Digital Communications
Telecommunications Engineering

Chapter 5

Multipulse modulations

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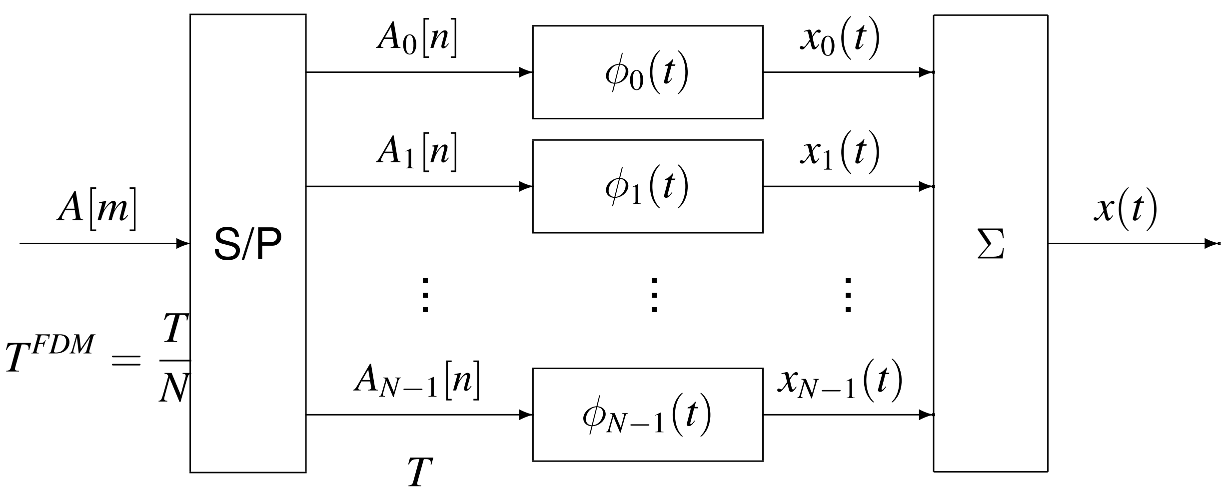
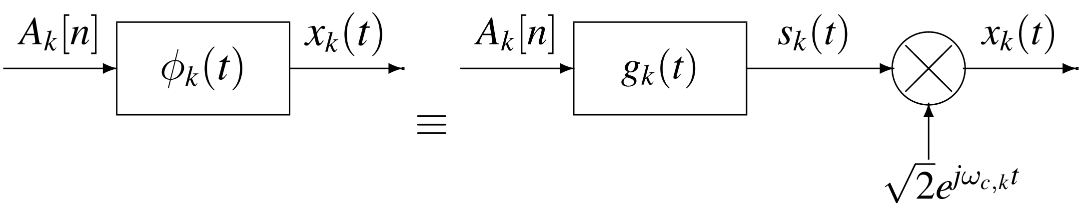
Modulation using multiple carriers - FDM

