Universidad Carlos III de Madrid

**OpenCourseWare**

# Digital Communications
Telecommunications Engineering

# Chapter 4

## Design of digital communications receivers in the presence of intersymbol interference (ISI)

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## Detection under ISI - Problem statement

- Receiver: $f(t) = g(-t)$, and $r_g(t)$ fulfills Nyquist
  - $z[n]$ white and Gaussian: $\sigma_z^2 = \begin{cases} N_0/2, & A[n] \in \boldsymbol{R} \\ N_0, & A[n] \in \mathcal{C} \end{cases}$
- Sequence of symbols $A[n]$: constellation with $M$ symbols
  - Stationnary white sequence with mean energy $E_s = E\left[|A[n]|^2\right]$

$$R_A[k] = E\left[A[n+k] \cdot A^*[n]\right] = E_s \cdot \delta[n]$$
$$S_A\left(e^{j\omega}\right) = E_s$$

- Response $p(t)$ is causal and time-limited ($T_p$ seconds)
  - $p[n]$ causal of length $K_p + 1$, $\Rightarrow K_p = \lfloor T_p/T \rfloor$
  - Observation at the output of the demodulator

$$q[n] = A[n] * p[n] + z[n] = o[n] + z[n]$$

  - ★ Noiseless output of the equivalent discrete channel

$$o[n] = A[n] * p[n] = \sum_{k=0}^{K_p} p[k] \cdot A[n-k]$$

$K_p \equiv$ memory of the equivalent discrete channel



## Symbol by symbol memoryless detection - Delay $d$

- Ideal channel with delay $d$ lags: $p[n] = C \cdot \delta[n-d]$
- Observation used to decide $A[n-d]$ from $q[n]$ (delay $d$)

$$q[n] = \underbrace{p[d] \cdot A[n-d]}_{\text{desired term}} + \underbrace{\sum_{k \neq d} p[k] \cdot A[n-k]}_{\text{ISI}} + \underbrace{z[n]}_{\text{noise}}$$

- Optimal choice for delay $d$
  - Normalization of observation to compensate gain $p[d]$

$$q'[n] = \frac{q[n]}{p[d]} = A[n-d] + \sum_{k \neq d} \frac{p[k]}{p[d]} \cdot A[n-k] + \frac{z[n]}{p[d]}$$

  - Once $d$ is selected, term $p[d]$ divides ISI and noise
    - ★ Optimal choice: select $d$ such that $|p[d]| \geq |p[n]|$ for all $n$
      - Minimizes the joint effect of ISI and noise

## Choice for optimal delay - Example

- Transmission of 2-PAM through $p[n] = \frac{1}{2}\delta[n] + \delta[n-1] + \frac{1}{4}\delta[n-2]$
- $o[n] = A[n] * p[n] = \frac{1}{2} \cdot A[n] + A[n-1] + \frac{1}{4} \cdot A[n-2]$

Delay $d = 0$

| $A[n]$ | $A[n-1]$ | $A[n-2]$ | $o[n]$ |
|---|---|---|---|
| $+1$ | $+1$ | $+1$ | $+\frac{7}{4}$ |
| $+1$ | $+1$ | $-1$ | $+\frac{5}{4}$ |
| $+1$ | $-1$ | $+1$ | $-\frac{1}{4}$ |
| $+1$ | $-1$ | $-1$ | $-\frac{3}{4}$ |
| $-1$ | $+1$ | $+1$ | $+\frac{3}{4}$ |
| $-1$ | $+1$ | $-1$ | $+\frac{1}{4}$ |
| $-1$ | $-1$ | $+1$ | $-\frac{5}{4}$ |
| $-1$ | $-1$ | $-1$ | $-\frac{7}{4}$ |

Delay $d = 1$

| $A[n]$ | $A[n-1]$ | $A[n-2]$ | $o[n]$ |
|---|---|---|---|
| $+1$ | $+1$ | $+1$ | $+\frac{7}{4}$ |
| $+1$ | $+1$ | $-1$ | $+\frac{5}{4}$ |
| $+1$ | $-1$ | $+1$ | $-\frac{1}{4}$ |
| $+1$ | $-1$ | $-1$ | $-\frac{3}{4}$ |
| $-1$ | $+1$ | $+1$ | $+\frac{3}{4}$ |
| $-1$ | $+1$ | $-1$ | $+\frac{1}{4}$ |
| $-1$ | $-1$ | $+1$ | $-\frac{5}{4}$ |
| $-1$ | $-1$ | $-1$ | $-\frac{7}{4}$ |

Blue : $A[n-d] = +1$ Red : $A[n-d] = -1$

- Symbol associated to highest $|p[n]|$ has the highest contribution in $o[n]$
  - Sign of $o[n]$ depends in this case on $A[n-1]$





## ISI level

- ISI level quantifies the ISI distortion introduced by a channel

$$\gamma_{ISI} = \frac{D_{peak}}{\eta} \geq 0$$

  - $D_{peak}$: peak distortion for a delay $d$

$$D_{peak} = \sum_{k \neq d} \frac{|p[k]|}{|p[d]|} \geq 0$$

    Depends on the equivalent discrete channel and selected delay for decision ($d$)
  - $\eta$: constellation efficiency

$$\eta = \frac{(d_{min}/2)}{|A|_{max}} \geq 0$$

    Depends on the constellation used to transmit data
    - $|A|_{max}$: maximum modulus of a symbol in the constellation

$$|A|_{max} = \text{máx}\{|A[n]|\}$$

    - $d_{min}$: minimum distace between two symbols in the constellation

$$d_{min} = \min_{A[n] \neq A[k]} |A[n] - A[k]|$$

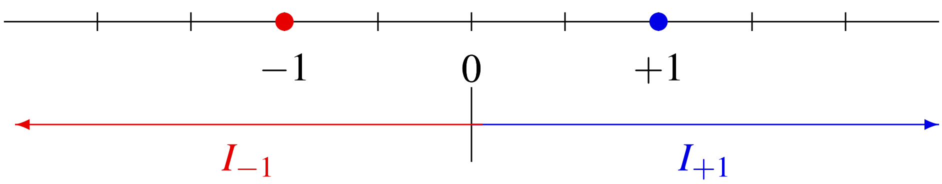
## ISI level - Measuring effect of ISI on the decision regions

- ISI level measures the effect of ISI in terms of how it affects to the received constellation (extended constellation generated by ISI)
- Value $\gamma_{ISI} = 1$ indicates the point where the extended constellation achieves the limits of the original decision regions
  - $\gamma_{ISI} < 1$: ISI does not move symbols out of its decision region
    - ★ Without noise unmodified symbol by symbol detector does not make erroneous decisions
  - $\gamma_{ISI} > 1$: ISI does move symbols out of its decision region
    - ★ Unmodified symbol by symbol detector makes erroneous decisions even without noise
    - ★ In this case, a re-definition of the decision regions, taking into account the underlying ISI, is necessary to guarantee a minimum performance using memoriless symbol by symbol detectors



## Example

- ISI level will be presented for the following case
  - Transmitted constellation: 2-PAM ($A[n] \in \{\pm 1\}$)
    - ★ Efficiency $\eta = 1$
    - ★ Constellation and original decision regions ($I_{+1},\ I_{-1}$)




  - Equivalent discrete channel

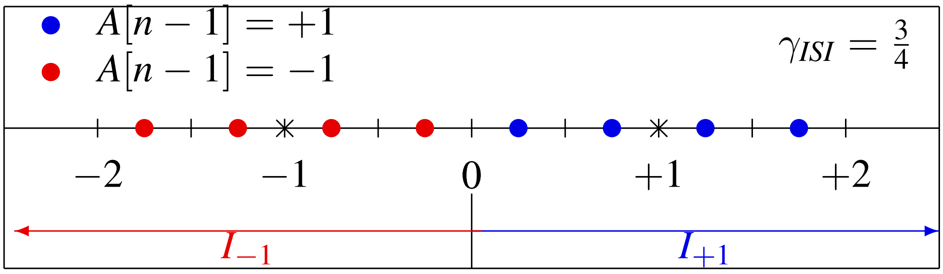
$$p[n] = \frac{1}{2}\delta[n] + \delta[n-1] + c \cdot \delta[n-2]$$

    - ★ Several values for $c$ will be tested: $c = \frac{1}{4}$, $c = \frac{1}{2}$, and $c = \frac{3}{4}$
      - In all cases, optimal delay is $d = 1$.
- Points of the extended constellation generated by ISI

Plot of values for $o[n] = A[n] * p[n] = \frac{1}{2}A[n] + A[n-1] + c \cdot A[n-2]$

# Example - $c = \frac{1}{4}$

| $A[n]$ | $A[n-1]$ | $A[n-2]$ | $o[n]$ |
|---|---|---|---|
| $+1$ | $+1$ | $+1$ | $+\frac{7}{4}$ |
| $+1$ | $+1$ | $-1$ | $+\frac{5}{4}$ |
| $+1$ | $-1$ | $+1$ | $-\frac{1}{4}$ |
| $+1$ | $-1$ | $-1$ | $-\frac{3}{4}$ |
| $-1$ | $+1$ | $+1$ | $+\frac{3}{4}$ |
| $-1$ | $+1$ | $-1$ | $+\frac{1}{4}$ |
| $-1$ | $-1$ | $+1$ | $-\frac{5}{4}$ |
| $-1$ | $-1$ | $-1$ | $-\frac{7}{4}$ |

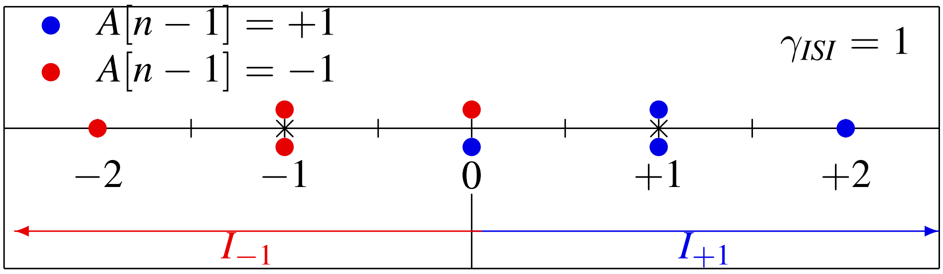



- Points are still inside of its corresponding decision region



# Example - $c = \frac{1}{2}$

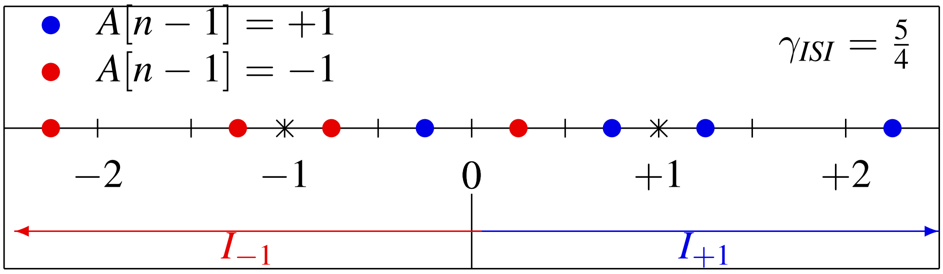
| $A[n]$ | $A[n-1]$ | $A[n-2]$ | $o[n]$ |
|---|---|---|---|
| $+1$ | $+1$ | $+1$ | $+2$ |
| $+1$ | $+1$ | $-1$ | $+1$ |
| $+1$ | $-1$ | $+1$ | $0$ |
| $+1$ | $-1$ | $-1$ | $-1$ |
| $-1$ | $+1$ | $+1$ | $+1$ |
| $-1$ | $+1$ | $-1$ | $0$ |
| $-1$ | $-1$ | $+1$ | $-1$ |
| $-1$ | $-1$ | $-1$ | $-2$ |




- Points now achieve the limits of its corresponding decision region

## Example - $c = \frac{3}{4}$

| $A[n]$ | $A[n-1]$ | $A[n-2]$ | $o[n]$ |
|---|---|---|---|
| $+1$ | $+1$ | $+1$ | $+\frac{9}{4}$ |
| $+1$ | $+1$ | $-1$ | $+\frac{3}{4}$ |
| $+1$ | $-1$ | $+1$ | $+\frac{1}{4}$ |
| $+1$ | $-1$ | $-1$ | $-\frac{5}{4}$ |
| $-1$ | $+1$ | $+1$ | $+\frac{5}{4}$ |
| $-1$ | $+1$ | $-1$ | $-\frac{1}{4}$ |
| $-1$ | $-1$ | $+1$ | $-\frac{3}{4}$ |
| $-1$ | $-1$ | $-1$ | $-\frac{9}{4}$ |

