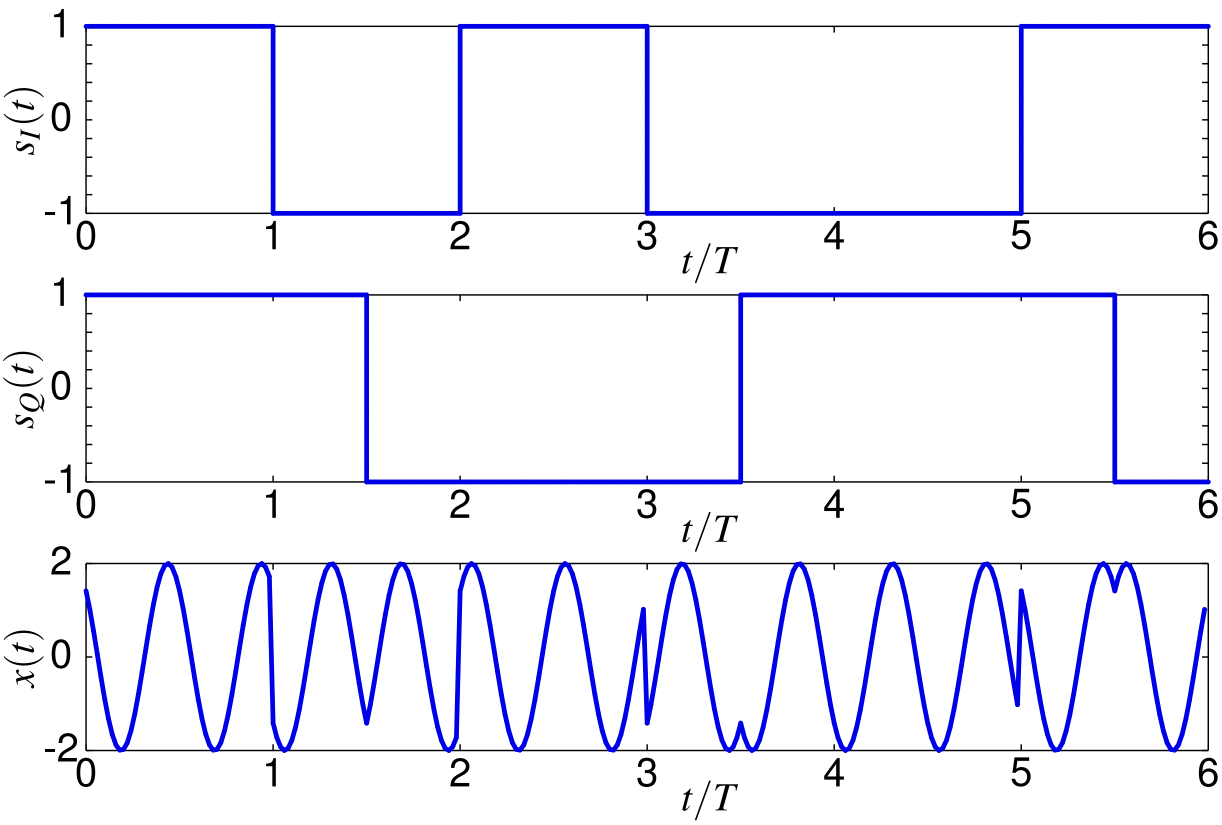
$$s_I(t) = \sum_n \mathcal{R}e\{A[n]\} \cdot g(t - nT)$$

$$s_Q(t) = \sum_n \mathcal{I}m\{A[n]\} \cdot g(t - nT - T/2)$$

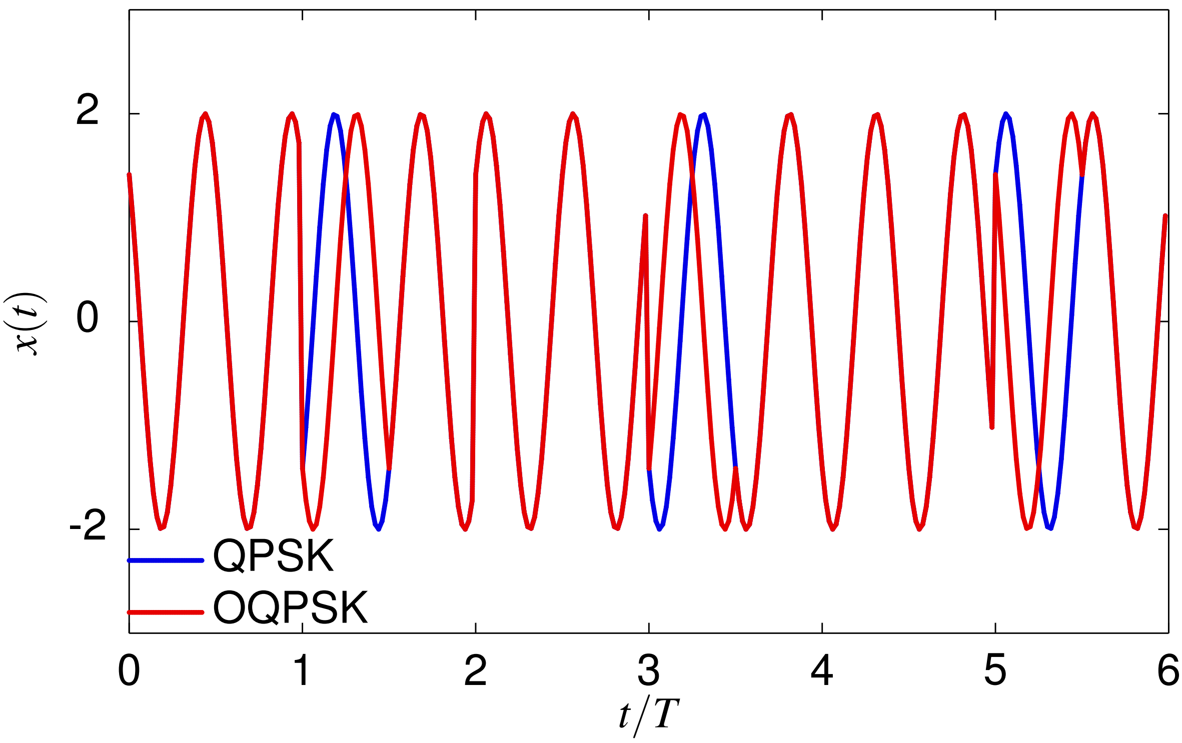
Waveforms for QPSK modulation



Waveforms for OQPSK - Delay of $s_Q(t)$



Modulation waveforms - QPSK vs OQPSK



Spectrum of OQPSK modulation

- Definitions

$$x_I(t) = \sqrt{2} \cdot s_I(t) \cdot \cos(\omega_c t),\ x_Q(t) = \sqrt{2} \cdot s_Q(t) \cdot \sin(\omega_c t)$$

- Spectrum for each component ($s_k,\ k \in \{I, Q\}$)

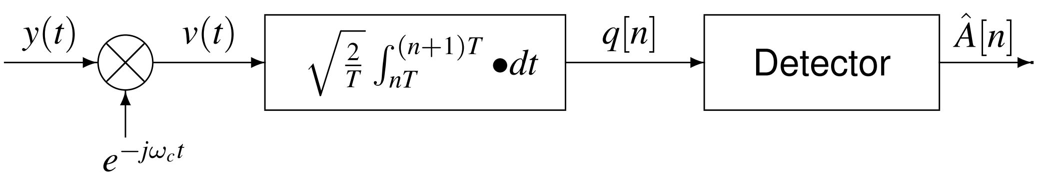
$$S_{x_k}(j\omega) = \frac{1}{2}\left[S_{s_k}(j\omega - j\omega_c) + S^*_{s_k}(-j\omega - j\omega_c)\right]$$