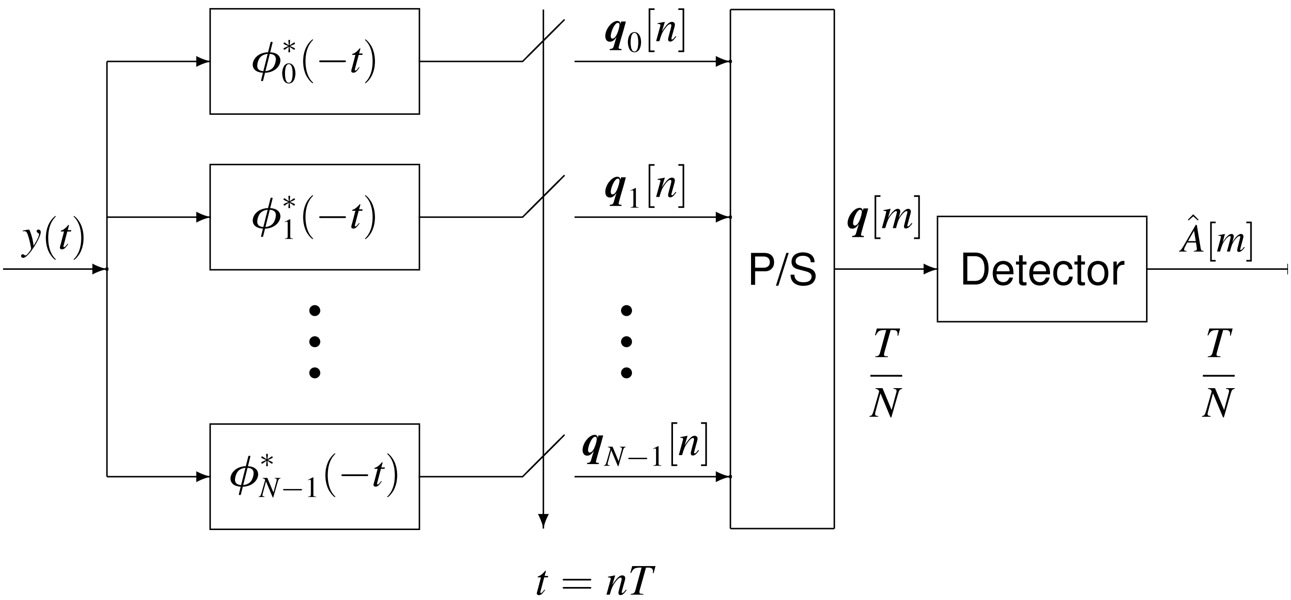
- FDM - Frequency division multiplex
- Available bandwidth (W^{FDM} rad/s) is divided in N subchannels
  - Data sequence $A[n]$ is divided in N sequences
  - Transmission of a different modulated signal in each subchannel
  - Subchannel symbol rate: $R_s = \frac{1}{T}$ bauds
  - Bandwidth for each subchannel: $W = \frac{W^{FDM}}{N} = \frac{2\pi}{T} \cdot (1+\alpha)$ rad/s
  - Total system rate: $R_s^{FDM} = N \times R_s = \frac{1}{T^{FDM}}$ bauds ($T^{FDM} = \frac{T}{N}$)
- Transmitter
  - Serial / parallel conversion: $A[m] \rightarrow \{A_0[n], \cdots, A_{N-1}[n]\}$
    - ★ FDM system rate $R_s^{FDM} \rightarrow$ channel rate R_s
  - N branches with bandpass PAM signals
    - ★ Transmission filter in k-th branch is $\phi_k(t)$, $k = 0, \cdots, N-1$
      - Parameters: shaping filter $g_k(t)$, central frequency $\omega_{c,k}$
    - ★ The modulated signal in k-th branch is $s_k(t)$
- Receiver
  - Set of N matched filters
  - Parallel / serial conversion: $\{\hat{A}_0[n], \cdots, \hat{A}_{N-1}[n]\} \rightarrow \hat{A}[m]$
    - ★ Channel rate $R_s \rightarrow$ FDM system rate R_s^{FDM}