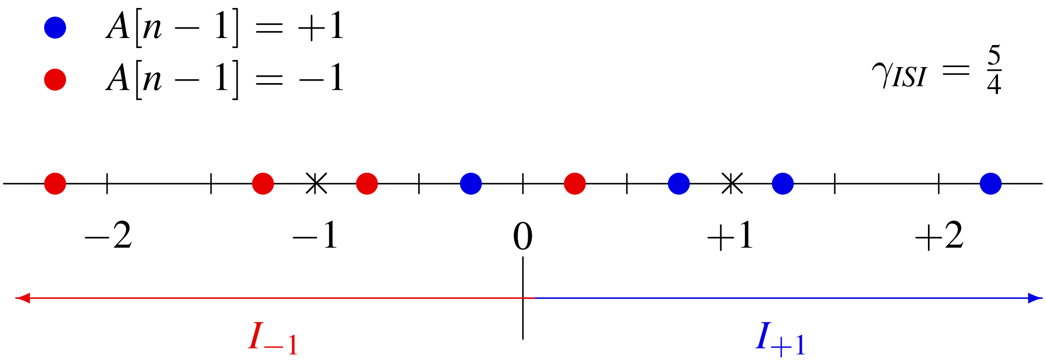


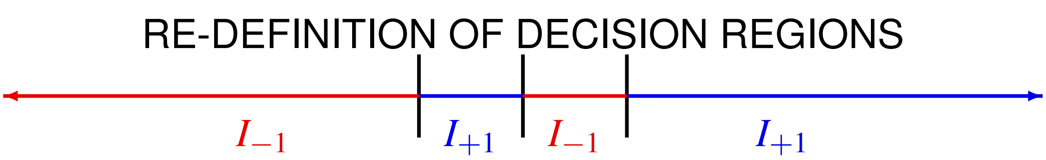

- Some points are now out of its corresponding decision region



## Re-defining decision regions

- Decision regions must be re-defined based on the location of points of the extended constellation defined by ISI



## Maximum likelihood sequence detection (MLSD)

- Optimal detection under ISI: MLSD
- Sequence to be detected: $L$ symbols ($M^L$ possible sequences)

$$\boldsymbol{A} = [A[0], A[1], \cdots, A[L-1]]^T$$

- Channel: $\boldsymbol{p} = [p[0], p[1], \cdots, p[K_p]]^T$
- Sufficient statistic for detection: $N_q = L + K_p$ observations

$$\boldsymbol{q} = [q[0], q[1], \cdots, q[N_q-1]], \quad N_q = L + K_p$$

$$\begin{aligned} o[0] =& p[0] \cdot A[0] + p[1] \cdot A[-1] + p[2] \cdot A[-2] + \cdots \cdot p[K_p] \cdot A[-K_p] \\ o[1] =& p[0] \cdot A[1] + p[1] \cdot A[0] + p[2] \cdot A[-1] \cdots + p[K_p] \cdot A[-K_p] \\ & \cdots \\ o[L+K_p-1] =& p[0] \cdot A[L+K_p-1] + p[1] \cdot A[L+K_p-2] + \cdots \\ & + p[K_p-1] \cdot A[L] + p[K_p] \cdot A[L-1] \end{aligned}$$

- Additional information that is necessary:

$A[-1], A[-2], \cdots, A[-K_p]$ and $A[L], A[L+1], \cdots, A[L+K_p-1]$

  - Previous $K_p$ symbols and posterior $K_p$ symbols



## Maximum likelihood sequence

- $M^L$ possible sequences

$$\boldsymbol{a}_i = [a_i[0], a_i[1], \cdots, a_i[L-1]]^T, \ i = 0, 1, \cdots, M^L - 1$$

- Sequence with highest likelihood:

$$\hat{\boldsymbol{A}} = \boldsymbol{a}_i = [a_i[0], a_i[1], \cdots, a_i[L-1]]^T$$

fulfilling the following condition

$$f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_i) \geq f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_j), \ j = 0, 1, \cdots, M^L - 1, \ \forall j \neq i.$$

## Estimation of the maximum likelihood sequence

- Analytic expression of likelihood

$$f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_i) = \prod_{n=0}^{N_q-1} f_{q[n]|\boldsymbol{A}}(q[n]|\boldsymbol{a}_i)$$

- Conditional distribution for each observation

$$f_{q[n]|\boldsymbol{A}}(q[n]|\boldsymbol{a}_i) = \mathcal{N}\left(o_i[n], \sigma_z^2\right) = \frac{1}{\sqrt{2\pi}\sigma_z} \cdot \exp\left\{ -\frac{1}{2\sigma_z^2} \left| q[n] - \sum_{k=0}^{K_p} p[k] \cdot a_i[n-k] \right|^2 \right\}$$

- Total likelihood

$$f_{\boldsymbol{q}|\boldsymbol{A}}(\boldsymbol{q}|\boldsymbol{a}_i) = \frac{1}{(2\pi\sigma_z^2)^{N_q/2}} \cdot \exp\left\{ -\frac{1}{2\sigma_z^2} \sum_{n=0}^{Nq-1} |q[n] - o_i[n]|^2 \right\}$$

- Maximum likelihood sequence

$$\hat{\boldsymbol{A}} = \text{arg } \min_{\boldsymbol{a}_i} \sum_{n=0}^{N_q-1} |q[n] - o_i[n]|^2 , \; o_i[n] = \sum_{k=0}^{K_p} p[k] \cdot a_i[n-k]$$



## Example: 2-PAM $K_p = 1$, $L = 3$

- Constellation: 2-PAM (BPSK): $A[n] \in \{\pm 1\}$
- Channel: $p[n] = \delta[n] + 0{,}5 \cdot \delta[n-1]$, $K_p = 1$
- Sequence to be estimated: $\boldsymbol{A} = [A[0], A[1], A[2]]$, $L = 3$
- Statistic for detection: $\boldsymbol{q} = [q[0], q[1], q[2], q[3]]$

$$q[-1] = A[-1] + 0{,}5 \cdot A[-2] + z[-1]$$

$$q[0] = A[0] + 0{,}5 \cdot A[-1] + z[0]$$

$$q[1] = A[1] + 0{,}5 \cdot A[0] + z[1]$$

$$q[2] = A[2] + 0{,}5 \cdot A[1] + z[2]$$

$$q[3] = A[3] + 0{,}5 \cdot A[2] + z[3]$$

$$q[4] = A[4] + 0{,}5 \cdot A[3] + z[4]$$

- Assumption: the following values are known $A[-1] = A[3] = +1$
- Problem: to decide the ML sequence when the sequence of observations is

$$q[0] = 1{,}4 \text{ - } q[1] = -0{,}4 \text{ - } q[2] = 0{,}6 \text{ - } q[3] = 1{,}6$$

## Brute force detection: comparison with noiseless outputs

$$q[0] = 1{,}4 \text{ - } q[1] = -0{,}4 \text{ - } q[2] = 0{,}6 \text{ - } q[3] = 1{,}6$$

- Evaluation of the noiseless output sequences, $o[n]$, generated by the 8 possible sequences, and its corresponding likelihood metric

$$\sum_{n=0}^{N_q-1} |q[n] - o_i[n]|^2$$

| $A[0]$ | $A[1]$ | $A[2]$ | $o[0]$ | $o[1]$ | $o[2]$ | $o[3]$ | Likelihood Metric |
|---|---|---|---|---|---|---|---|
| $-1$ | $-1$ | $-1$ | $-0{,}5$ | $-1{,}5$ | $-1{,}5$ | $+0{,}5$ | 10,44 |
| $-1$ | $-1$ | $+1$ | $-0{,}5$ | $-1{,}5$ | $+0{,}5$ | $+1{,}5$ | 4,84 |
| $-1$ | $+1$ | $-1$ | $-0{,}5$ | $+0{,}5$ | $-0{,}5$ | $+0{,}5$ | 6,84 |
| $-1$ | $+1$ | $+1$ | $-0{,}5$ | $+0{,}5$ | $+1{,}5$ | $+1{,}5$ | 5,24 |
| $+1$ | $-1$ | $-1$ | $+1{,}5$ | $-0{,}5$ | $-1{,}5$ | $+0{,}5$ | 5,64 |
| $+1$ | $-1$ | $+1$ | $+1{,}5$ | $-0{,}5$ | $+0{,}5$ | $+1{,}5$ | 0,04 |
| $+1$ | $+1$ | $-1$ | $+1{,}5$ | $+1{,}5$ | $-0{,}5$ | $+0{,}5$ | 6,04 |
| $+1$ | $+1$ | $+1$ | $+1{,}5$ | $+1{,}5$ | $+1{,}5$ | $+1{,}5$ | 4,44 |

- Sequence with the “most similar” noiseless output (ML sequence):
$+1\ -1\ +1$



## Efficient estimation - Definition of system state $\psi[n]$

- To compute the likelihood metric for each possible sequence is highly uneffient
- Noiseless output is a finite state machine

$$o[n] = A[n] \cdot p[0] + \sum_{k=1}^{K_p} p[k] \cdot A[n-k]$$

- Definition of system state at discrete instant $n$
Set of $K_p$ previous symbols contributing to the value of $o[n]$

$$\psi[n] = [A[n-1], A[n-2], \cdots, A[n-K_p]]^T$$

Number of possible estates is $M^{K_p}$

- Some dependencies

$$o[n] = f(A[n], \psi[n])$$

$$o[n] = g(\psi[n], \psi[n+1])$$

$$\psi[n+1] = f(\psi[n], A[n])$$

## State diagram

Drawing of the evolution of system state under ISI

$$\psi[n] = [A[n-1], A[n-2], \cdots, A[n-K_p+1], A[n-K_p]]^T$$
$$\psi[n+1] = [A[n], A[n-1], A[n-2], \cdots, A[n-K_p+1]]^T$$

- There are $M^{K_p}$ possible states
- $M$ arrows go out of each state, one for each possible value of $A[n]$
- $M$ arrows arrive at each state, all of them associated to the same value of $A[n]$
- Each arrow is labelled with the following information

$$A[n]|o[n]$$

i.e., the value of current symbol that forces the state transition, and noiseless output that is generated in that case



## State diagram - Example A

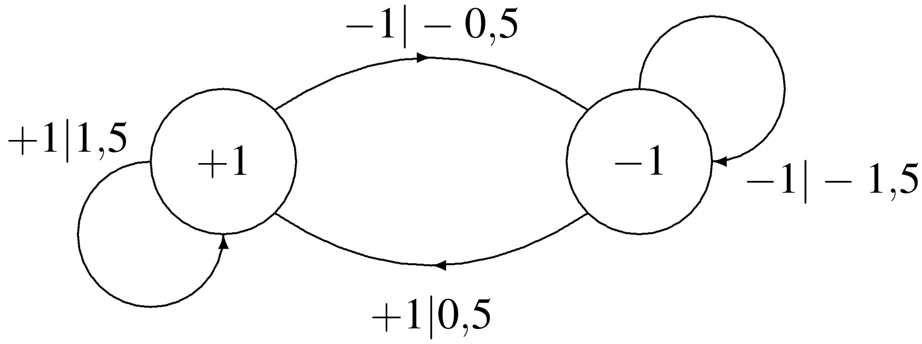
- $A[n] \in \{\pm 1\}$, $p[n] = \delta[n] + \frac{1}{2}\delta[n-1]$
  - Noiseless output

$$o[n] = A[n] + \frac{1}{2}A[n-1]$$

  - State

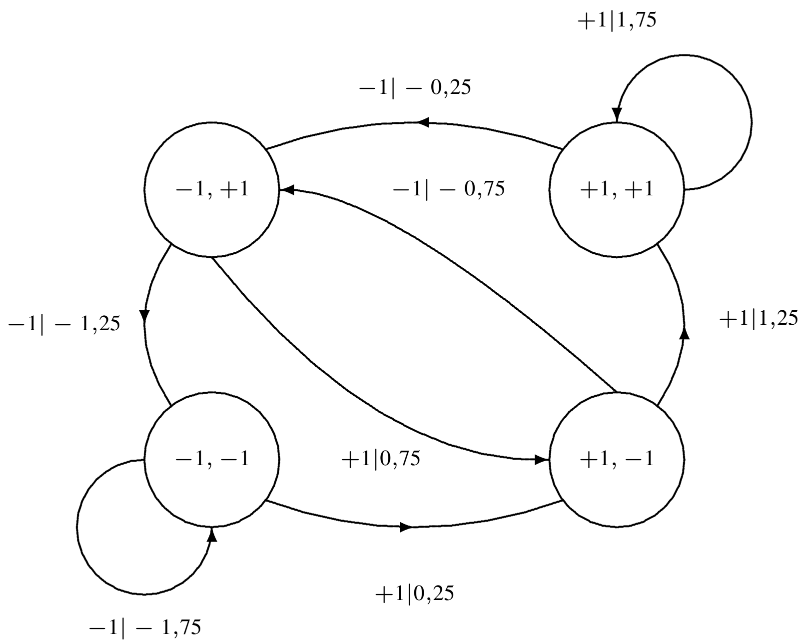
$$\psi[n] = A[n-1], \quad \psi[n+1] = A[n]$$