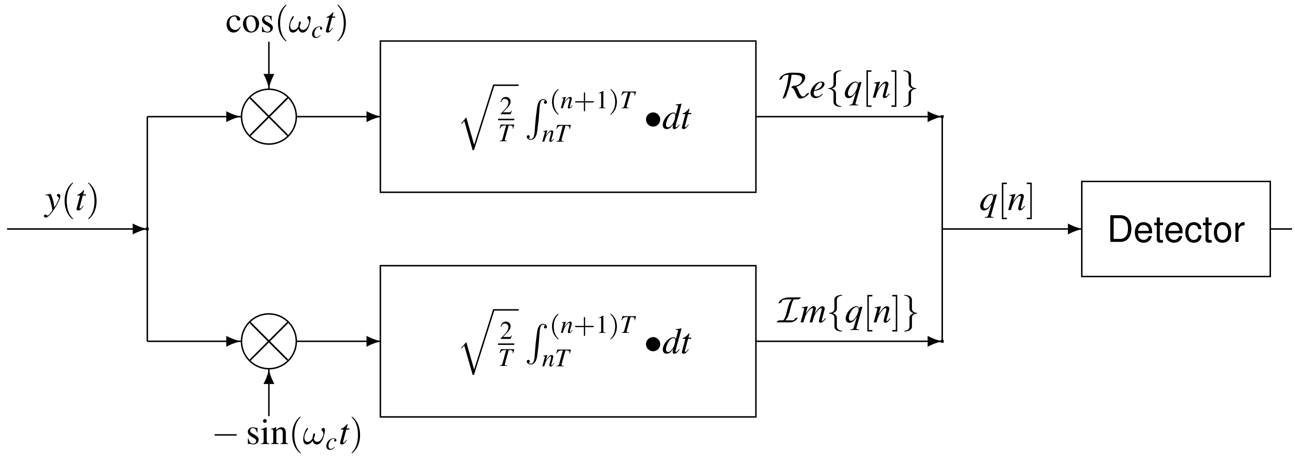
$$S_{s_I}(j\omega) = \frac{\mathcal{E}\{\mathcal{R}e\{A[n]\}\}}{T}\,|G(j\omega)|^2\,, S_{s_Q}(j\omega) = \frac{\mathcal{E}\{\mathcal{I}m\{A[n]\}\}}{T}\,|G(j\omega)|^2$$

- Spectrum of OQPSK modulation

$$S_x(j\omega) = \frac{E_s}{2T}\left[|G(j\omega - j\omega_c)|^2 + |G(-j\omega - j\omega_c)|^2\right]$$

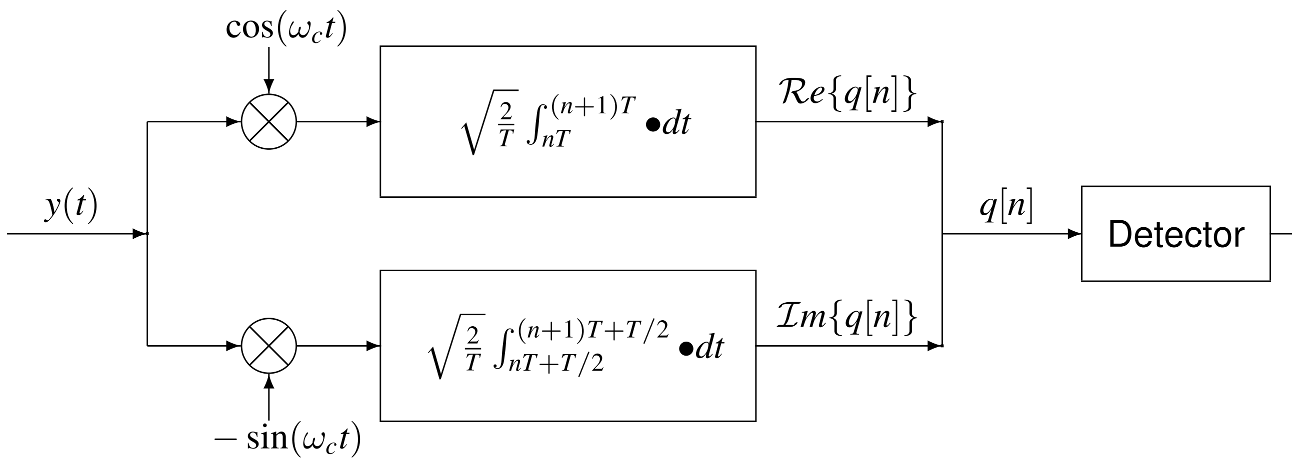
Receivers for PSK modulations





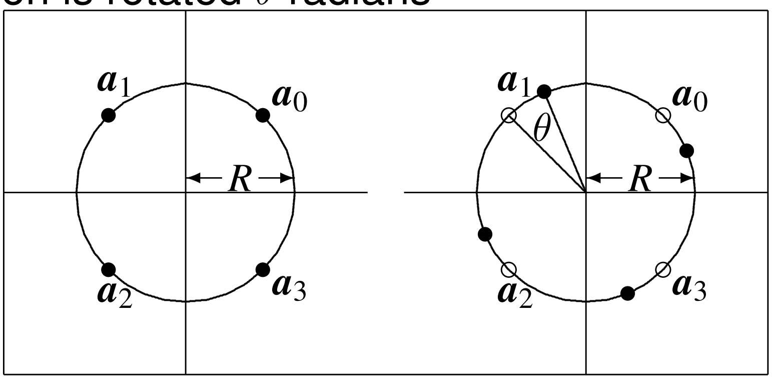
Receiver for OQPSK

The $T/2$ delay in the quadrature component is taken into account (delay in the correlator)



Effect of non-coherent receiver in PSK modulations

- In a non-coherent receiver, phase of carriers used at receiver to demodulate is different from phase of carriers used at the transmitter to modulate
  - Difference of θ radians
- The effect of this phase difference is that received constellation is rotated θ radians

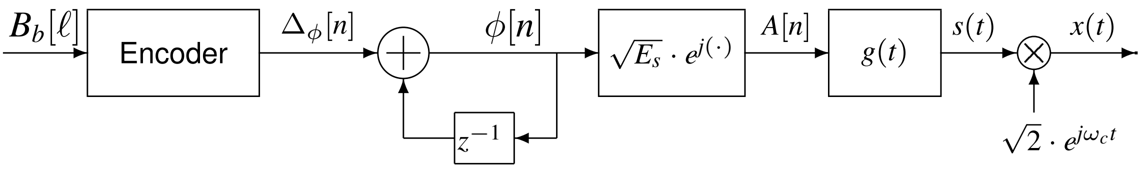


  - This effect can seriously affect performance
  - However, non-coherent receivers have a lower cost
    - ★ Differential PSK modulation allows the use of non-coherent receivers

Differential phase modulations

- They do not require a coherent demodulation
- PSK with differential phase enconding

$$\phi[n] = \phi[n-1] + \Delta_\phi[n]$$



- Encoder for M-ary modulation

$$\Delta_\phi[n] \in \left\{0, \frac{2\pi}{M}, \cdots, \frac{2\pi(M-1)}{M}\right\}$$

Bit assignment is performed through $\Delta_\phi[n]$

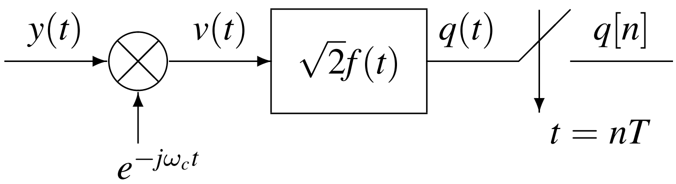
Example: 4-PSK

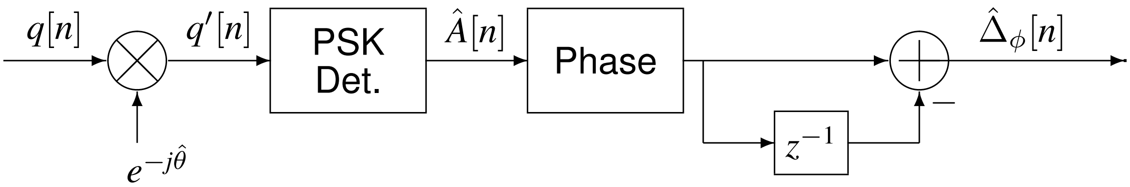
$\Delta_\phi[n]$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
Bits	00	10	11	01

(Gray encoding)

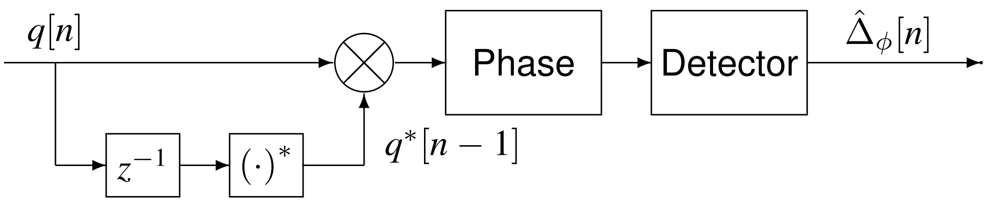
- Initialization: selection of an arbitrary (known) value for $\phi[-1]$
  - No error propagation because of initialization

Differential PSK demodulators





Coherent Receiver



Non-coherent Receiver

Non-coherent DPSK demodulator

- Observation

$$q[n] = \sqrt{E_s} \cdot e^{j(\phi[n]+\theta)} + z[n]$$

$$q^*[n-1] = \sqrt{E_s} \cdot e^{-j(\phi[n-1]+\theta)} + z^*[n-1]$$

- Multiplier

$$q[n] \cdot q^*[n-1] = E_s \cdot e^{j(\phi[n]-\phi[n-1])} + \sqrt{E_s} \cdot e^{j(\phi[n]+\theta)} \cdot z^*[n-1] + \sqrt{E_s} \cdot e^{-j(\phi[n-1]+\theta)} \cdot z[n] + z[n] \cdot z^*[n-1]$$