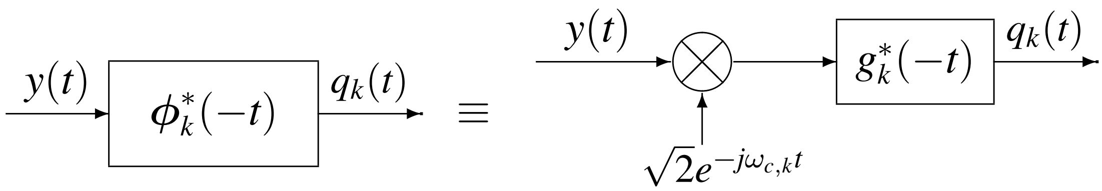
FDM modulator





FDM demodulator

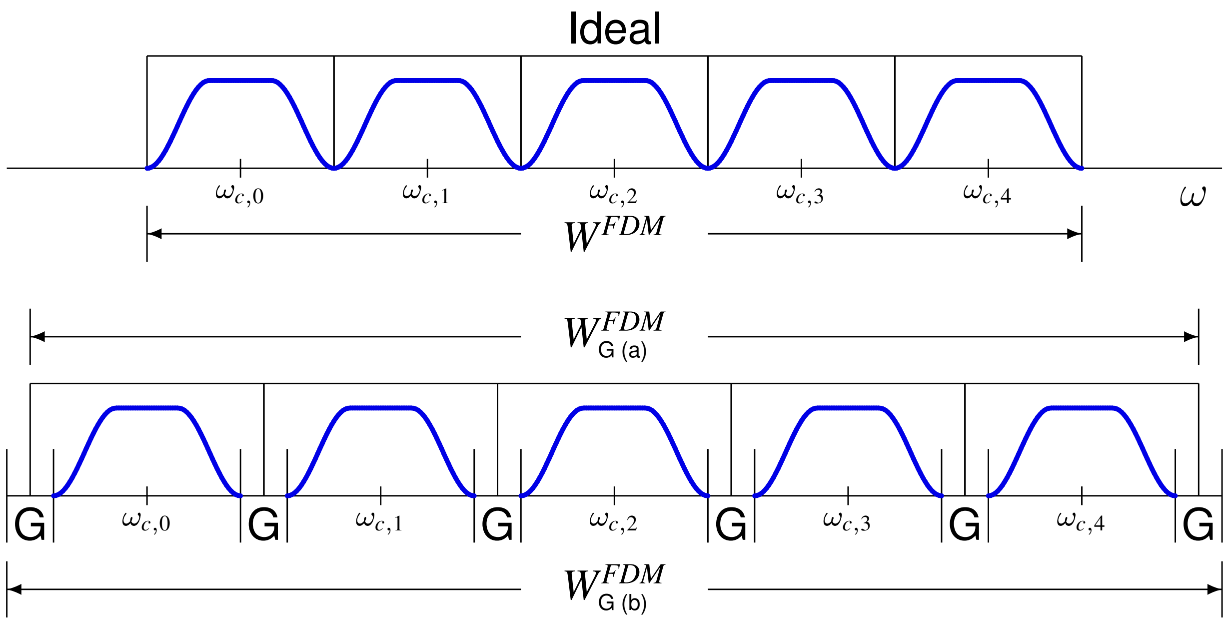




Drawbacks of FDM solution

- Hardware complexity
  - N transmission filters (bandpass: in phase and quadrature components)
  - N modulators / demodulators (bandpass)
  - N receiver filters (bandpass)
  - N synchronous samplers (bandpass)
- Ideal filters are required to optimize bandwidth use
  - Without ideal filters, guard intervals must be introduced to separate channels
    - Loss of spectral efficiency
- Alternative solution:
  - Orthogonal FDM modulation (OFDM)
    - N orthogonal pulses (allowing spectral overlapping)
    - Efficient use of available bandwidth
    - Efficient implementation : low hardware complexity

FDM - Guard bands



$$W_{\text{G (a)}}^{FDM} = W^{FDM} + N \times \text{G}$$

$$W_{\text{G (b)}}^{FDM} = W^{FDM} + (N+1) \times \text{G}$$

NOTE: in some systems, guards at both extremes of the band are half size (a)

Continuous time OFDM

- Modulated signal in terms of complex baseband signal

$$x(t) = \sqrt{2} \cdot \mathcal{Re}\{s(t) \cdot e^{j\omega_c t}\}$$

Usual notation for bandpass modulated signals

- Complex baseband signal
  - Addition of N signals, one for each data sequence $A_k[n]$

$$s(t) = \sum_{k=0}^{N-1} \underbrace{\sum_n A_k[n] \cdot \phi_k(t - nT)}_{s_k(t)}$$