  - State diagram



## State diagram - Example B

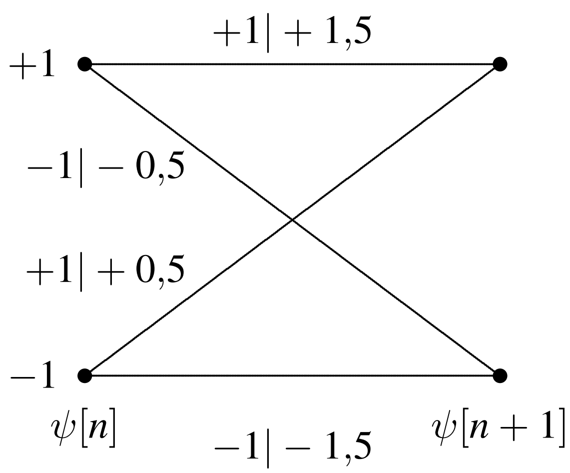
- $A[n] \in \{\pm 1\}$, $p[n] = \delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]$
- $o[n] = A[n] + \frac{1}{2}A[n-1] + \frac{1}{4}A[n-2]$
- $\psi[n] = [A[n-1], A[n-2]]^T$, $\psi[n+1] = [A[n], A[n-1]]^T$



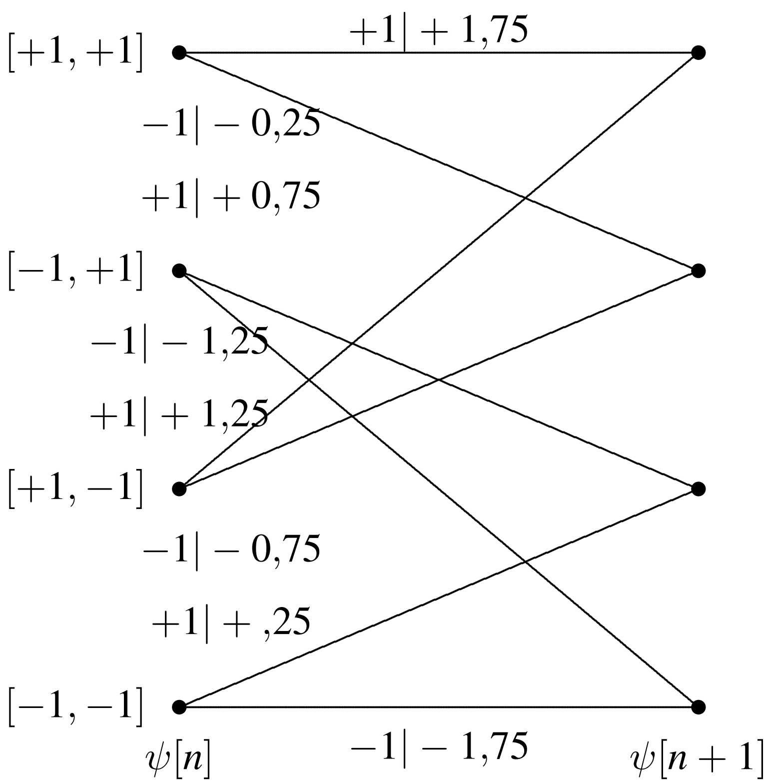



## Trellis diagram - Example A

- Represents the time evolution of states
- Example: $A[n] \in \{\pm 1\}$, $p[n] = \delta[n] + \frac{1}{2}\delta[n-1]$
- Definition of state: $\psi[n] = A[n-1]$
- Transition from states: $\psi[n] = A[n-1] \rightarrow \psi[n+1] = A[n]$
- Labels: $A[n]|o[n]$. In this case $o[n] = A[n] * p[n] = A[n] + \frac{1}{2}A[n-1]$



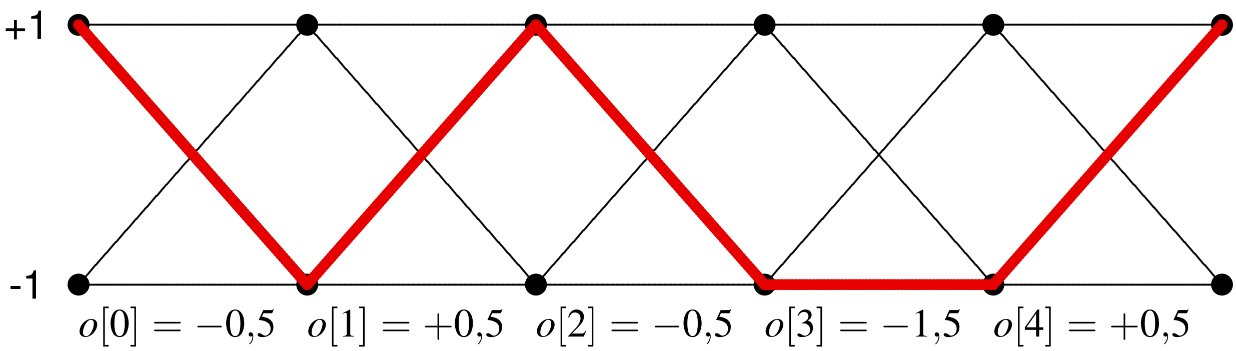
## Trellis diagram - Example B






## Trellis diagram - Representation of a sequence $A$

- A given sequence can be drawn as a path through the trellis
  - Example $\boldsymbol{A} = [-1, +1, -1, -1, +1]$
  - Initial state = $\psi[0] = +1$

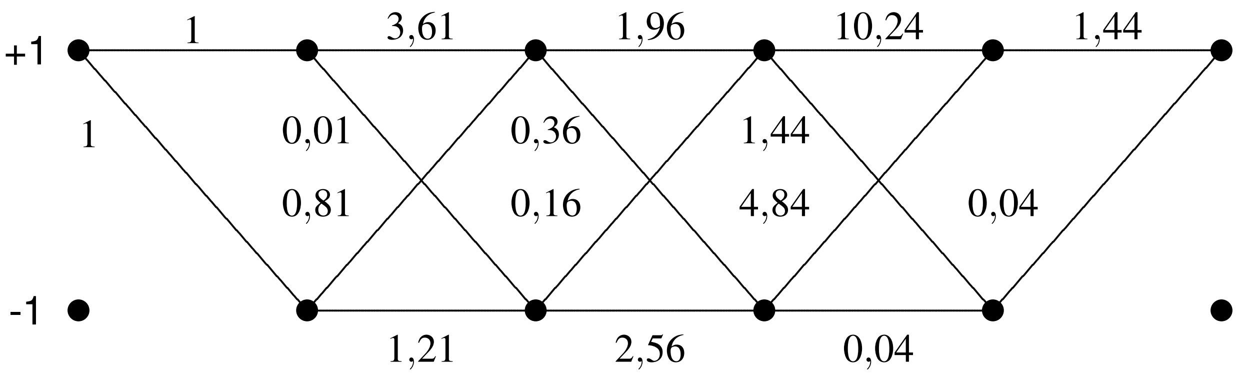


# MLSD using the trellis diagram

- Maximum likelihood sequence

$$\hat{\boldsymbol{A}} = \arg \min_{\boldsymbol{a}_i} \sum_{n=0}^{N_q-1} \left| q[n] - \underbrace{\sum_{k=0}^{N} p[k] \cdot a_i[n-k]}_{o_i[n]} \right|^2$$

  - New labels in the trellis - branch metric: $|q[n] - o_i[n]|^2$
  - Likelihood metric for a sequence: addition of branch metrics of its path through the trellis




$q[0] = +0{,}5 \quad q[1] = -0{,}4 \quad q[2] = +0{,}1 \quad q[3] = -1{,}7 \quad q[4] = +0{,}3$

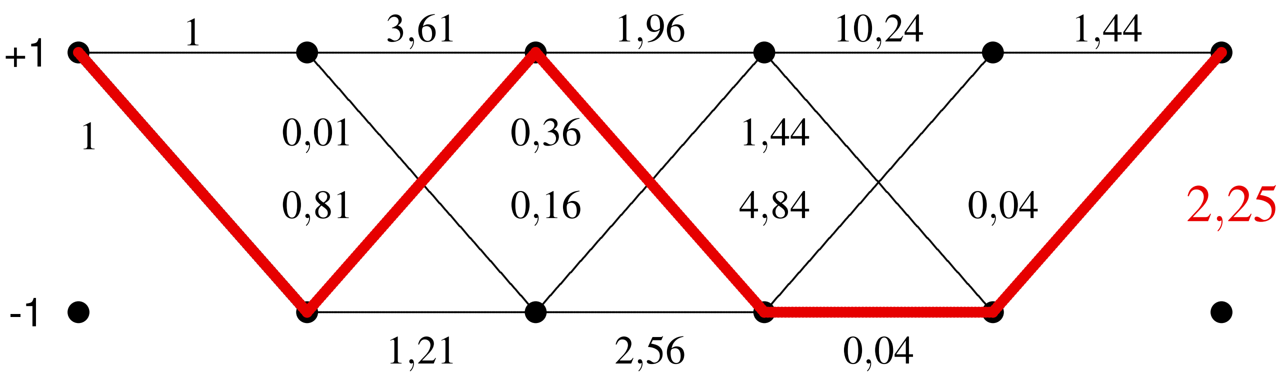


# MLSD using the trellis diagram

- Maximum likelihood sequence

$$\hat{\boldsymbol{A}} = \arg \min_{\boldsymbol{a}_i} \sum_{n=0}^{N_q-1} \left| q[n] - \underbrace{\sum_{k=0}^{N} p[k] \cdot a_i[n-k]}_{o_i[n]} \right|^2$$

  - New labels in the trellis - branch metric: $|q[n] - o_i[n]|^2$
  - Likelihood metric for a sequence: addition of branch metrics of its path through the trellis



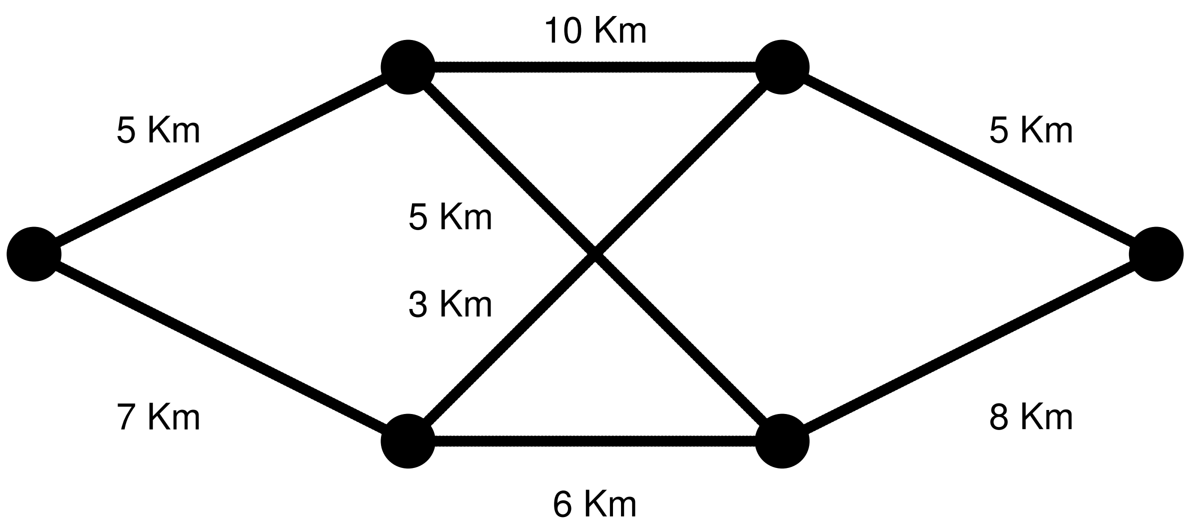

$q[0] = +0{,}5 \quad q[1] = -0{,}4 \quad q[2] = +0{,}1 \quad q[3] = -1{,}7 \quad q[4] = +0{,}3$