- Detector

$$\hat{\Delta}_\phi[n] = \angle\{q[n] \cdot q^*[n-1]\}$$

Probability of error for DPSK

- Probability of error using coherent receivers

$$P_e \approx 2 \cdot P_e^{PSK}$$

An erroneous decided symbol $\hat{A}[n]$ affects two increments $\Delta_\phi[n]$ and $\Delta_\phi[n+1]$

- Probability of error with non-coherent receivers
  - Statistic used for detection

$$\frac{q[n] \cdot q^*[n-1]}{\sqrt{E_s}} = \sqrt{E_s} \cdot e^{j(\phi[n]-\phi[n-1])} + e^{j(\phi[n]+\theta)} \cdot z^*[n-1] + e^{-j(\phi[n-1]+\theta)} \cdot z[n] + \frac{z[n] \cdot z^*[n-1]}{\sqrt{E_s}}$$

  - Three terms of noise
    - ★ Last one can be negligible for high E_s/σ_z^2
    - ★ The other two terms: independent, circularly symmetric
  - Signal to noise ratio: 3 dB loss (double noise power)
    - ★ Signal: E_s
    - ★ Noise: $2\sigma_z^2$

Frequency shift keying (FSK) modulation

- Information: discrete frequency changes in the frequency of a carrier
- Definition: M pulses (to map M symbols)

$$g_i(t) = \sin(\omega_i t) \cdot w_T(t), \quad i = 0, 1, \cdots, M-1$$

- Encoder: defines index of transmitted pulse at discrete time n

$$A[n] \in \{i = 0, 1, \cdots, M-1\}$$

- FSK signal in the time domain

$$x(t) = K \cdot \sum_n g_{A[n]}(t - nT)$$

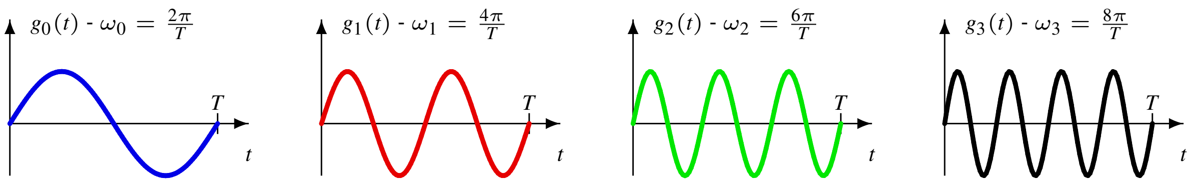
- Continuous phase FSK (CPFSK)
  - Phase continuity: pulses with an integer number of periods in T seconds

Frequencies: $\omega_i = \frac{2\pi}{T} \cdot N_i, \qquad N_i \in \mathbb{Z}, \qquad i = 0, \cdots, M-1$

  - Minimum bandwidth: consecutive N_i (spectrum of $g_i(t)$ is centered at ω_i)

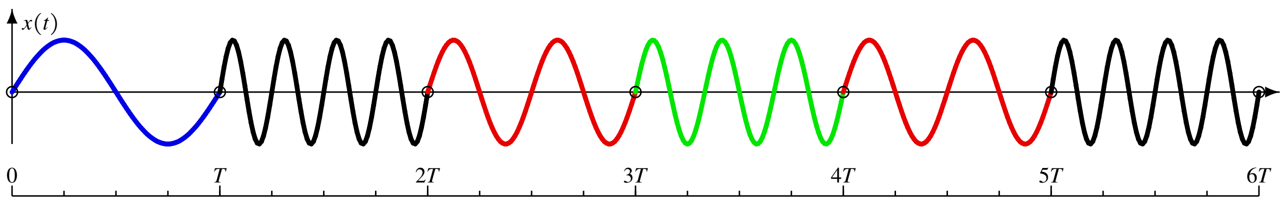
CPFSK waveforms - Example for $M = 4$

- CPFSK pulses for $M = 4$ (just one example)



- Waveform for data sequence

n	0	1	2	3	4	5
$A[n]$	0	3	1	2	1	3



Orthogonality of CPFSK

- Inner product of two pulses

$$\langle g_i(t), g_\ell(t)\rangle = \int_0^T \sin(\omega_i t)\cdot \sin(\omega_\ell t)\, dt$$
$$= \frac{1}{2}\int_0^T \cos(\underbrace{(\omega_i-\omega_\ell)}_{(N_i-N_\ell)\frac{2\pi}{T}}\cdot t)\, dt - \frac{1}{2}\int_0^T \cos(\underbrace{(\omega_i+\omega_\ell)}_{(N_i+N_\ell)\frac{2\pi}{T}}\cdot t)\, dt$$
$$= \frac{T}{2}\cdot \delta[i-\ell]$$

Pulses of CPFSK are orthogonal

- Definition of an orthonormal base of dimension M

$$\phi_i(t) = \sqrt{\frac{2}{T}}\sin(\omega_i t)\cdot w_T(t), \quad i = 0, 1, \cdots, M-1$$

- CPFKS signal as an expansion in the orthonormal base

$$x(t) = \sqrt{E_s}\cdot \sum_n \phi_{A[n]}(t-nT)$$

FSK spectrum

- Mean of the signal is periodic
- Discrete spectrum (spectrum of the periodic mean)

$$S_{Xd}(j\omega) = \frac{2E_s}{T}\frac{1}{(MT)^2}\left|\sum_{i=0}^{M-1} G_i(j\omega)\right|^2 \cdot \sum_k \delta\left(\omega - \frac{2\pi k}{T}\right)$$

- Continuous spectrum (spectrum of the signal without the mean)

$$S_{Xc}(j\omega) = \frac{2E_s}{T}\frac{1}{MT}\left\{\sum_{i=0}^{M-1}|G_i(j\omega)|^2 - \frac{1}{M}\left|\sum_{i=0}^{M-1} G_i(j\omega)\right|^2\right\}$$

- FSK - Power spectral density

$$S_X(j\omega) = S_{Xc}(j\omega) + S_{Xd}(j\omega)$$

Receivers for FSK modulation

- Coherent receiver with matched filters or correlators

$$P_e = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

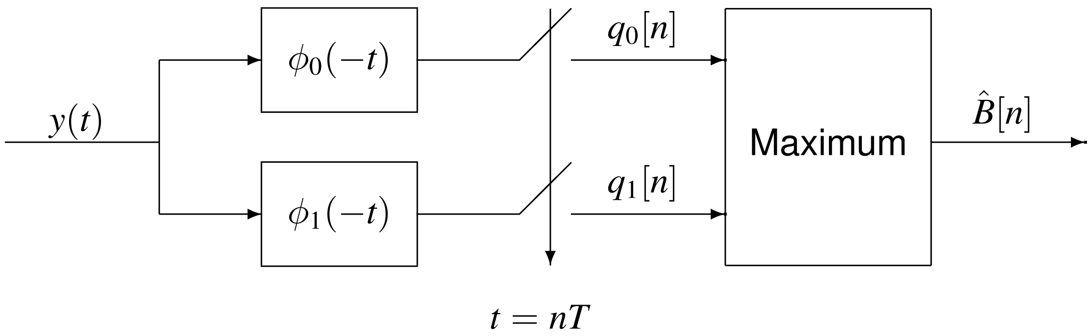
- Efect of phase error - Example: $n = 0$, $A[n] = i$, phase error θ

$$y(t) = \sqrt{\frac{2E_s}{T}} \cdot \sin(\omega_i t + \theta) \cdot w_T(t)$$

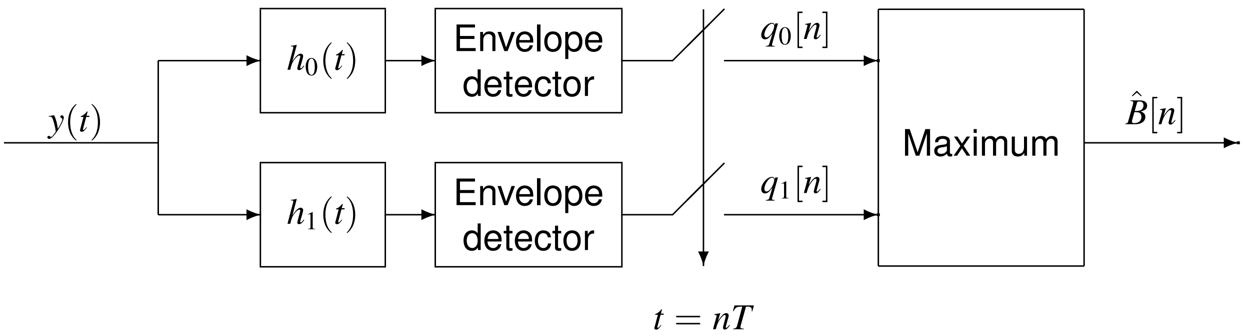
$$\begin{aligned} q_\ell[0] &= \int_0^T y(t) \cdot \phi_\ell(t)\, dt = \int_0^T \sqrt{\frac{2E_s}{T}} \cdot \sin(\omega_i t + \theta) \cdot \sqrt{\frac{2}{T}} \cdot \sin(\omega_\ell t)\, dt \\ &= \frac{\sqrt{E_s}}{T} \int_0^T [\cos((\omega_i - \omega_\ell)t + \theta) - \cos(\omega_i + \omega_\ell)t + \theta)]\, dt \\ &= \sqrt{E_s} \cdot \cos(\theta) \cdot \delta[i - \ell] \end{aligned}$$