Each signal $s_k(t)$ is a PAM signal with transmission filter $\phi_k(t)$

- N transmission filters: prototype filter $\times$ N different carriers

$$\phi_k(t) = \frac{1}{\sqrt{T}} \cdot w_T(t) \cdot e^{j\frac{2\pi k}{T} \cdot t}$$

$w_T(t)$: continuous time causal window of T seconds $w_T(t) = \begin{cases} 1 & 0 \leq t < T \\ 0 & \text{other case} \end{cases}$

Orthonormality of pulses

- OFDM pulses can be seen as an orthonormal basis
  Inner product is

$$
\begin{aligned}
\langle \phi_k, \phi_\ell \rangle =& \frac{1}{T}\int_0^T e^{j\frac{2\pi k}{T}\cdot t} \cdot e^{-j\frac{2\pi \ell}{T}\cdot t}\, dt \\
=& \frac{1}{T}\int_0^T \cos\left(\frac{2\pi(k-\ell)}{T}\cdot t\right) dt + j\frac{1}{T}\int_0^T \operatorname{sen}\left(\frac{2\pi(k-\ell)}{T}\cdot t\right) dt \\
=& \delta[k-\ell]
\end{aligned}
$$

- Relationship of pulses with prototype filter $\phi_0(t)$

$$\phi_k(t-nT) = \phi_0(t-nT)\cdot e^{\frac{2\pi k}{T}\cdot(t-nT)} = \phi_0(t-nT)\cdot e^{\frac{2\pi k}{T}\cdot t}$$

Spectrum of continuous time OFDM

- Frequency response for the pulses

$$|\Phi_k(j\omega)|^2 = T\cdot \text{sinc}^2\left(\frac{\left(\omega-\frac{2\pi k}{T}\right)T}{2\pi}\right),\; k=0,\cdots,N-1.$$

- $A_k[n]$ and $A_\ell[n]$ are not correlated and $A_k[n]$ is assumed to be white $\forall k$

$$S_s(j\omega) = \frac{1}{T}\cdot\sum_{k=0}^{N-1} E_{s,k}\cdot|\Phi_k(j\omega)|^2$$