## Obtention of the ML sequence - Viterbi algorithm

- Evaluation of the likelihood metric for the $M^L$ possible sequences
  - Analyticaly, or by means of the path metric of $M^L$ path through the trellis
  - Computationally expensive
- Efficient obtention of the ML sequence - Viterbi algorithm
  - Efficient obtention of the shortest path through a trellis
- Basic foundations of Viterbi algorithm
  - A trellis includes a set of nodes (states in our problem) and branches linking nodes
  - Branch metric: defines the metric associated to each branch
  - Path metric: the addition of branch metrics for all branches in a path
  - Survival path for a node: the path arriving at that node having the lowest path metric
  - Accumulated metric of a state: the metric of its survival path

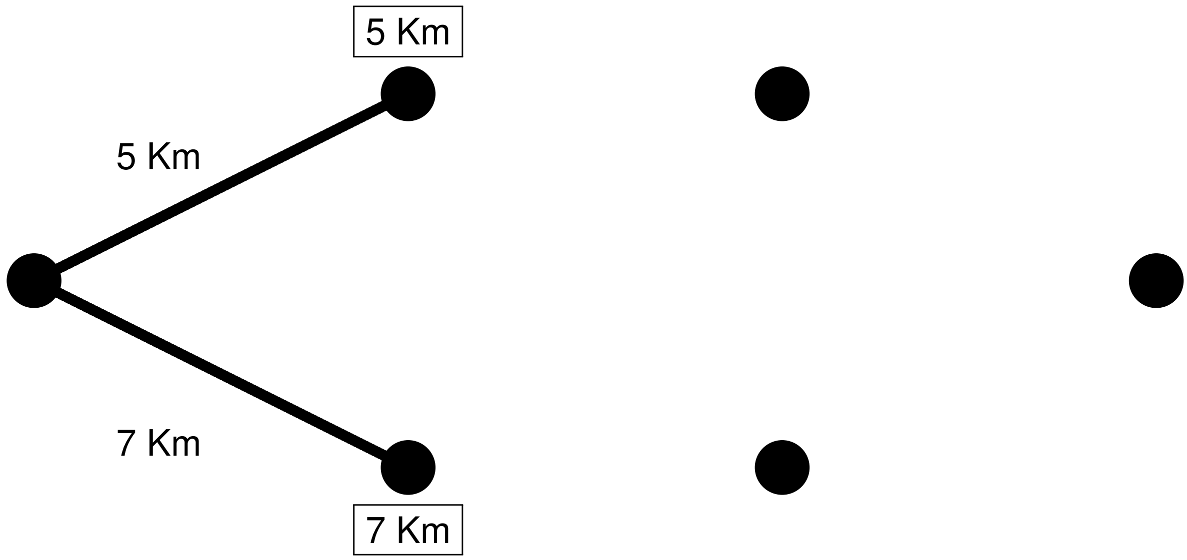


## Viterbi algorithm - A simple example




- Goal: finding the shortest path through a trellis
- Some example metrics

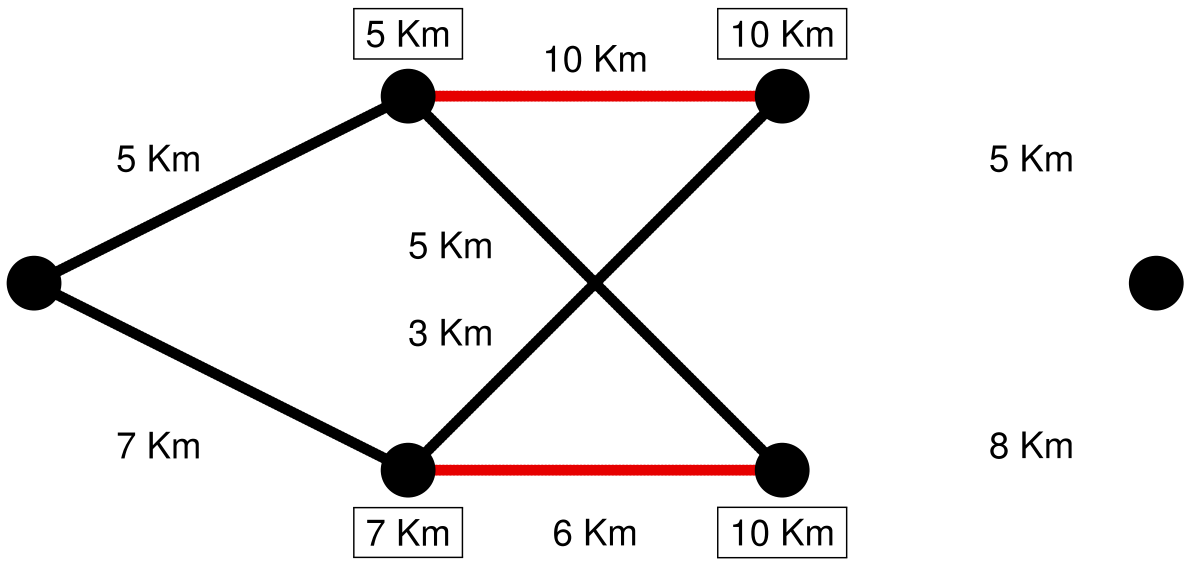
## Viterbi algorithm - A simple example




- First step
  - Computing the metrics at first stage after opening the trellis
  - Accumulated metric for the nodes are framed in a squared box

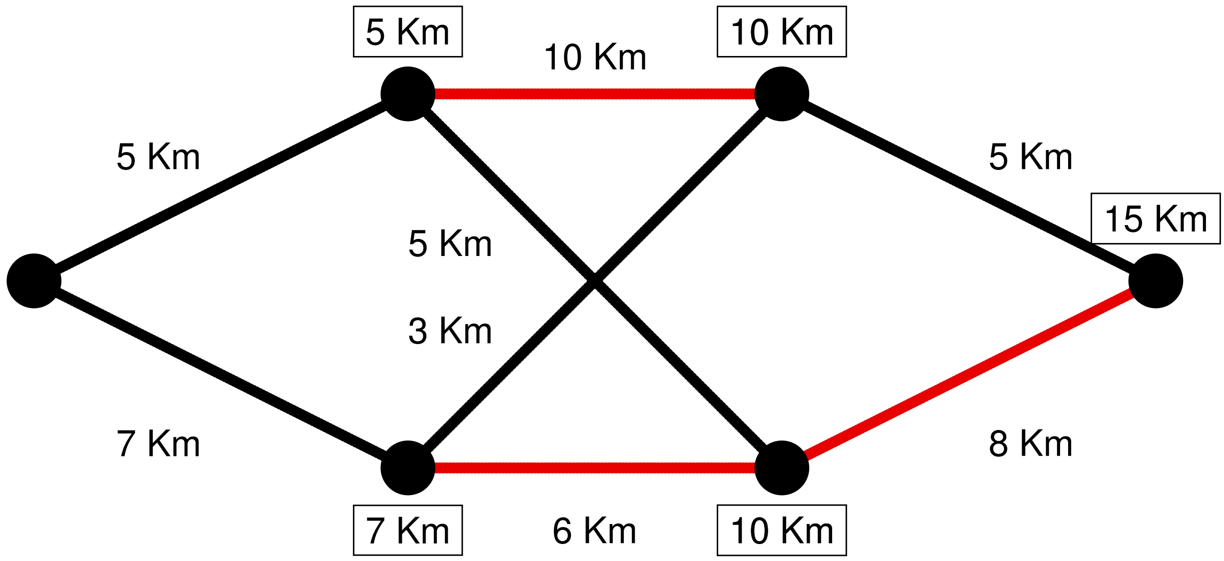


## Viterbi algorithm - A simple example




- Second step
  - Computing survival paths (black) for each node in next stage
    - Upper node: 7 Km + 3 Km is shorter than 5 Km + 10 Km
    - Lower node: 5 Km + 5 Km is shorter than 7 Km + 6 Km

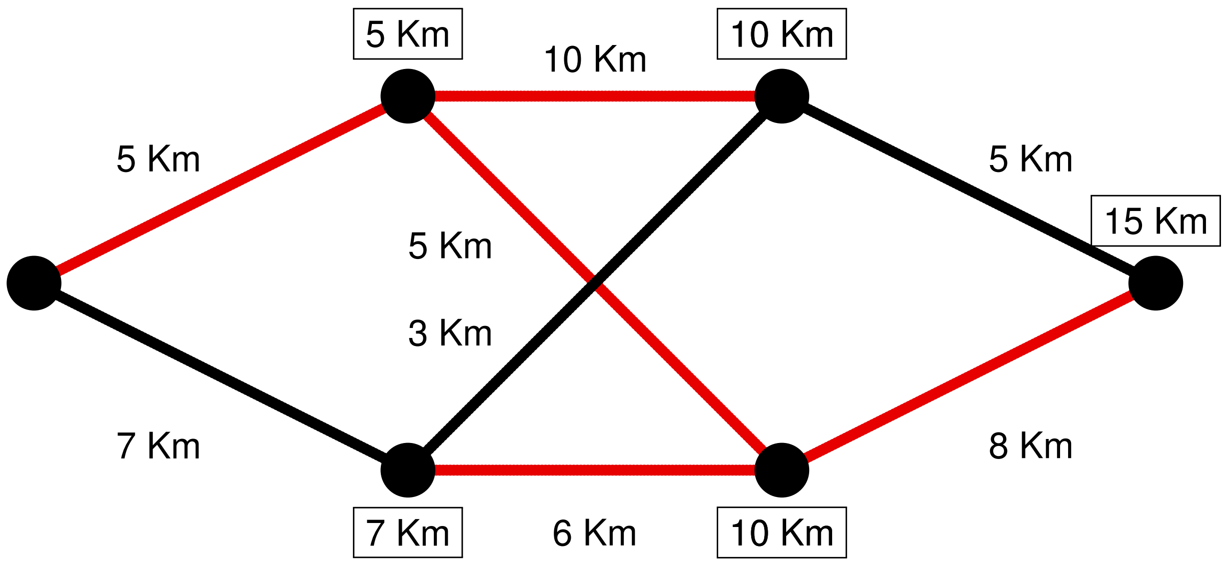
## Viterbi algorithm - A simple example




- Third step
  - Computing survival path (black) for node in final stage
    - ⋆ 10 Km + 5 Km is shorter than 10 Km + 8 Km

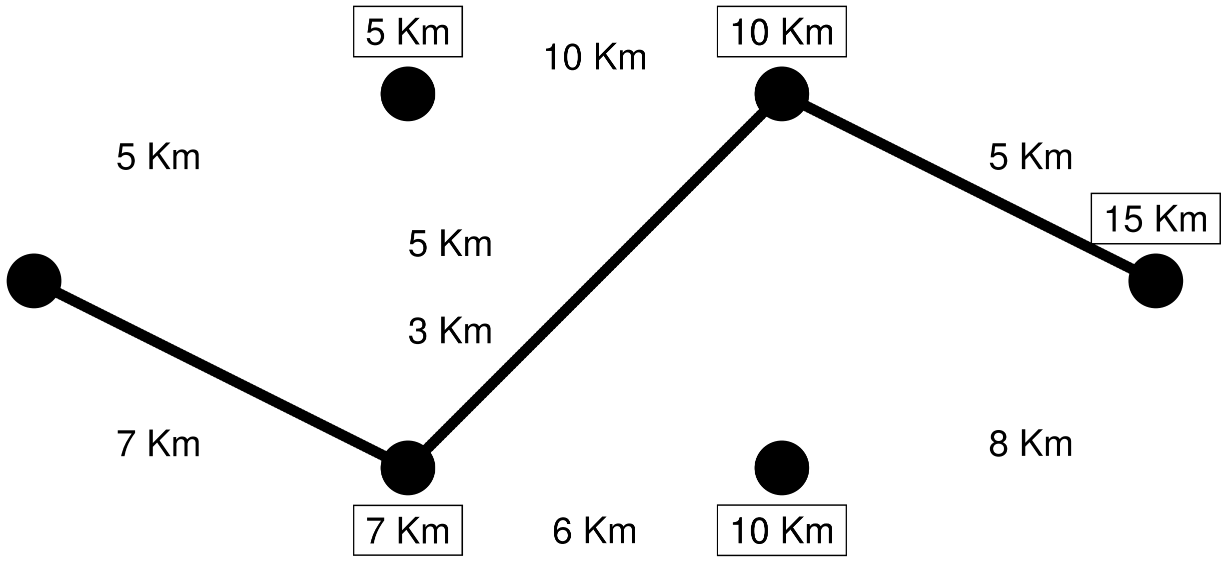


## Viterbi algorithm - A simple example




- Fourth step
  - Going back through the survival path
    - ⋆ Identification of the only survival path (black)
    - ⋆ Remove branches linking forward non-survival paths (red)