  - Atenuation term: $\cos(\theta)$

Coherent receiver for binary FSK

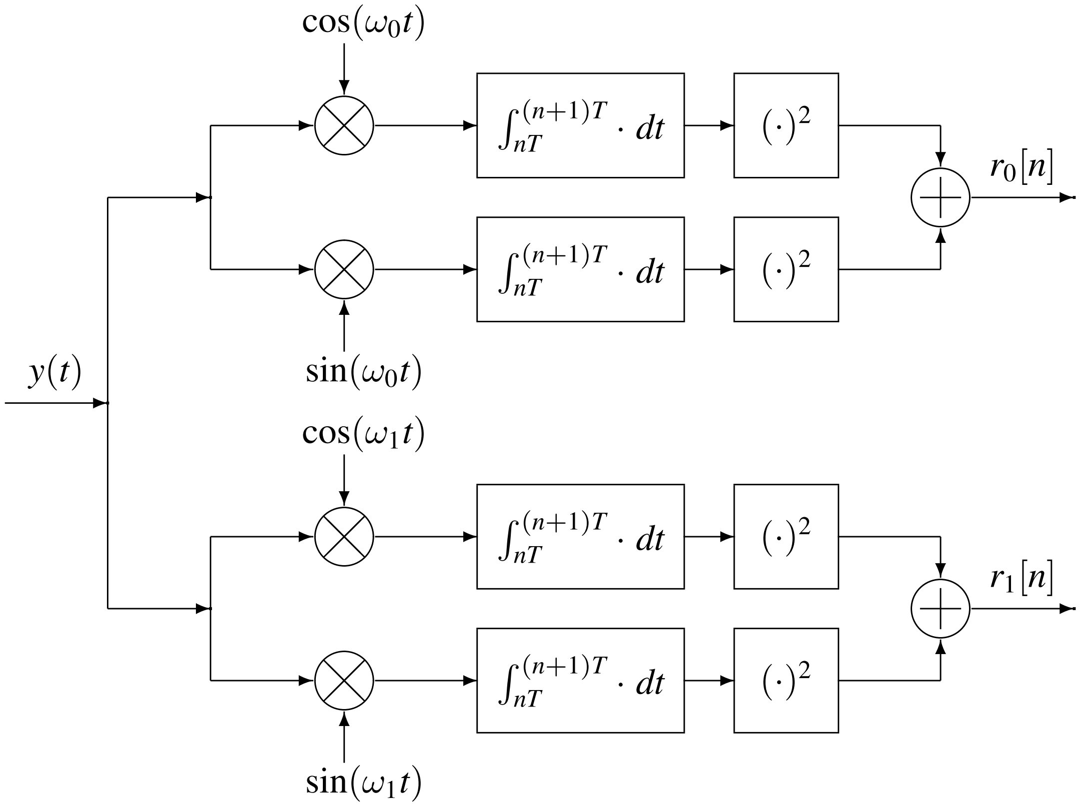


Coherent Receiver



Non-coherent Receiver

Non-coherent (quadratic law) receiver for FSK



FSK seem as frequency shift from a central frequency

- Definition of central frequency

$$\omega_c = \frac{\omega_0 + \omega_{M-1}}{2} = \frac{\pi}{T} \times C,\ \ C \in \mathbb{Z},\ \ C \text{ odd}$$

  - Value of central frequency: $\omega_c = \frac{\pi}{T} \times$ odd integer
  - Frecuencies of the pulse for symbol of discrete index n

$$\omega_c + I[n] \cdot \frac{\pi}{T}$$

- Encoder

$$I[n] \in \{\pm 1, \pm 3, \cdots, \pm(M-1)\}$$

- FSK analytic expression as shift from ω_c

$$x(t) = \sqrt{\frac{2E_s}{T}} \cdot \sum_n \sin\left(\omega_c t + I[n] \cdot \frac{\pi t}{T}\right) \cdot w_T(t - nT)$$

Minimum shift keying (MSK) modulation

- Information: discrete frequency changes in the frequency of a carrier
- Orthogonality of carriers with minimum frequency separation
- Inner product of pulses $g_i(t)$

$$\begin{aligned}\langle \boldsymbol{g}_i, \boldsymbol{g}_\ell \rangle &= \int_0^T \sin(\omega_i t) \cdot \sin(\omega_\ell t)\, dt \\ &= \frac{1}{2}\int_0^T \cos[(\omega_i - \omega_\ell)\cdot t]\, dt - \frac{1}{2}\cos[(\omega_i + \omega_\ell)\cdot t]\, dt \\ &= \frac{1}{2}\frac{\sin[(\omega_i - \omega_\ell)\cdot T]}{(\omega_i - \omega_\ell)} - \frac{1}{2}\frac{\sin[(\omega_i + \omega_\ell)\cdot T]}{(\omega_i + \omega_\ell)}\end{aligned}$$

- Minimum required frequency separation (in narrow band systems)
  - Assumption: $\frac{\sin[(\omega_i+\omega_\ell)\cdot T]}{(\omega_i+\omega_\ell)}$ can be neglected (high denominator)

$$\omega_i - \omega_\ell = \frac{\pi}{T}\cdot N_{i,\ell}, \qquad i,j = 0, 1, \cdots, M-1, \qquad i \neq \ell$$

Minimum shift keying (MSK) modulation (II)

- Key differences with CPFSK modulation
  - Separation between consecutive frequencies is half for MSK
    - ★ MSK: $\Delta_\omega = \omega_i - \omega_{i-1} = \frac{\pi}{T}$
    - ★ CPFSK: $\Delta_\omega = \omega_i - \omega_{i-1} = \frac{2\pi}{T}$
  - Values for ω_i are not constrained to be integer multiples of $\frac{2\pi}{T}$ as in CPFSK (neither integer multiples of $\frac{\pi}{T}$)
    - ★ Frequency selection does not automatically provides phase continuity
    - ★ Memory must be introduced to provide phase continuity
- MSK signal using central frequency notation

$$x(t) = \sqrt{\frac{2E_s}{T}} \cdot \sum_n \sin\left(\omega_c t + I[n]\frac{\pi t}{2T} + \theta[n]\right) \cdot w_T(t - nT)$$

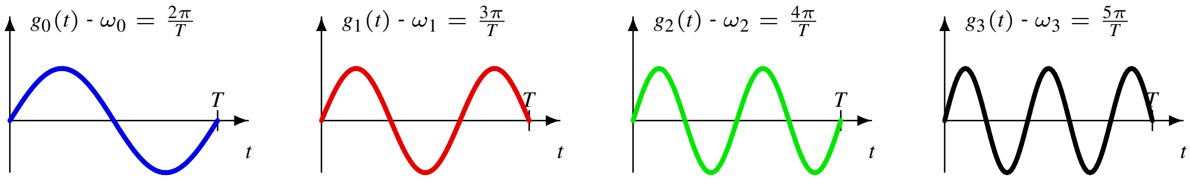
  - Encoder: $I[n] \in \{\pm 1, \pm 3, \cdots, \pm(M-1)\}$
  - Phase continuity is achieved by introducing memory term $\theta[n]$

$$\theta[n] = \theta[n-1] + \frac{\pi n}{2}\cdot\left(I[n-1] - I[n]\right), \qquad \text{mod } 2\pi$$

Recursive estimation of accumulated phase at the end of symbol intervals

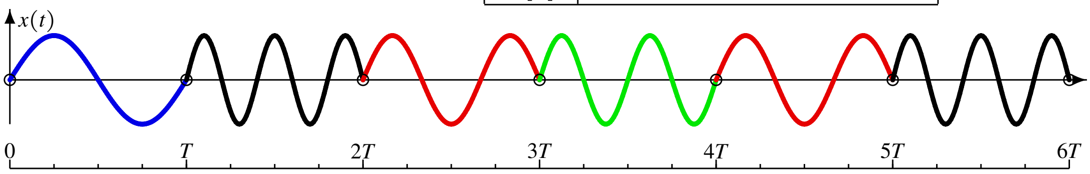
MSK waveforms - Example for $M = 4$

- Pulses for $M = 4$ (just one example)