  $E_{s,k}$: mean energy per symbol of constellation for $A_k[n]$

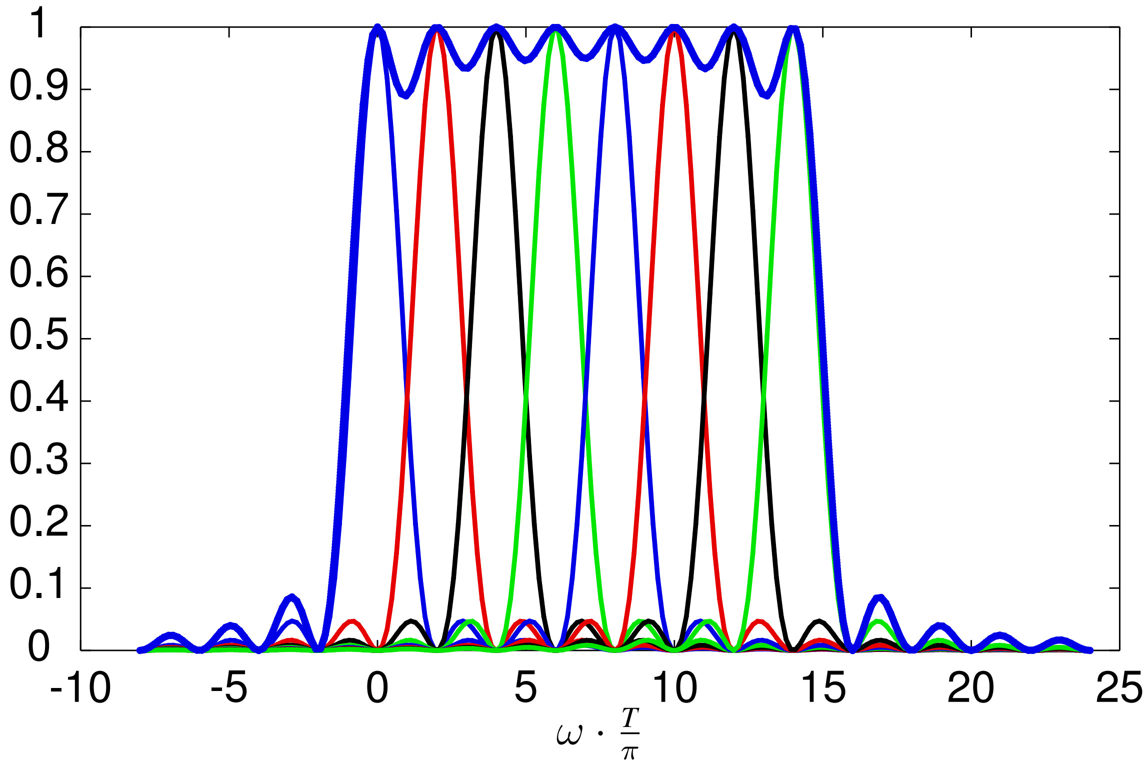
- Power of the transmitted signal

$$P_S = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_s(j\omega)\, d\omega = \frac{1}{T}\sum_{k=0}^{N-1} E_{s,k}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}|\Phi_k(j\omega)|^2 d\omega = \frac{1}{T}\sum_{k=0}^{N-1} E_{s,k}$$

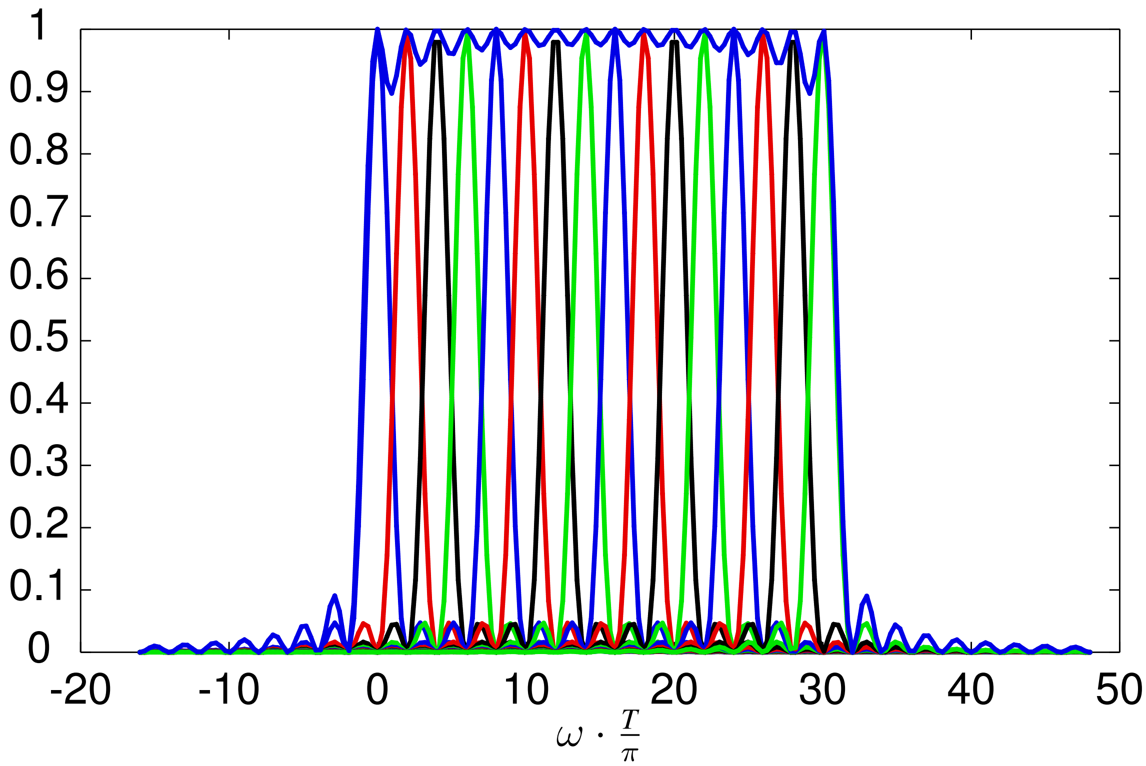
- When constellatios of all N sequences are identical