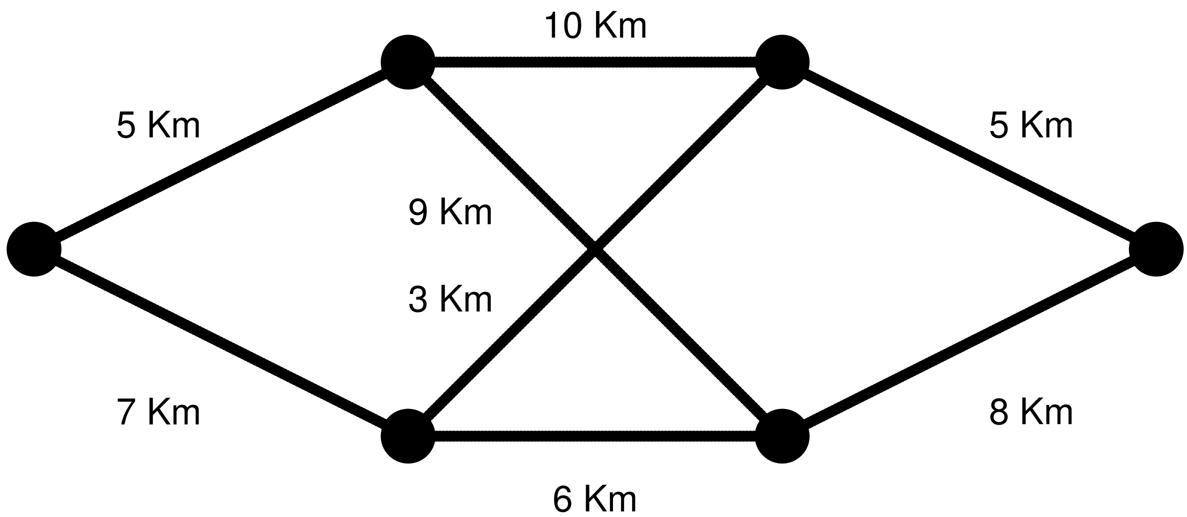
# Viterbi algorithm - A simple example




- Final step
  - The shortest path is identified

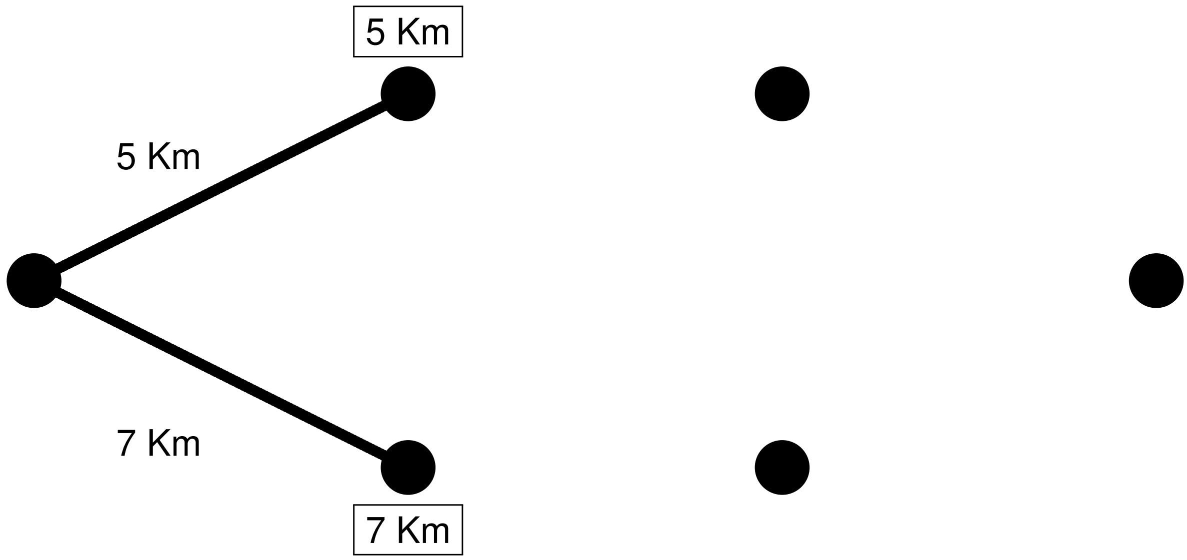


# Viterbi algorithm - A simple example - Different metrics




- In previous example, identification of each branch of the survival path requires to process up to the final node
  - Partial paths can be identified previously under some conditions

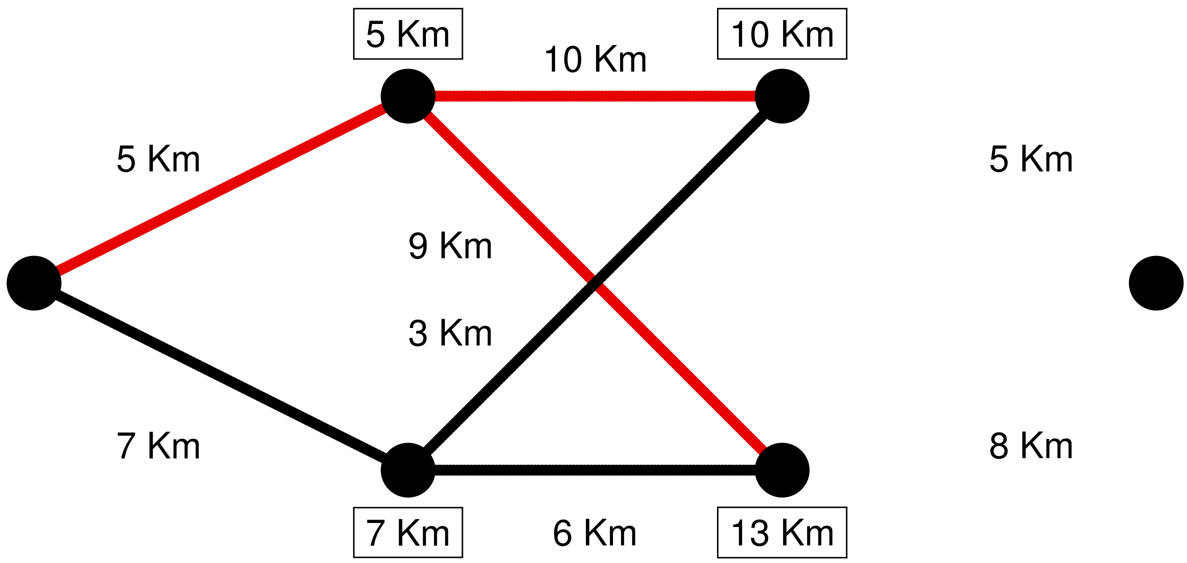
# Viterbi algorithm - A simple example - Different metrics




- First step
  - Computing the metrics at first stage after opening the trellis

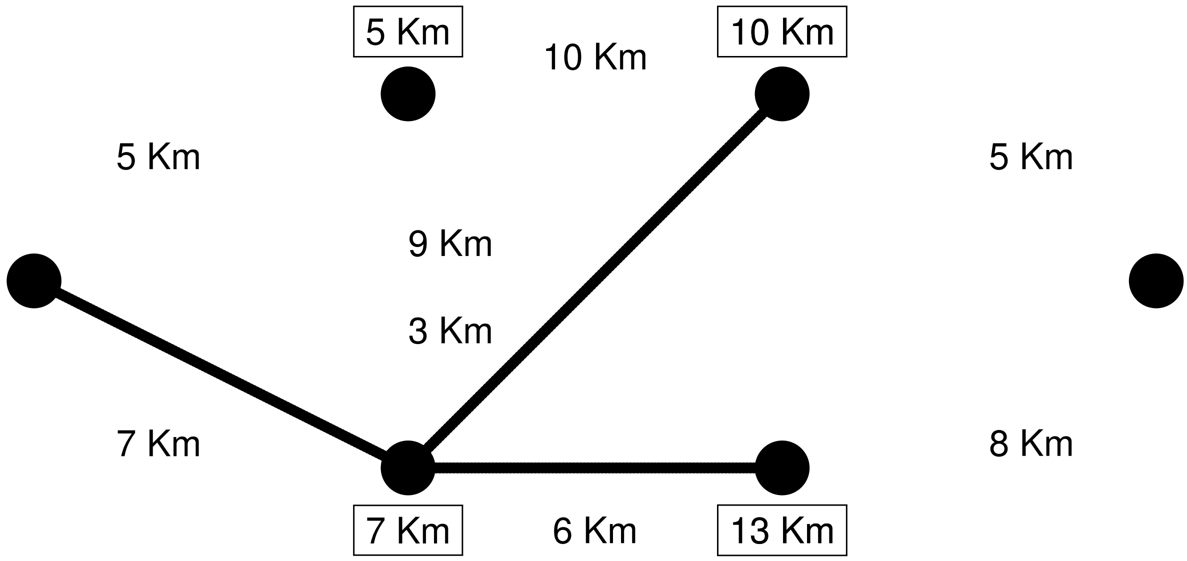


# Viterbi algorithm - A simple example - Different metrics




- Second step
  - Computing survival paths (black) for each node in next stage
    - Upper node: 7 Km + 3 Km is shorter than 5 Km + 10 Km
    - Lower node: 7 Km + 6 Km is shorter than 5 Km + 9 Km

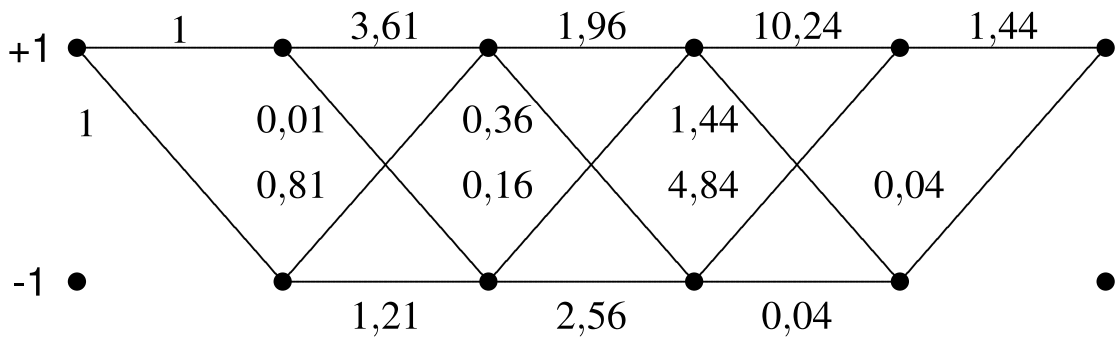
# Viterbi algorithm - A simple example - Different metrics




- At this point the survival path are fused at the lowest node of first stage (the one with metric 7 Km)
  - We now know which is the first branch of the shortest path !!!
  - This is known without processing the last stage

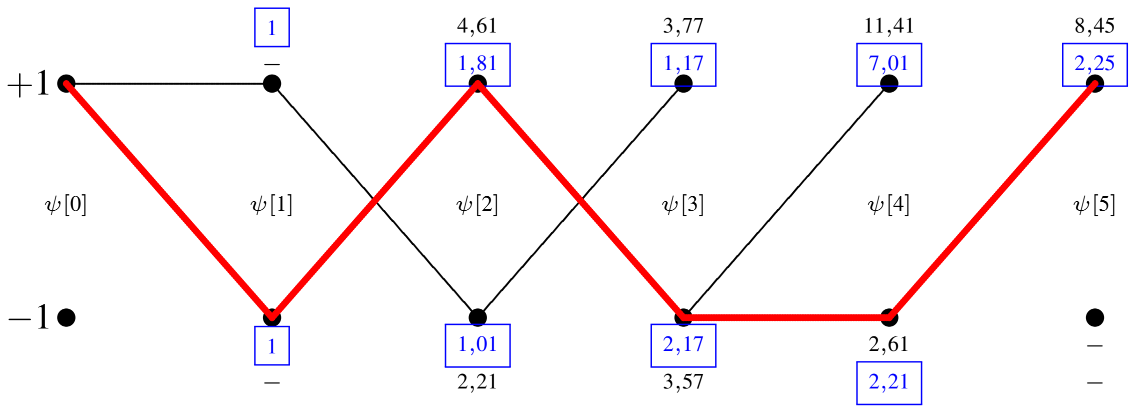


# Viterbi applied to ISI receiver - Example

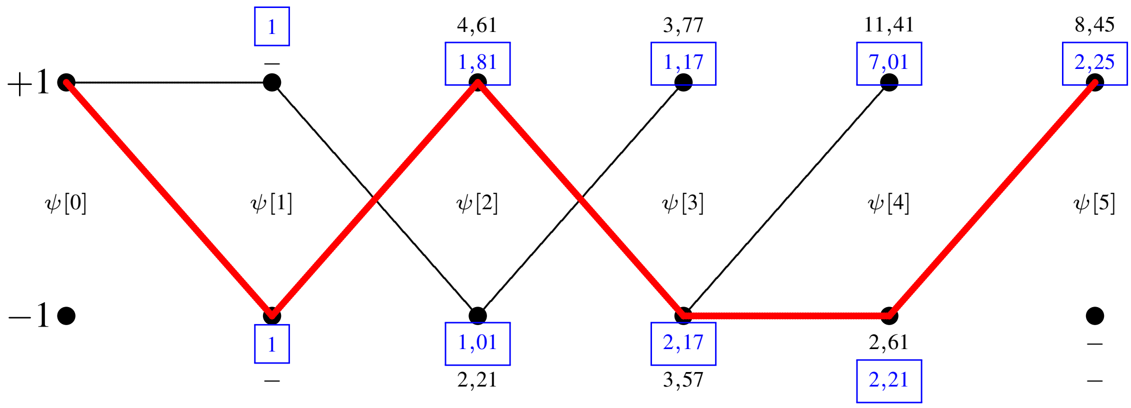



$q[0] = +0{,}5 \; q[1] = -0{,}4 \; q[2] = +0{,}1 \; q[3] = -1{,}7 \; q[4] = +0{,}3$