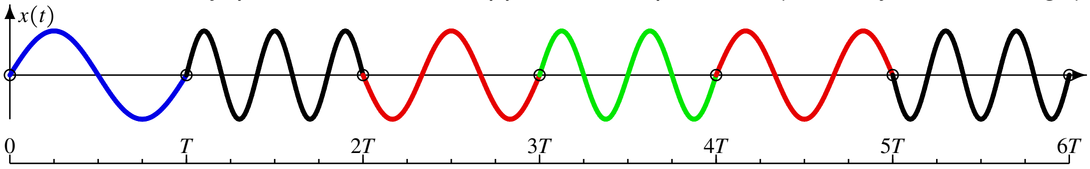


- Waveform for data sequence

n	0	1	2	3	4	5
$A[n]$	0	3	1	2	1	3



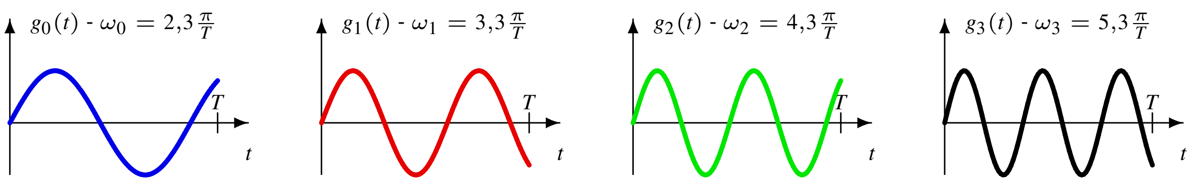
Without memory, phase shifts can happen at multiples of T (when symbols change)



Idenfification of phase at the end of each symbol interval ($\theta[n]$) allows phase continuity

MSK waveforms - Example for $M = 4$ (II)

- Another example with frequencies not being integer multiples of $\frac{\pi}{T}$



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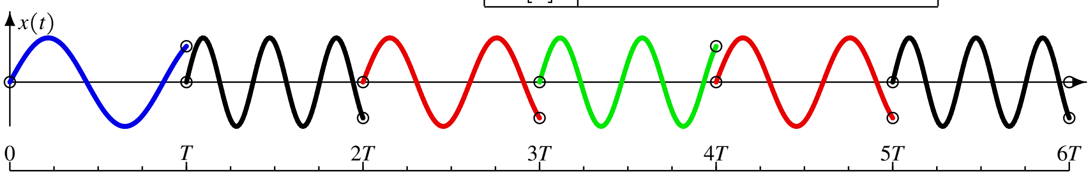
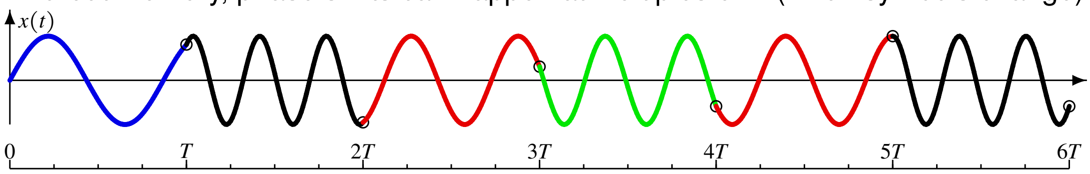


Without memory, phase shifts can happen at multiples of T (when symbols change)



Idenfification of phase at the end of each symbol interval ($\theta[n]$) allows phase continuity

MSK spectrum

- Alternative expression for MSK

$$x(t) = \sqrt{2E_s}\cos(\omega_c t)\sum_{\text{even } n} I[n]\cos(\theta[n])(-1)^{n/2} g(t-nT)$$
$$+ \sqrt{2E_s}\sin(\omega_c t)\sum_{\text{even } n} \cos(\theta[n])(-1)^{n/2} g(t-nT+T)$$

- Similar to OQPSK
  - Modified symbols
  - Pulse:

$$g(t) = \sqrt{\frac{1}{T}}\sin\left(\frac{\pi t}{2T}\right)\cdot w_{2T}(t),\ |G(j\omega)|^2 = 16T\pi^2\left(\frac{\cos(\omega T)}{\pi^2 - 4\omega^2 T^2}\right)^2$$

- MSK spectrum

$$S_X(j\omega) = 8E_s\pi^2\left(\frac{\cos[(\omega-\omega_c)T]}{\pi^2 - 4(\omega-\omega_c)^2T^2}\right)^2 + 8E_s\pi^2\left(\frac{\cos[(\omega+\omega_c)T]}{\pi^2 - 4(\omega+\omega_c)^2T^2}\right)^2$$

Receiver for MSK modulation

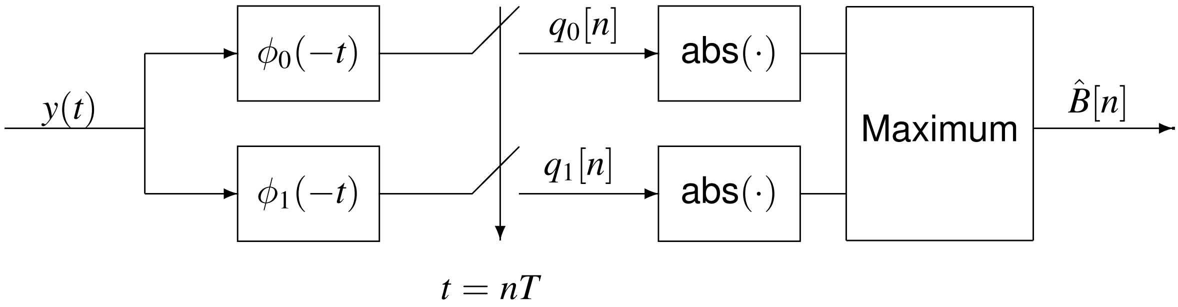
- Demodulator based on the ML receiver for FSK
- Demodulator based on the ML receiver for OQPSK
- Probability of error

$$P_e = 2\cdot Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

  - Memory is not taken into account at the receiver
  - Optimum receiver has a higher complexity

Receiver for binary MSK

- Sub-optimal MSK binary receiver based on a FSK receiver where the absolute value evaluation for each possible frequency is introduced to consider different possible initial phases



Continuous phase (CPM) modulations

- Family of modulations including CPFSK and MSK modulations
- Basic characteristics
  - Constant envelope
  - Phase continuity
  - Bandwith reduction: smoothing the evolution of the instantaneous phase
- CPM signal: analytic expression in the time domain

$$x(t) = \sqrt{\frac{2E_s}{T}} \cdot \sin\left[\omega_c t + \theta_0 + \theta(t, \boldsymbol{I})\right]$$

  - $\boldsymbol{I}$: Sequence of transmitted symbols
  - ω_c: nominal carrier frequency
  - θ_0: initial carrier phase
  - E_s: mean energy transmitted in a symbol period

Generation of the CPM signal

- Encoder: $I[n] \in \{\pm 1, \pm 3, \cdots, \pm(M-1)\}$
- Base band PAM signal

$$s(t) = \sum_n I[n] \cdot g(t - nT)$$

- Pulse $g(t)$ is causal, of length T, and normalized

Normalization: $\int_{-\infty}^{\infty} g(t)\, dt = \frac{1}{2}$

- CPM signal: instantaneous frequency $\omega_c + 2 \cdot \omega_d \cdot T \cdot s(t)$
- Instantaneous phase is obtained by integrating this frequency

$$\theta(t, \boldsymbol{I}) = 2 \cdot \omega_d \cdot T \cdot \int_{-\infty}^{t} s(\tau)\, d\tau$$

  - ω_d: peak frequency deviation

Time domain expression for CPM

$$x(t) = \sqrt{\frac{2E_s}{T}} \cdot \sin\left[\omega_c t + \theta_0 + \overbrace{2 \cdot \omega_d \cdot T \cdot \int_{-\infty}^{t} \underbrace{\sum_n I[n] \cdot g(\tau - nT)}_{s(\tau)}\, d\tau}^{\theta(t,\boldsymbol{I})}\right]$$

- Phase value $\theta(t, \boldsymbol{I})$ inside interval $[nT, (n+1)T]$ (symbol interval for $I[n]$)

$$\theta(t, \boldsymbol{I}) = 2 \cdot \omega_d \cdot T \cdot \int_{-\infty}^{t} s(\tau)\, d\tau = \theta[n] + \theta(t, n)$$

  - $\theta[n]$: phase that has been accumulated up to $t = nT$:
    - ★ Due to previous transmitted symbols (up to $I[n-1]$)

$$\theta[n] = \omega_d \cdot T \cdot \sum_{m=-\infty}^{n-1} I[m]$$

  - $\theta(t, n)$: incremental phase starting from $t = nT$:
    - ★ Due only to current symbol $I[n]$