$$P_S = \frac{E_s}{T}\times N = E_s \times R_s \times N \text{ Watts}$$

Spectrum of continuous time OFDM - $N = 8$



Spectrum of continuous time OFDM - $N = 16$



Spectrum is asymptotically flat

- We consider infinite carriers and identical constellations

$$S_s(j\omega) = E_s \sum_{k=-\infty}^{\infty} \text{sinc}^2\left(\frac{\left(\omega - \frac{2\pi k}{T}\right)T}{2\pi}\right)$$
$$= E_s \cdot \text{sinc}^2\left(\frac{\omega T}{2\pi}\right) * \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

- This PSD is flat if the following condition is fulfilled

$$\frac{E_s}{T} \cdot (\phi_0(t) * \phi_0(-t)) \cdot \sum_{k=-\infty}^{\infty} \delta(t - kT) = C \cdot \delta(t)$$

Discrete time OFDM modulation

- Approximation: bandwidth can be considered limited
  Approximated bandwidth is $2\pi N/T$ rad/s
- Alternative for signal generation
  - Synthesis of samples of the signal, $s[m]$, at sampling rate given by Nyquist
    With the assumed approximation this means to sample at T/N s
  - Digital / analog conversion (reconstruction filter at T/N)
- Analytic expression for samples in interval $0 \le t < T$ (first N samples)

$$s[m] = \sum_{k=0}^{N-1} A_k[0] \cdot \phi_k(mT/N), \qquad m = 0, \cdots, N-1$$

- Equivalent expression for these samples

$$s[m] = \frac{1}{\sqrt{T}} \cdot \sum_{k=0}^{N-1} A_k[0] \cdot e^{j\frac{2\pi k}{N} \cdot m}, \qquad m = 0, \cdots, N-1$$

  - Inverse DFT of N samples of $A_k[0]$, with $k = 0, 1, \cdots, N-1$

$$\text{IDFT}_N\left(\{A_0[0], A_1[0], \cdots, A_{N-1}[0]\}\right) \rightarrow s^{(0)}[m] = \{s[0], s[1], \cdots, s[N-1]\}$$

General expressions for samples and reconstruction filter

- Samples of OFDM signal

$$s[m] = \sum_n \sum_{k=0}^{N-1} A_k[n] \cdot \phi_k(mT/N - nT)$$

$$= \frac{1}{\sqrt{T}} \sum_n \sum_{k=0}^{N-1} A_k[n] \cdot e^{j\frac{2\pi k}{N}\cdot(m-nN)} \cdot w_N[m-nN]$$

$w_N[m]$: discrete time causal window of N samples $w_N[m] = \begin{cases} 1 & 0 \leq m \leq N-1 \\ 0 & \text{other case} \end{cases}$

  - Samples are generated in blocks of N samples

$$\text{IDFT}_N\left(\{A_0[n], A_1[n], \cdots, A_{N-1}[n]\}\right) \rightarrow \{s[nN], s[nN+1], \cdots, s[(n+1)N-1]\}$$