- Application: survival paths and accumulated metrics for each state



## Viterbi applied to ISI receiver - Example (II)




- Metric of survival path for each state is highlighted
- Path associated to maximum likelihood sequences is in red
  - This defines the associated transmitted sequence

$$\hat{A}[0] = -1,\ \hat{A}[1] = +1,\ \hat{A}[2] = -1,\ \hat{A}[3] = -1$$



## Truncated Viterbi algorithm

- To ensure proper decoding of $L$ data symbols
  - Initial and final states have to be known
    - ★ Cyclic header of $K_p$ known symbols between data sequences of $L$ symbols (which defines initial and final state)
    - ★ $L + K_p$ transitions of trellis are processed to decode $L$ data symbols
    - ★ Delay and memory constraints
- Decision for symbol $A[n]$ before processing the $L + K_p$ observations requires the merging of all survival paths at $\psi[n+1]$
  - Introduction of arbitrary delay
  - Need for information storing
- Truncated algorithm with truncation length (depth) $d$
  - After processing observation at discrete instant $n$ (transition from $\psi[n]$ and $\psi[n+1]$ in the trellis), a decision is made for estimation of symbol $A[n-d]$
    - ★ Choice of symbol associated to survival path with lowest length

## Probability of error - Erroneous event

- A path through the trellis (associated to a given symbol sequence) can be described by a sequence of states through the trellis

$$\boldsymbol{\psi} = [\psi[0], \psi[1], \psi[2], \cdots]$$

- Erroneous event: different sequence of states for transmitted sequence and detected sequence

$$\boldsymbol{e} = (\boldsymbol{\psi}, \hat{\boldsymbol{\psi}})$$

- Each erroneous event has two associated parameters
  - Length of the event ($\ell(\boldsymbol{e})$)
  - Number of erroneous symbols in the event ($w(\boldsymbol{e})$)
- Definition of erroneous event of length $\ell(\boldsymbol{e})$
  - $\psi[m] = \hat{\psi}[m]$
  - $\psi[m+\ell+1] = \hat{\psi}[m+\ell+1]$
  - $\psi[n] \neq \hat{\psi}[n]$ para $m < n \leq m+\ell$
- The number of errors of an event, $w(\boldsymbol{e})$, fulfills $1 \leq w(\boldsymbol{e}) \leq \ell$



## Probability of detecting an erroneous sequence

- Approximation

$$P\{\text{erroneous sequence}\} \approx k \cdot Q\left(\frac{D_{min}/2}{\sqrt{N_0/2}}\right)$$

  - $D_{min}$: minimum euclidean distance between noiseless outputs of two different symbol sequencesdistancia euclídea mínima entre las salidas sin ruido de dos $\{o_i[n], o_j[n]\}$, $j \neq i$
  - $k$: maximum number of sequences whose noiseless outputs are at distance $D_{min}$ of the noiseless output of the noiseless output of a symbol sequence
- When $L$ grows up, $k$ grows up, and therefore the probability of detecting an erroneous sequence goes to infinity when $L$ grows up

## Probability of symbol errors

- Usually is more usefull to estimate $P_e = P\{\hat{A}[n] \neq A[n]\}$

$$P_e = \frac{1}{L} \cdot \sum_{\boldsymbol{e} \in \mathcal{E}} w(\boldsymbol{e}) \cdot P\{\boldsymbol{e}\}$$

  where

  - $P\{\boldsymbol{e}\}$ is the probability of erroneous event $\boldsymbol{e}$
  - $\mathcal{E}$ is the set of all erroneous events that can happen in the trellis

- Denoting an erroneous event as $\boldsymbol{e} = (\psi, \hat{\psi})$, its probability of error is

$$P\{\boldsymbol{e}\} = P\{\hat{\psi}|\psi\} \cdot P\{\psi\}$$

- Difficult to evaluate $\rightarrow$ Bounds and approximations for $P_e$





## Bounds for probability of symbol error

- Bounds for the probability error for symbols

$$k_2 \cdot Q\left(\frac{D_{min}}{2\sqrt{N_0/2}}\right) \leq P_e \leq k_1 \cdot Q\left(\frac{D_{min}}{2\sqrt{N_0/2}}\right)$$

  - $k_2$ is the ratio of trellis paths having associated an erroneous event at distance $D_{min}$. It always fulfills $k_2 \leq 1$
  - $k_1$ averages the number of errors produced in the erroneous events with minimum distance $k_1 = \sum_{\boldsymbol{e} \in \mathcal{E}_{min}} w(\boldsymbol{e}) \cdot P\{\psi\}$

- Aproximation for $P_e$

$$P_e \approx k_0 \cdot Q\left(\frac{D_{min}}{2\sqrt{N_0/2}}\right)$$

  - $k_0$: constant such that $k_2 \leq k_0 \leq k_1$. Both $k_1$ and $k_2$ are independent of the noise variance

## Minimum euclidean distance with respect to the noiseless output of a given sequence

- Reference sequence $\boldsymbol{A} = \boldsymbol{a}_i$

$$D_{min}(\boldsymbol{a}_i) = \arg \min_{\substack{\boldsymbol{a}_j \\ j \neq i}} \sqrt{\sum_{n=0}^{N_q-1} \left| o_i[n] - \underbrace{\sum_{k=0}^{K_p} p[k] \cdot a_j[n-k]}_{o_j[n]} \right|^2}$$

- It can be found by using Viterbi algorithm
  - Branch metric: $|o_i[n] - o_j[n]|^2$
  - Reference: $o_i[n]$
  - Algorithm looks for the erroneous event including the reference sequence having lowest distance





## Minimum distance $D_{min}$

- Is the minimum value of $D_{min}(\boldsymbol{a}_i)$, for $i = 0, 1, \cdots, M^L - 1$

$$D_{min} = \arg \min_{\substack{\boldsymbol{a}_i, \boldsymbol{a}_j \\ j \neq i}} \sqrt{\sum_{n=0}^{N_q-1} \left| \sum_{k=0}^{K_p} p[k] \cdot (a_i[n-k] - a_j[n-k]) \right|^2}$$

i.e., the minimum distance between the noiseless output of any two different sequences

- If trellis diagram is symmetric
  - Viterbi with a single reference sequence
- In generall: Viterbi algorithm is applied for the sequence of errors

$$\xi[n] = a_i[n] - a_j[n]$$

  - Reference: sequence of zeros (associated to sequence detection without errors)

## Matched filter bound

- Provides a bound for probability of symbol error under ISI

$$P_e \geq k \cdot Q\left(\frac{d_{min}}{2} \cdot \frac{||\boldsymbol{p}||}{\sqrt{N_0/2}}\right)$$

  - $k$: maximun number of symbols at minimun distance $d_{min}$ of any symbol in the constellation
  - Channel norm is defined as follows: $||\boldsymbol{p}|| = \sqrt{\sum_{k=0}^{K_p} |p[k]|^2}$
- This allows to bound $D_{min}$

$$D_{min} \leq d_{min} \cdot ||\boldsymbol{p}||$$

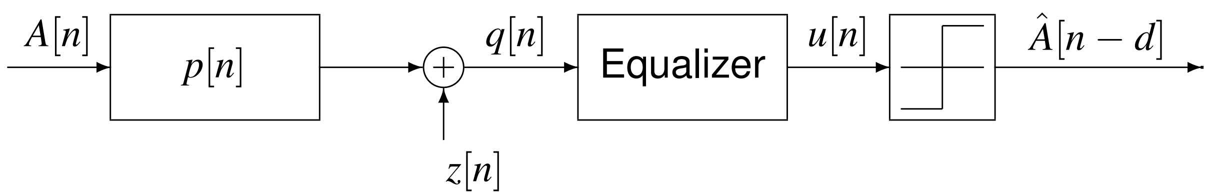
- The increase in signal to noise ration required to achieve the same $P_e$ than in a system without ISI es

$$\Delta\text{SNR} = 20 \log_{10} \frac{d_{min}||\boldsymbol{p}||}{D_{min}}$$



## Channel equalizers

- Complexity of Viterbi algorithm is exponential with $M$ and $K_p$
  - There are $M^{K_p}$ states
  - $M$ branches go out of each state
  - $M$ branches arrive to each state
- Sub-optimal alternative: channel equalizer + memoryless symbol by symbol detector



## Equalizer structure

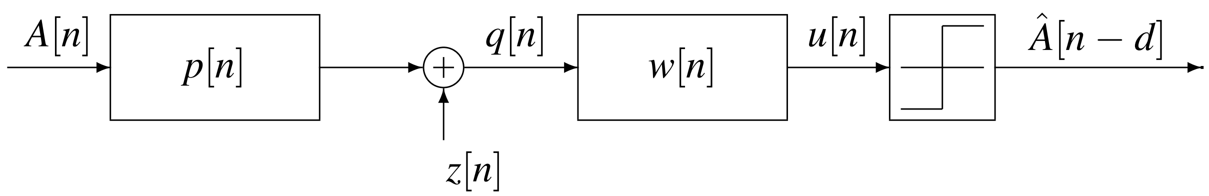
- Linear equalizer
  - LTE: *Linear Transversal Equalizer*
  - Equalizer is a linear filter
- Nonlinear equalizer with decision feedback
  - DFE: *Decision Feedback Equalizer*
- Other nonlinear structures
  - Bayesian equalizer
  - Neural networks (MLP. RBF, etc.)
  - Support vector machines
  - ...