$\theta(t, n) = 2 \cdot \omega_d \cdot T \cdot I[n] \cdot q_g(t - nT)$, being $q_g(t) = \int_{-\infty}^{t} g(\tau)\, d\tau$

Time domain expression for CPM - Modulation index

- Alternative time domain expression introducing a different parameter (replacing peak frequency deviation)
- Definition of modulation index h:

$$h = \omega_d \cdot \frac{T}{\pi}$$

- Phase value in the symbol interval associated to $I[n]$:
  - $\theta[n]$: accumulated phase up to $t = nT$:

$$\theta[n] = \pi \cdot h \cdot \sum_{m=-\infty}^{n-1} I[m]$$

  - $\theta(t,n)$: incremental phase from $t = nT$:

$$\theta(t,n) = 2 \cdot \pi \cdot h \cdot I[n] \cdot q_g(t - nT)$$

Identification of binary CPFSK modulation as a CPM

- Analytic expression for a CPFSK modulation

$$x(t) = \sqrt{\frac{2E_s}{T}} \cdot \sum_n \sin\left(\omega_c t + I[n] \cdot \frac{\pi t}{T}\right) \cdot w_T(t - nT)$$

- Binary CPFSK as a CPM: $\omega_d = \frac{\pi}{T}, \qquad h = 1$
- Considering $\theta[0] = 0$

$$\theta(t, \boldsymbol{I}) = \pi \sum_{m=0}^{n-1} I[m] + 2\pi \cdot I[n] \cdot \frac{(t - nT)}{2T} = \pi \sum_{m=0}^{n-1} I[m] - n \cdot \pi \cdot I[n] + \frac{\pi t}{T} \cdot I[n]$$

  - Taking into account that

$$\pi \sum_{m=0}^{n-1} I[m] - n \cdot \pi \cdot I[n] = K \cdot 2\pi, \quad K \in \mathbb{Z}$$

  - Phase $\theta(t, \boldsymbol{I})$ is, 2π modulus

$$\theta(t, \boldsymbol{I}) = \frac{\pi t}{T} \cdot I[n] = \pm \frac{\pi t}{T}$$

Identification of MSK modulation as a CPM

- MSK signal in the time domain

$$x(t) = \sqrt{\frac{2E_s}{T}} \sum_n \sin\left(\omega_c t + I[n] \cdot \frac{\pi t}{2T} + \theta[n]\right) \cdot w_T(t - nT)$$

- Parameters identifying MSK as a CPM

$$\omega_d = \frac{\pi}{2T}, \qquad h = \frac{1}{2}$$

Phase trees in CPM modulations

- Drawing of possible phase evolution in time starting from an initial phase
- Transitions in a symbol interval
  - Phase increment in each symbol interval

$$\theta((n+1)T) - \theta(nT) = \theta[n+1] - \theta[n] = \pi \cdot h \cdot I[n]$$

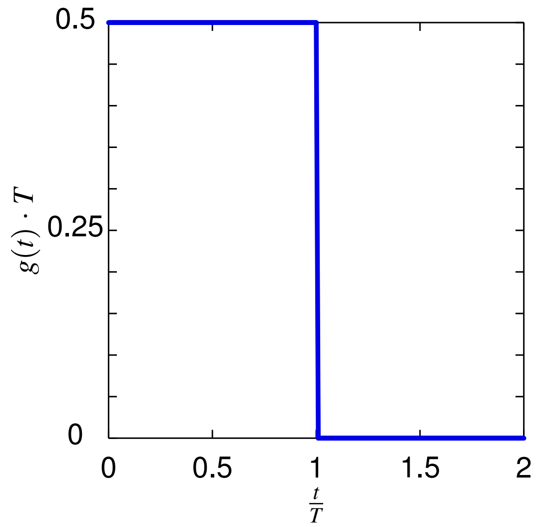
  - Shape for moving from the value of phase at the begining of symbol interval to the value of phase at the end of symbol interval
    - ★ Proportional to the integral of pulse $g(t)$, i.e., $q_g(t)$

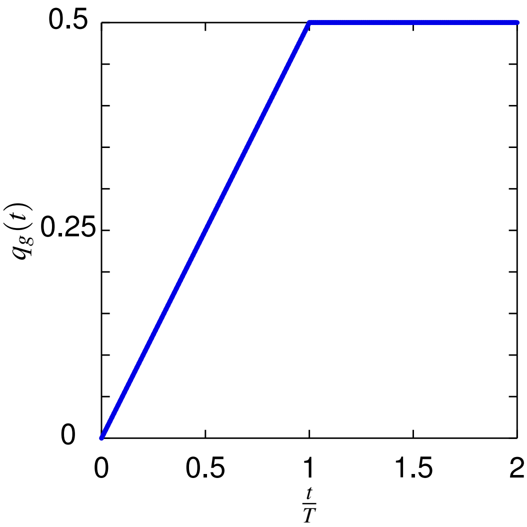
$$\theta(t, n) = 2 \cdot \pi \cdot h \cdot I[n] \cdot q_g(t - nT)$$

Phase tree - Example - Squared pulse

- Example: squared pulse

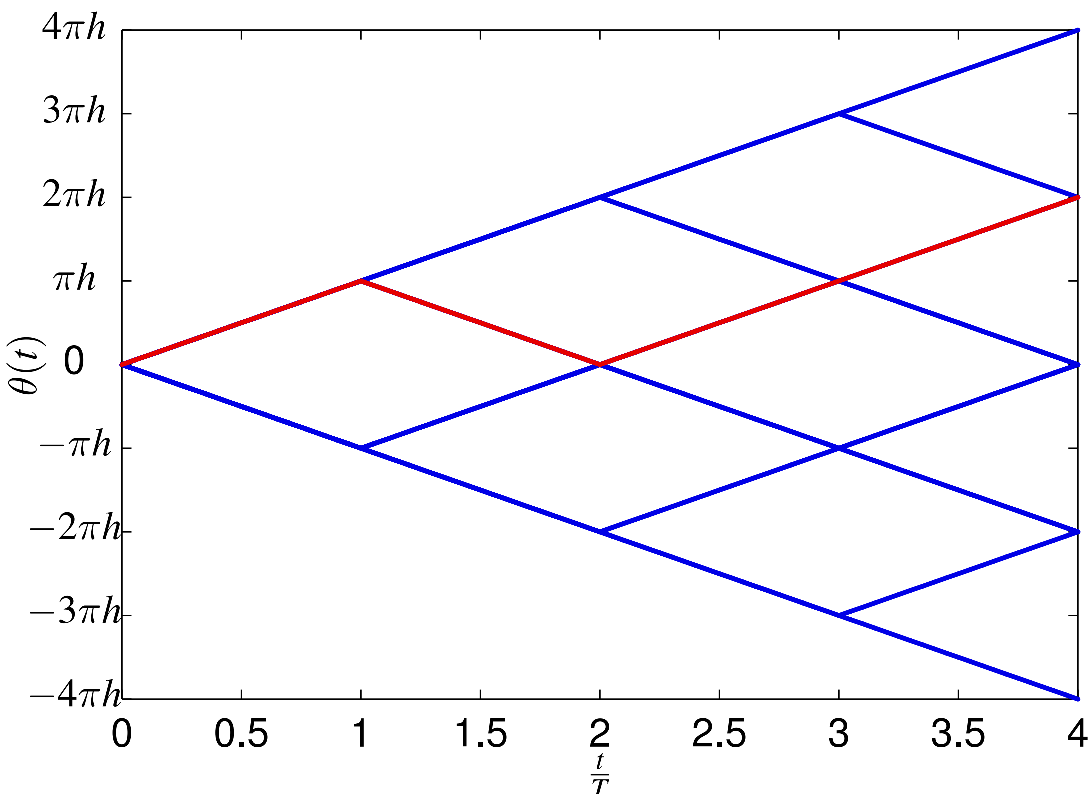
$$g(t) = \begin{cases} \frac{1}{2T}, & 0 \leq t < T \\ 0, & \text{en otro caso} \end{cases}, \quad q_g(t) = \int_{-\infty}^{t} g(t)\, dt = \begin{cases} 0, & t < 0 \\ \frac{t}{2T}, & 0 \leq t < T \\ 1/2, & t \geq T \end{cases}$$





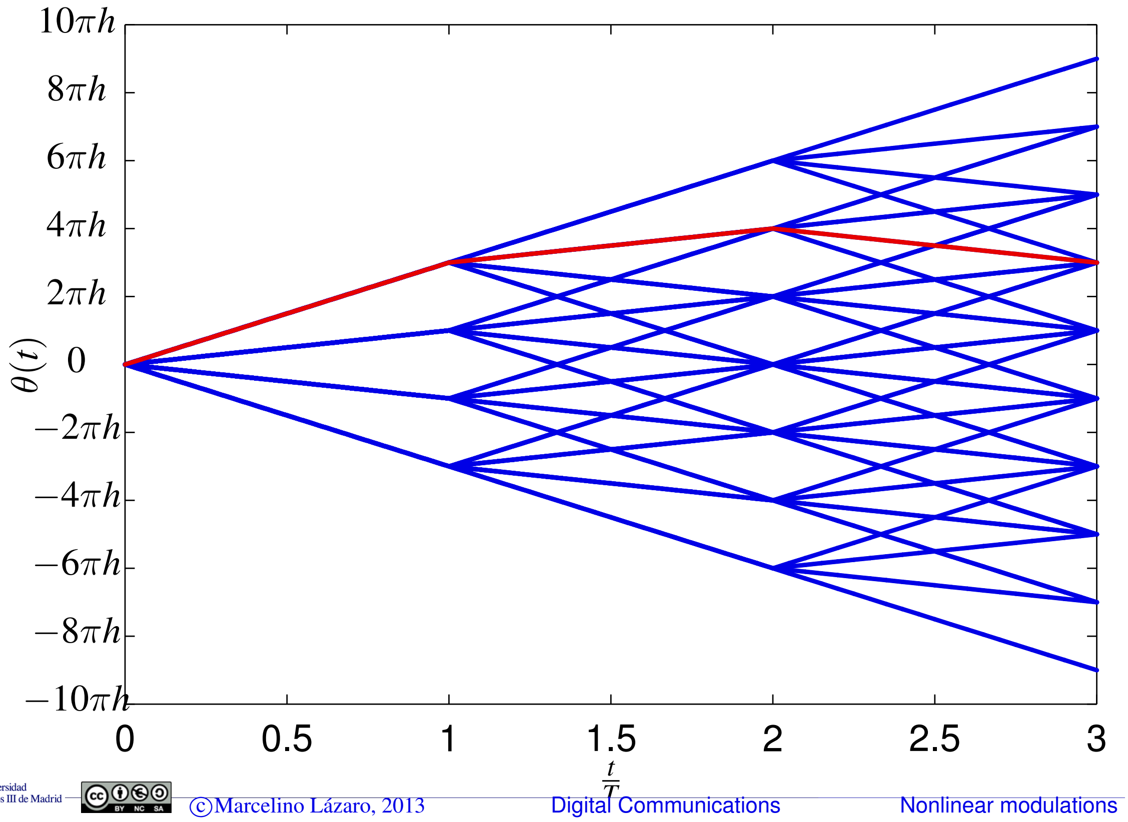
Example of phase tree - squared pulse - binary

- Highlighted sequence: $I[0] = +1, \ I[1] = -1, \ I[2] = +1, \ I[3] = +1$



Example of phase tree - squared pulse - 4-ary

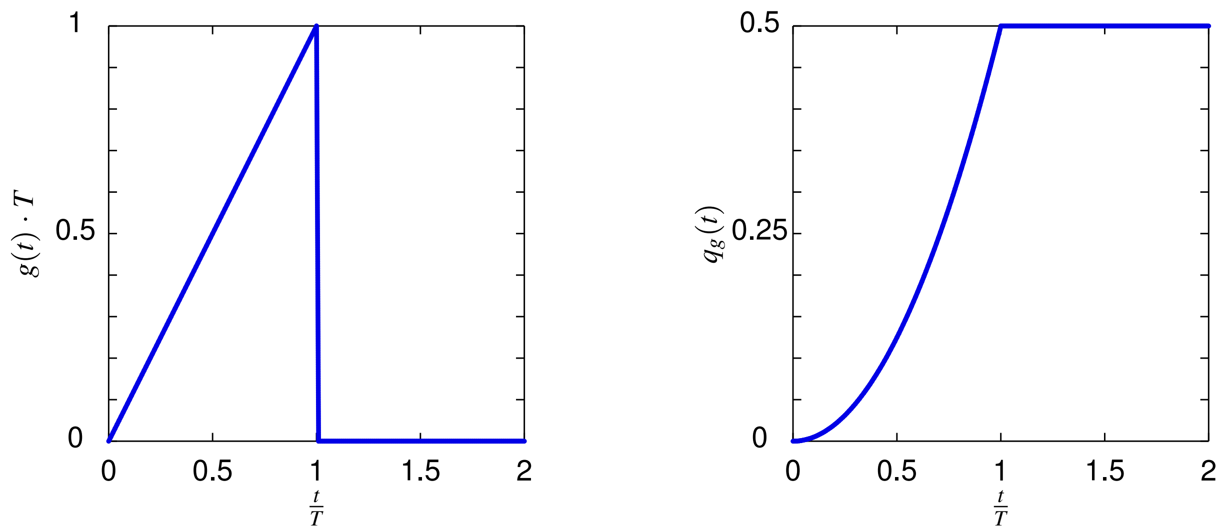
- Highlighted sequence: $I[0] = +3,\ I[1] = +1,\ I[2] = -1$



Phase tree - Example - Triangle pulse

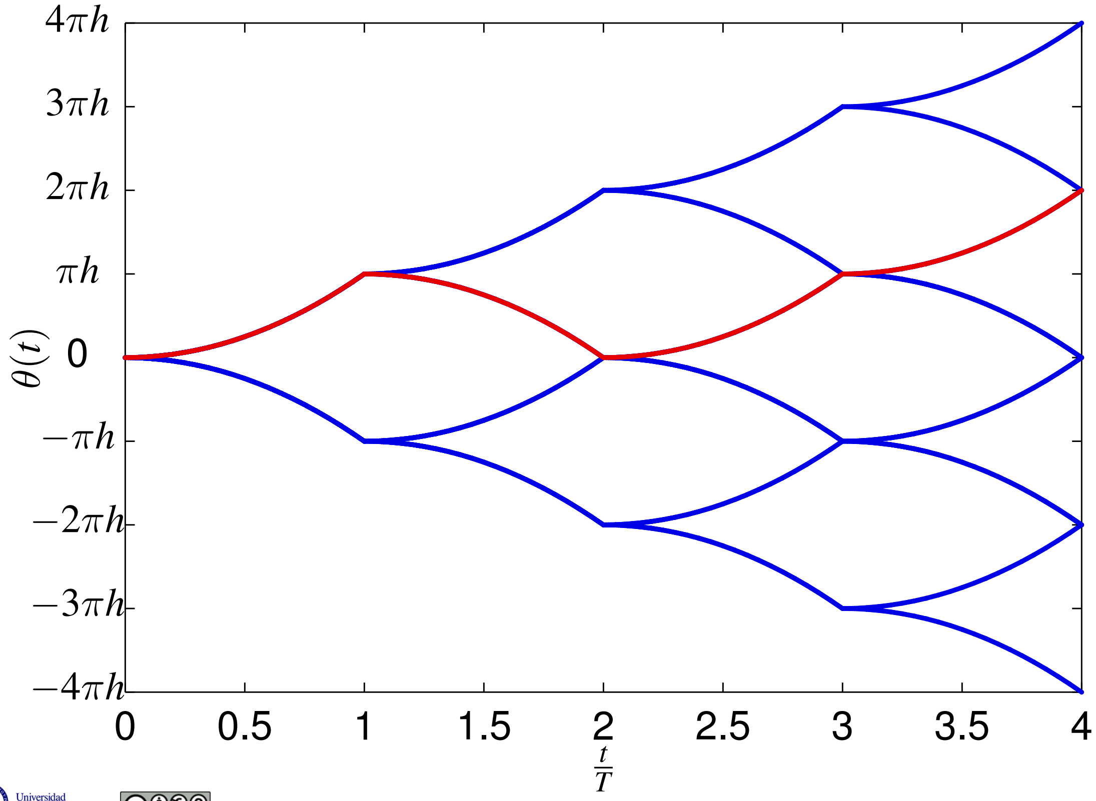
- Example: triangle pulse

$$g(t) = \begin{cases} \frac{t}{T^2}, & 0 \leq t < T \\ 0, & \text{en otro caso} \end{cases}, \quad q_g(t) = \int_{-\infty}^{t} g(t)\, dt = \begin{cases} 0, & t < 0 \\ \frac{t^2}{2T^2}, & 0 \leq t < T \\ 1/2, & t \geq T \end{cases}$$



Phase tree - Example - Triangle pulse - binary

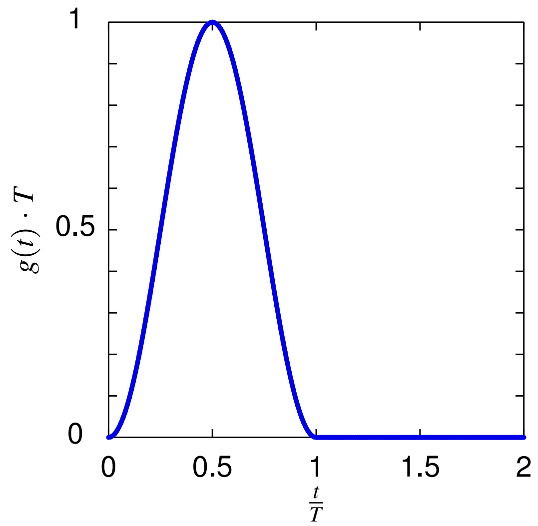
- Highlighted sequence: $I[0] = +1,\ I[1] = -1,\ I[2] = +1,\ I[3] = +1$