## Blind / Non-blind equalization

- Non-blind equalization
  - Channel is known ($p[n]$ is known)
  - A reference sequence for transmitted symbols is available
- Blind equalization
  - Channel is unknown
  - A reference sequence for transmitted symbols is not available
  - Available information is reduced to statistical information about $A[n]$

## Non-blind linear equalization




$$u[n] = \sum_{k=0}^{K_w} w[k] \cdot q[n-k] = \mathbf{w}^T \cdot \boldsymbol{q}_n$$

- The channel is assumed to be known $p[n]$
- Criteria for equalizer design (to obtain the equalizer parameters)
  - Zero forcing (ZF)
  - Minimum mean squared error (MMSE)



## Linear equalizer

- Equalizer is a causal discrete time linear filter of $K_w + 1$ coefficients, $w[n]$

$$u[n] = \sum_{k=0}^{K_w} w[k] \cdot q[n-k] = \sum_{k=0}^{K_w} w[k] \cdot \left( \sum_{\ell=0}^{K_p} p[\ell] \cdot A[n-k-\ell] + z[n-k] \right)$$

- Definition: joint channel-equalizer response ($K_p + K_w + 1$ non-null coefficients)

$$c[n] = w[n] * p[n], \quad 0 \leq n \leq K_p + K_w$$

$$u[n] = \sum_{k=0}^{K_p+K_w} c[k] \cdot A[n-k] + \sum_{k=0}^{K_w} w[k] \cdot z[n-k]$$

- Output of the equalizer - delay $d$ in the decision

$$u[n] = \underbrace{c[d] \cdot A[n-d]}_{\text{desired term}} + \underbrace{\sum_{\substack{k=0 \\ k \neq d}}^{K_p+K_w} c[k] \cdot A[n-k]}_{\text{residual ISI}} + \underbrace{\sum_{k=0}^{K_w} w[k] \cdot z[n-k]}_{\text{filtered noise}}$$

## Design criteria for linear equalizers

- Zero forcing (ZF) criterion
  - Aims at eliminating intersymbol interference (ISI)
  - Mathematically, this means to look for an ideal joint channel-equalizer response

$$c[n] = p[n] * w[n] = \delta[n-d], \text{ for some arbitrary delay } d$$

- Minimim mean squared error (MMSE) criterion
  - Aims at minimizing the joint effect of resigual ISI and filtered noise
  - Mathematically, MMSE minimizes expected energy of observation error, which for a delay $D$ is defined as

$$e_d[n] = A[n-d] - u[n]$$

Difference between equalizer output and the transmitted symbol (considering $d$)



## Design using ZF criterion with unlimited equalizer length

- Ideal response (in time domain)

$$c[n] = p[n] * w[n] = \delta[n-d]$$

- Equalizer ideal response can easily be obtained in frequency domain

$$C(e^{j\omega}) = P(e^{j\omega}) \cdot W(e^{j\omega}) = e^{-j\omega d} \rightarrow W(e^{j\omega}) = \frac{e^{-j\omega d}}{P(e^{j\omega})}$$

- Selection of delay $d$ - Decomposition of $P(z)$ in minimum and maximum phase systems

$$P(z) = P_0 \cdot \underbrace{\prod_{k=1}^{K_1} \left(1 - \alpha_k \cdot z^{-1}\right)}_{P_{min}(z)} \cdot \underbrace{\prod_{\ell=1}^{K_2} \left(1 - \beta_\ell \cdot z^{-1}\right)}_{P_{max}(z)}$$

$$|\alpha_k| < 1, \text{ para } 1 \leq k \leq K_1, \; |\beta_\ell| > 1, \text{ para } 1 \leq \ell \leq K_2$$

  - $P_{min}(z)$, a minimum phase system, has an stable causal inverse
  - Stable inverse of $P_{max}(z)$, a maximum phase system, in non-causal
  - Choice of $d$ to have a causal stable inverse of the channel

## Design using ZF criterion with unlimited equalizer length (II)

- Procedure to obtain the ZF unlimited linear equalizer:
  - Obtain the the inverse (frequency) response of channel $p[n]$

$$W^{(0)}\left(e^{j\omega}\right) = \frac{1}{P\left(e^{j\omega}\right)}$$

  - Obtain its corresponding time response

$$w^{(0)}[n] = \mathcal{FT}^{-1}\left\{W^{(0)}\left(e^{j\omega}\right)\right\}$$

    - ⋆ If this is a non-causal response, evaluate the length of non-causal side, i.e., look for the maximum value for $k$ such that $w^{(0)}[-k] \neq 0$
    - ⋆ Then, choose delay as $d = k$.
  - Obtain the causal stable ZF equalizer as follows

$$w[n] = w^{(0)}[n-d]$$





## Main drawback of ZF equalizer

- ZF equalizer basically inverts the channel frequency response
  - The equalizer affects the transmitted data signal but also has an effect on the noise
- Power espectral density of the filtered noise $z[n]$

$$S_z^{ZF}(e^{j\omega}) = S_z(e^{j\omega}) \cdot \left|W(e^{j\omega})\right|^2 = \frac{\sigma_z^2}{|P(e^{j\omega})|^2}$$

- Power of noise sequence $z[n]$ is

$$\sigma_z^2|_{ZF} = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_z^{ZF}(e^{j\omega})\, d\omega = \frac{\sigma_z^2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|P(e^{j\omega})|^2}\, d\omega$$

  - Noise amplification can happen if channel has strong attenuation for some frequencies

    NOTE: Infinite noise power is analitycaly obtained for channels with spectral nulls

## Design using ZF criterion with $K_w + 1$ coefficients

- Equation system given by the joint channel-equalizer response

$$c[n] = \sum_{k=0}^{K_p} p[k] \cdot w[n-k] = \delta[n-d]$$

  - There are $K_p + K_w + 1$ equations, one for each value of $c[n]$

- Equation system in matrix notation

$$\underbrace{\begin{bmatrix} c[0] \\ c[1] \\ \vdots \\ c[K_p+K_w] \end{bmatrix}}_{\boldsymbol{c}} = \underbrace{\begin{bmatrix} p[0] & 0 & 0 & \cdots & 0 \\ p[1] & p[0] & 0 & \cdots & 0 \\ p[2] & p[1] & p[0] & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p[K_p] & p[K_p-1] & p[K_p-2] & \cdots & 0 \\ 0 & p[K_p] & p[K_p-1] & \cdots & p[0] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p[K_p] \end{bmatrix}}_{\boldsymbol{P}} \cdot \underbrace{\begin{bmatrix} w[0] \\ w[1] \\ \vdots \\ w[K_w] \end{bmatrix}}_{\mathbf{w}}$$

Matrix $\boldsymbol{P}$ is called CHANNEL CONVOLUTION MATRIX



## Design using ZF criterion with $K_w + 1$ coefficients (II)

- Desired response for joint response with delay $d$

$$c[n] = \delta[n-d] \rightarrow \mathbf{c}_d = [\underbrace{00\cdots0}_{d}10\cdots0]^T$$

Equation system for this ideal response $\mathbf{c}_d = \boldsymbol{P} \cdot \mathbf{w}$

- This is an overdetermined equation system
  - $K_p + K_w + 1$ equations
  - $K_w + 1$ unknowns
- Least squares (LS) solution

$$\mathbf{w}_d^{ZF} = \arg \min_{\mathbf{w}} ||\mathbf{c}_d - \boldsymbol{P} \cdot \mathbf{w}||^2 = \boldsymbol{P}^{\#} \cdot \mathbf{c}_d$$

  - Solution is provided by Moore-Penrose pseudo-inverse

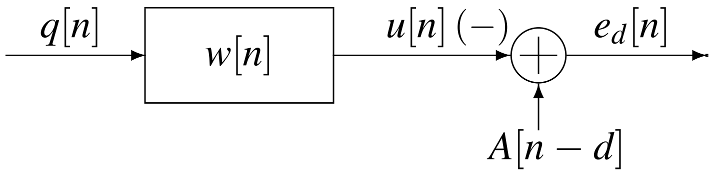
$$P^{\#} = (\boldsymbol{P}^H \cdot \boldsymbol{P})^{-1} \cdot \boldsymbol{P}^H$$

The obtained solution does not fulfills all equations, i.e.

Joint response $\boldsymbol{c}_d^{ZF} = \boldsymbol{P} \cdot \mathbf{w}_d^{ZF} \neq \boldsymbol{c}_d$

Some residual ISI is still present because of limitation in number of coefficients

# Design using MMSE criterion with unlimited equalizer length




- Error sequence at the equalizer output for delay $d$

$$e_d[n] = A[n-d] - u[n]$$

- MMSE optimal linear filtering: minimization of $E\left[|e_d[n]|^2\right]$
- MMSE solution - Orthogonality principle:
  - Error sequence $e_d[n]$ is orthogonal to system output $u[n]$
  - Error sequence $e_d[n]$ is orthogonal to system input $q[n]$

$$E[\underbrace{(A[n-d] - u[n])}_{e_d[n]} \cdot q^*[\ell]] = 0, \ \forall \ell$$

This equation can be written as follows

$$E[A[n-d] \cdot q^*[\ell]] = E[u[n] \cdot q^*[\ell]], \ \forall \ell$$



# Orthogonality principle - first term

- Initial assumptions
  - Data sequence $A[n]$ is white: $R_A[k] = E_s \cdot \delta[k]$
  - Noise sequence $z[n]$ is white: $R_z[k] = \sigma_z^2 \cdot \delta[k]$
  - Data and noise sequences, $A[n]$ and $z[n]$, are independent

    This means that $R_{A,z}[k] = E\left[A[n+k] \cdot z^*[n]\right] = 0, \ \forall k$
- Development of the first term of orthogonality principle

$$\begin{aligned} E[A[n-d] \cdot q^*[\ell]] =& E\left[A[n-d] \cdot \left(\sum_{k=0}^{K_p} p[k] \cdot A[\ell-k] + z[\ell]\right)^*\right] \\ =& \sum_{k=0}^{K_p} p^*[k] \cdot \underbrace{E\left[A[n-d] \cdot A^*[\ell-k]\right]}_{R_A[n-d-\ell+k]} \\ &+ \underbrace{E\left[A[n-d] \cdot z^*[\ell]\right]}_{R_{A,z}[n-d-\ell]} \\ =& E_s \cdot p^*[\ell+d-n] \end{aligned}$$

Note that because $R_A[k] = E_s \cdot \delta[n]$, then $R_A[n-d-\ell+k] \neq 0$ just for index $k = \ell + d - n$

# Orthogonality principle - Second term

$$E\left[u[n]\cdot q^*[\ell]\right] = E\left[\left(\sum_{k=0}^{K_w} w[k]\cdot q[n-k]\right)\cdot q^*[\ell]\right]$$

$$= \sum_{k=0}^{K_w} w[k]\cdot \underbrace{E[q[n-k\cdot q^*[\ell]]}_{R_q[n-k-\ell]} = \left.(w[k]*R_q[k])\right|_{k=n-\ell}$$

- Autocorrelation function for observations $q[n]$

$$R_q[n] = E\left[q[\ell+n]\cdot q^*[\ell]\right]$$

$$= E\left[\left(\sum_{k=0}^{K_p} p[k]\cdot A[\ell+n-k]+z[\ell+n]\right)\cdot\left(\sum_{j=0}^{K_p} p[j]\cdot A[\ell-j]+z[\ell]\right)^*\right]$$

$$= \sum_{k=0}^{K_p}\sum_{j=0}^{K_p} p[k]\cdot p^*[j]\,\underbrace{E\left[A[\ell+n-k]\cdot A^*[\ell-j]\right]}_{R_A[n-k+j]} + \underbrace{E\left[z[\ell+n]\cdot z^*[\ell]\right]}_{R_z[n]}$$

$$= E_s\cdot\sum_{k=0}^{K_p} p[k]\cdot p^*[k-n]+\sigma_z^2\cdot\delta[n] = E_s\cdot(p[n]*p^*[-n])+\sigma_z^2\cdot\delta[n]$$

Note that because $R_A[k]=E_s\cdot\delta[n]$, then $R_A[n-k+j]\neq 0$ just for index $j=k-n$



# Orthogonality principle - MMSE equalizer

- Combining both terms

$$E_s\cdot p^*[\underbrace{\ell+d-n}_{-(n-\ell-d)}] = w[n]*\left[E_s\cdot\left.(p[k]*p^*[-k])\right|_{k=n-\ell}+\sigma_z^2\cdot\delta[n-\ell]\right]$$

- Making change of variable $k=n-\ell$, and dividing by $E_s$

$$p^*[-(k-d)] = w[k]*\left[(p[k]*p^*[-k])+\frac{\sigma_z^2}{E_s}\cdot\delta[k]\right]$$

  - This is equivalent in the frequency domain to

$$P^*(e^{j\omega})\cdot e^{-j\omega d} = W(e^{j\omega})\times\left[P(e^{j\omega})\cdot P^*(e^{j\omega})+\frac{\sigma_z^2}{E_s}\right]$$

- The expression of equalizer in the frequency domain becomes

$$W(e^{j\omega}) = \frac{P^*(e^{j\omega})\cdot e^{-j\omega d}}{P(e^{j\omega})\cdot P^*(e^{j\omega})+\frac{\sigma_z^2}{E_s}}$$

## Design using MMSE criterion with $K_w + 1$ coefficients

- Orthogonality principle can be written as follows

$$R_{A,q}[n-d] = \sum_{k=0}^{K_w} w[k] \cdot R_q[n-k]$$

  - System with $K_w + 1$ equations for the $K_w + 1$ unknowns

$$\boldsymbol{r}_{A,q}^{d} = \boldsymbol{R}_q \cdot \mathbf{w} \rightarrow \mathbf{w}_d^{MMSE} = (\boldsymbol{R}_q)^{-1} \cdot \boldsymbol{r}_{A,q}^{d}$$