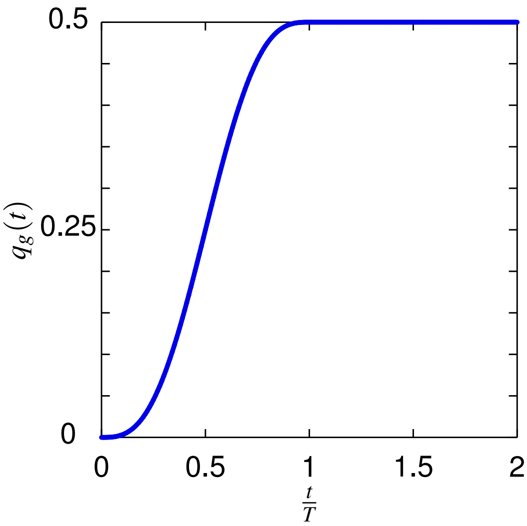


Phase tree - Example - smoother pulses

- Example: raised cosine pulse ($L = 1$)

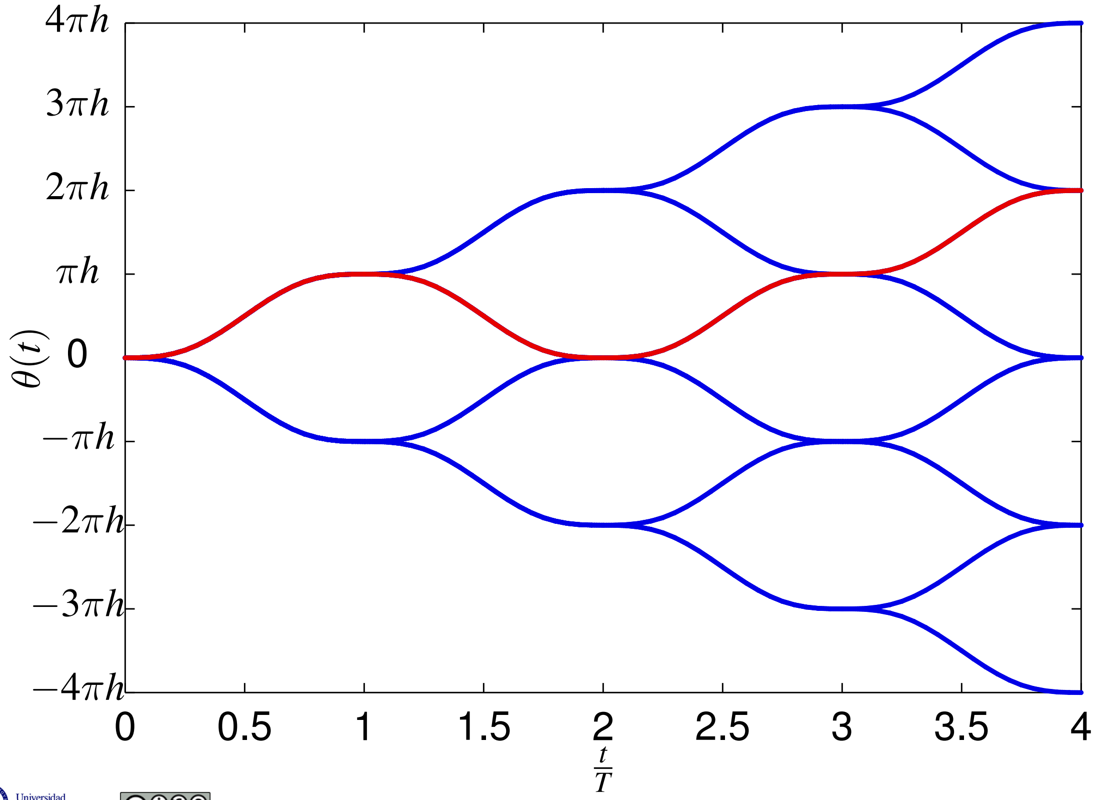
$$g(t) = \frac{1}{2T}\left[1 - \cos\left(\frac{2\pi t}{T}\right)\right] w_T(t),\ q_g(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2T}\left[t - \frac{T}{2\pi}\sin\left(\frac{2\pi t}{T}\right)\right] & 0 \leq t < T \\ 1/2, & t \geq T \end{cases}$$





Phase tree - Example - smoother pulses - binary

- Highlighted sequence: $I[0] = +1,\ I[1] = -1,\ I[2] = +1,\ I[3] = +1$



Partial response CPM

- Duration of pulse $g(t)$ is extended up to L symbol periods ($L > 1$)
- Phase $\theta(t, \boldsymbol{I})$ in symbol interval $[nT, (n+1)T]$ is now

$$\begin{aligned}\theta(t, \boldsymbol{I}) =& 2\pi h \sum_{m=-\infty}^{n} I[m] \cdot q_g(t - mT) \\ =& \theta[n] + \theta(t, n)\end{aligned}$$

  - $\theta[n]$: phase that is accumulated up to nT due to finished pulses

$$\theta[n] = \pi h \sum_{m=-\infty}^{n-L} I[m]$$

  - $\theta(t, n)$: contribution of pulses that have not finished at the begining of the interval

$$\theta(t, n) = 2\pi h \sum_{m=n-L+1}^{n} I[m] \cdot q_g(t - mT)$$

Pulses for partial response CPM

- Raised cosine pulses

$$g(t) = \frac{1}{2LT}\left[1 - \cos\left(\frac{2\pi t}{LT}\right)\right] \cdot w_{LT}(t)$$

  - Smoothing phase transitions

- Gaussian MSK (GMSK)

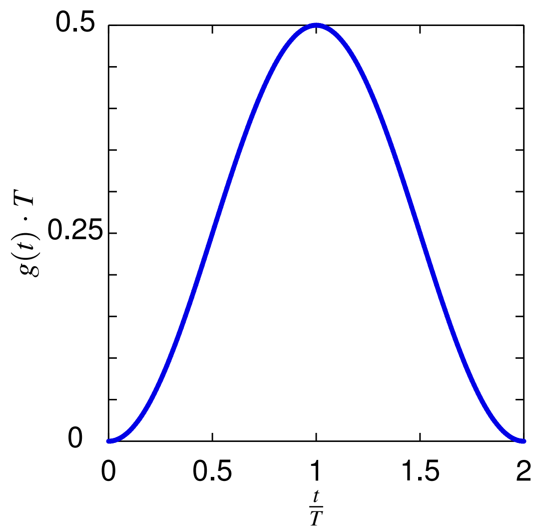
$$g(t) = \frac{1}{2T}\left[Q\left(\frac{2\pi\beta(t - T/2)}{\sqrt{\ln 2}}\right) - Q\left(\frac{2\pi\beta(t + T/2)}{\sqrt{\ln 2}}\right)\right]$$

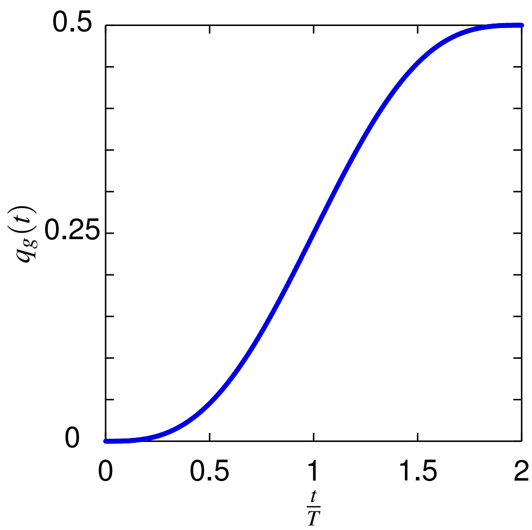
  - Employed in GSM ($\beta = 0{,}3$) and DECT ($\beta = 0{,}2$)
  - Squared pulse filtered with a Gaussian impulse response

Phase tree - Partial response CPM - Example

- Example: raised cosine pulse ($L = 2$)

$$g(t) = \frac{1}{4T}\left[1 - \cos\left(\frac{2\pi t}{2T}\right)\right] w_{2T}(t),\; q_g(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{4T}\left[t - \frac{2T}{2\pi}\sin\left(\frac{2\pi t}{2T}\right)\right] & 0 \le t < T \\ 1/2, & t \ge T \end{cases}$$





Phase tree - Partial response CPM - Example - Binary

- Highlighted sequence: $I[0] = +1,\ I[1] = -1,\ I[2] = +1,\ I[3] = +1$