NOTE: definition of involved vectors and matrix are in next slide

- Solution can also be found through channel matrix $\boldsymbol{P}$

$$\mathbf{w}_d^{MMSE} = (\boldsymbol{P}^H \cdot \boldsymbol{P} + \lambda \cdot \boldsymbol{I})^{-1} \cdot \boldsymbol{P}^H \cdot \mathbf{c}_d$$

$$\lambda = \frac{\sigma_z^2}{E_s}$$

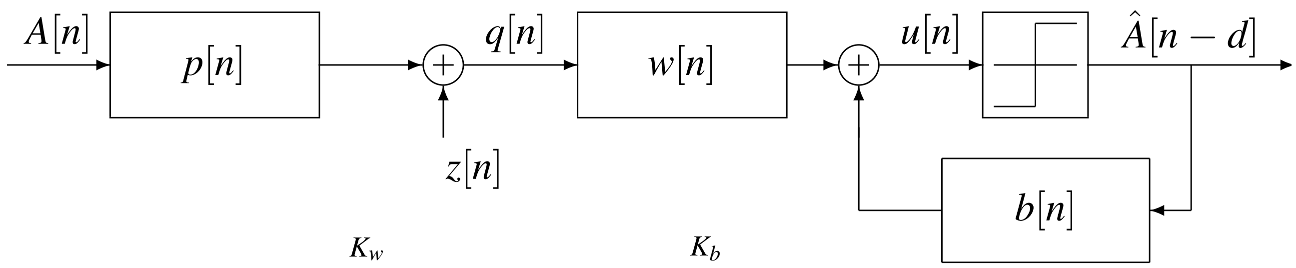


## Matrix equation system

$$\underbrace{\begin{bmatrix} R_{A,q}[-d] \\ R_{A,q}[-(d-1)] \\ R_{A,q}[-(d-2)] \\ \vdots \\ R_{A,q}[K_w-d] \end{bmatrix}}_{\boldsymbol{r}_{A,q}^{d}} =$$

$$\underbrace{\begin{bmatrix} R_q[0] & R_q^*[1] & R_q^*[2] & \cdots & R_q^*[K_W] \\ R_q[1] & R_q[0] & R_q^*[1] & \cdots & R_q^*[K_W-1] \\ R_q[2] & R_q[1] & R_q[0] & \cdots & R_q^*[K_W-2] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_q[K_W] & R_q[K_w-1] & R_q[K_w-2] & \cdots & R_q[0] \end{bmatrix}}_{\boldsymbol{R}_q} \cdot \underbrace{\begin{bmatrix} w[0] \\ w[1] \\ w[2] \\ \vdots \\ w[K_w] \end{bmatrix}}_{\mathbf{w}}$$

## Equalization with decision feedback




$$u[n] = \sum_{k=0}^{K_w} w[k] \cdot q[n-k] - \sum_{k=1}^{K_b} b[k] \cdot \hat{A}[n-d-k]$$

- Equalization structure: two causal linear filters
  - Forward filter: $Kw + 1$ coefficients, input is $q[n]$
  - Feedback filter: $K_b$ coefficients, input is the sequence of previous decisions
- Design criteria to obtain filter coefficients
  - Zero forcing (ZF)
  - Minimum mean squared error (MMSE)



## Equations with decision feedback

$$u[n] = \underbrace{c[d] \cdot A[n-d]}_{\text{cursor}} + \underbrace{\sum_{k=0}^{d-1} c[k] \cdot A[n-k]}_{\text{precursor ISI}} + \underbrace{\sum_{d+1}^{K_p+K_w} c[k] \cdot A[n-k]}_{\text{postcursor ISI}}$$

$$\underbrace{- \sum_{k=1}^{K_b} b[k] \cdot \hat{A}[n-d-k]}_{\text{feedback decisions}} + \underbrace{\sum_{k=0}^{K_w} w[k] \cdot z[n-k]}_{\text{filtered noise } z'[n]}$$

$$u[n] = \underbrace{c[d] \cdot A[n-d]}_{\text{cursor}} + \underbrace{\sum_{k=0}^{d-1} c[k] \cdot A[n-k]}_{\text{precursor ISI}} + \underbrace{\sum_{k=d+1}^{d+K_b} c[k] \cdot A[n-k]}_{\text{postcursor ISI (a)}}$$

$$+ \underbrace{\sum_{k=d+K_b+1}^{K_p+K_w} c[k] \cdot A[n-k]}_{\text{Postcursor ISI (b)}} \underbrace{- \sum_{k=1}^{K_b} b[k] \cdot \hat{A}[n-(d+k)]}_{\text{feedback decisions}} + \underbrace{z'[n]}_{\text{noise}}$$

## Coefficients for DFE filters

- Coefficients of the feedback filter ($b[n],\ n=1,\cdots,K_b$)
  - Goal: cancelation of $K_b$ terms of postcursor ISI (*postcursor ISI (a)*)
  - Identification of terms in *postcursor ISI (a)* and *feedback decisions*

$$b[k]=c[d+k],\quad k=1,2,\cdots,K_b$$

- Coefficients of forward filter ($w[n],\ n=0,\cdots,K_w$): ZF criterion
  - Reduced equation system: equations for terms of *postcursor ISI (a)*, which is handled with $b[k]$, are removed

$$\mathbf{c}'_d=\boldsymbol{P}'\cdot\mathbf{w}$$

$$\mathbf{c}'_d=[c[0],c[1],\cdots,c[d],c[d+K_b+1],\cdots,c[K_p+K_w]]^T$$

- Coeficientes of forward filter: MMSE criterion
  - Equations for terms in *postcursor ISI (a)* are decoupled (see $\boldsymbol{D}_d$)

$$\mathbf{w}_d|_{MMSE}=(\boldsymbol{P}^H\cdot\boldsymbol{D}_d\cdot\boldsymbol{P}+\lambda\cdot\boldsymbol{I})^{-1}\cdot\boldsymbol{P}^H\cdot\mathbf{c}_d$$

$$\boldsymbol{D}_d=\text{diag}(\underbrace{11\cdots1}_{d+1}\underbrace{00\cdots0}_{K_b}\ \underbrace{11\cdots1}_{K_p+K_w-d-K_b}\ )$$



## Asymptotic performance of linear equalizers

- Analysis based on the characterization of $e_d[n]=A[n-d]-u[n]$
- Equation output

$$u[n]=A[n]*p[n]*w[n]+z[n]*w[n]$$

- Error sequence at equalizer output

$$e_d[n]=A[n]*(\delta[n-d]-w[n]*p[n]))-z[n]*w[n]$$

- Power spectral density of of error sequence

$$S_{e_d}\left(e^{j\omega}\right)=S_A\left(e^{j\omega}\right)\cdot\left|e^{-j\omega d}-W\left(e^{j\omega}\right)\cdot P\left(e^{j\omega}\right)\right|^2+S_z\left(e^{j\omega}\right)\cdot\left|W\left(e^{j\omega}\right)\right|^2$$

- For white data sequences and white noise

$$S_{e_d}\left(e^{j\omega}\right)=E_s\cdot\left|e^{-j\omega d}-W\left(e^{j\omega}\right)\cdot P\left(e^{j\omega}\right)\right|^2+\sigma_z^2\cdot\left|W\left(e^{j\omega}\right)\right|^2$$

- Power of the error sequence

$$\sigma_{e_d}^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}S_{e_d}\left(e^{j\omega}\right)\,d\omega$$

## Asymptotic performance of linear equalizers (II)

- Replacing $W\left(e^{j\omega}\right)$ for ZF criterion

$$W(e^{j\omega}) = \frac{e^{-j\omega d}}{P(e^{j\omega})}$$

$$\sigma_{e_d}^2(ZF) = \frac{\sigma_z^2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\left|P\left(e^{j\omega}\right)\right|^2}\, d\omega$$

- Replacing $W\left(e^{j\omega}\right)$ for MMSE criterion

$$W(e^{j\omega}) = \frac{P^*(e^{j\omega}) \cdot e^{-j\omega d}}{P(e^{j\omega}) \cdot P^*(e^{j\omega}) + \frac{\sigma_z^2}{E_s}}$$

$$\sigma_{e_d}^2(MMSE) = \frac{\sigma_z^2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\left|P\left(e^{j\omega}\right)\right|^2 + \frac{\sigma_z^2}{E_s}}\, d\omega$$



## Asymptotic performance of DFE equalizers

- Power spectral density of noise is

$$S_{e_d}^{DFE}\left(e^{j\omega}\right) = S_{e_d}^{LIN}\left(e^{j\omega}\right) \cdot \left|1 + B\left(e^{j\omega}\right)\right|^2$$

- Approximations for power of the error sequence can be obtained assuming that correct decisions are been feedbacked
  - ZF criterion

$$\sigma_{e_d}^2(ZF - DFE) = \sigma_z^2 \cdot \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln \frac{1}{\left|P\left(e^{j\omega}\right)\right|^2}\, d\omega\right)$$

  - MMSE criterion

$$\sigma_{e_d}^2(MMSE - DFE) = \sigma_z^2 \cdot \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln \frac{1}{\left|P\left(e^{j\omega}\right)\right|^2 + \frac{\sigma_z^2}{E_s}}\, d\omega\right)$$

## Performance of linear equalizers with $K_w + 1$ coefficients

- Equalizer output is now

$$u[n] = \underbrace{c[d]}_{\text{gain}} \cdot A[n-d] + \underbrace{\sum_{\substack{k=0\\k\neq d}}^{K_p+K_w} c[k]\cdot A[n-k]}_{\text{residual ISI}} + \underbrace{\sum_{k=0}^{K_w} w[k]\cdot z[n-k]}_{\text{filtered noise } z'[n]}$$

- Assumptions
  - ISI and filtered noise are independent
  - Distribution for ISI is Gaussian
- Approximation for the probability of error

$$P_e \approx k \cdot Q\left(\frac{d_{min}\cdot |c[d]|}{2\sqrt{\sigma_{z'}^2 + \sigma_{ISI}^2}}\right)$$

$k$: maximum number of symbols at minimum distance of another symbol in the constellation



## Mean y variance for filtered noise $z'[n]$

- Mean of $z'[n]$

$$E\left[z'[n]\right] = \sum_{k=0}^{K_w} w[k]\cdot E[z[n-k]] = 0$$

- Variance of $z'[n]$

$$\sigma_{z'}^2 = E\left[\left(\sum_{k=0}^{K_w} w[k]\cdot z[n-k]\right)\cdot\left(\sum_{j=0}^{K_w} w^*[j]\cdot z^*[n-j]\right)\right]$$

$$= \sum_{k=0}^{K_w}\sum_{j=0}^{K_w} w[k]\cdot w^*[j]\cdot \underbrace{E\left[z[n-k]\cdot z^*[n-j]\right]}_{R_z[j-k]=\sigma_z^2\cdot\delta[j-k]}$$

$$= \sigma_z^2 \cdot \sum_{k=0}^{K_w} |w[k]|^2$$

## Mean and variance of ISI term

- Mean of ISI term

$$E\left[ISI\right] = \sum_{\substack{k=0\\k\neq d}}^{K_p+K_w} c[k]\cdot E[A[n-k]] = 0$$

- Variance of ISI term

$$\sigma^2_{ISI} = E\left[\left(\sum_{\substack{k=0\\k\neq d}}^{K_p+K_w} c[k]\cdot A[n-k]\right)\cdot\left(\sum_{\substack{j=0\\j\neq d}}^{K_p+K_w} c^*[j]\cdot A^*[n-j]\right)\right]$$

$$= \sum_{\substack{k=0\\k\neq d}}^{K_p+K_w}\sum_{\substack{j=0\\j\neq d}}^{K_p+K_w} c[k]\cdot c^*[j]\cdot \underbrace{E\left[A[n-k]\cdot A^*[n-j]\right]}_{R_A[j-k]=E_s\cdot\delta[j-k]}$$

$$= E_s\cdot\sum_{\substack{k=0\\k\neq d}}^{K_p+K_w} |c[k]|^2$$



## Performance of DFE with finite coefficient lengths

- Correct decisions in the feedback filter are assumed
- Mean of ISI term

$$E\left[ISI\right] = \sum_{\substack{k=0\\k\notin[d,d+K_b]}}^{K_p+K_w} c[k]\cdot E[A[n-k]] = 0$$

- Variance of ISI term

$$\sigma^2_{ISI} = E\left[\left(\sum_{\substack{k=0\\k\notin[d,d+K_b]}}^{K_p+K_w} c[k]\cdot A[n-k]\right)\cdot\left(\sum_{\substack{j=0\\j\notin[d,d+K_b]}}^{K_p+K_w} c^*[j]\cdot A^*[n-j]\right)\right]$$

$$= \sum_{\substack{k=0\\k\notin[d,d+K_b]}}^{K_p+K_w}\sum_{\substack{j=0\\j\notin[d,d+K_b]}}^{K_p+K_w} c[k]\cdot c^*[j]\cdot \underbrace{E\left[A[n-k]\cdot A^*[n-j]\right]}_{R_A[j-k]=E_s\cdot\delta[j-k]}$$

$$= E_s\cdot\sum_{\substack{k=0\\k\notin[d,d+K_b]}}^{K_p+K_w} |c[k]|^2$$