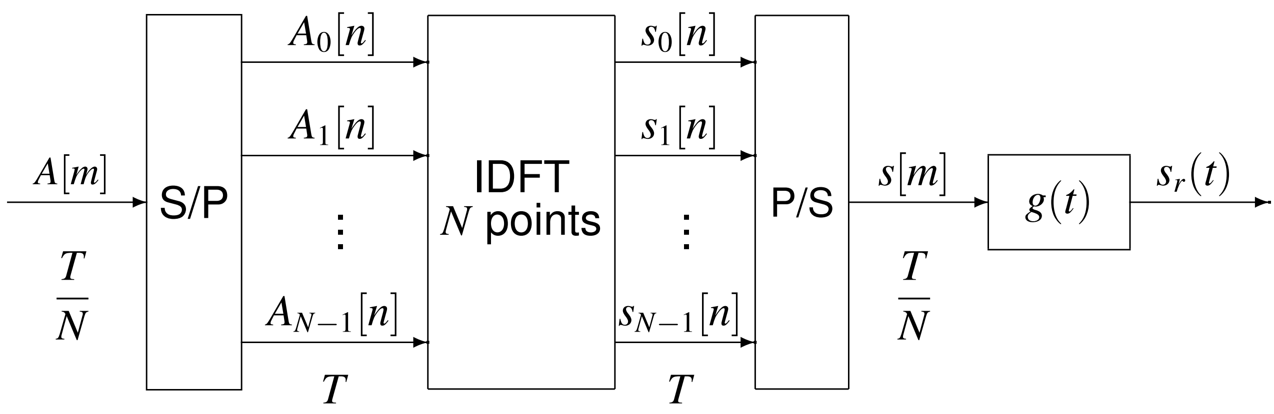
Notation: n-th data block $s^{(n)}[m] = s[nN+m]$

$$\text{IDFT}_N\left(\{A_k[n]\}_{k=0}^{N-1}\right) \rightarrow \left\{s^{(n)}[m]\right\}_{m=0}^{N-1}$$

- Reconstruction filter: interpolation at rate T/N

$$g(t) = \text{sinc}\left(\frac{N}{T}\cdot t\right), \qquad G(j\omega) = \frac{T}{N}\cdot \Pi\left(\frac{\omega T}{2\pi N}\right)$$

Modulator for discrete time OFDM



$$s_r(t) = \sum_m s[m] \cdot g(t - mT/N)$$

Discrete time orthonormal basis

- Discrete time basis functions

$$\xi_k[m] = \frac{1}{\sqrt{N}} \cdot e^{j\frac{2\pi k}{N} \cdot m} \cdot w_N[m], \qquad k = 0, \cdots, N-1$$

- Orthonormal basis

$$\langle \xi_k, \xi_\ell \rangle = \sum_m \xi_k[m] \cdot \xi_\ell^*[m] = \frac{1}{N} \cdot \sum_{m=0}^{N-1} e^{j\frac{2\pi(k-\ell)}{N} \cdot m} = \delta[k-\ell]$$

- Signal samples: expansion in the orthonormal basis

$$s[m] = \sqrt{\frac{N}{T}} \cdot \sum_n \sum_{k=0}^{N-1} A_k[n] \cdot \xi_k[m-nN]$$

Equivalent continuous time orthonormal basis

- Reconstructed signal is

$$s_r(t) = \sqrt{\frac{N}{T}} \cdot \sum_n \sum_{k=0}^{N-1} A_k[n] \cdot \sum_m \xi_k[m-nN] \cdot g(t - mT/N)$$

$$s_r(t) = \sum_n \sum_{k=0}^{N-1} A_k[n] \cdot \hat{\phi}_k(t-nT)$$

- Equivalent continuous time basis functions

$$\hat{\phi}_k(t) = \sqrt{\frac{N}{T}} \cdot \sum_m \xi_k[m] \cdot g(t - mT/N)$$

$$= \frac{1}{\sqrt{T}} \cdot \sum_{m=0}^{N-1} e^{j\frac{2\pi k}{N} \cdot m} \cdot \text{sinc}\left(\frac{N}{T} \cdot (t - mT/N)\right)$$

Orthonormality of equivalent basis functions

$$\langle \hat{\phi}_k, \hat{\phi}_\ell \rangle = \int_{-\infty}^{\infty} \hat{\phi}_k(t) \cdot \hat{\phi}_\ell^*(t) dt$$

$$= \frac{1}{T} \sum_{m=0}^{N-1} \sum_{i=0}^{N-1} e^{j\frac{2\pi k}{N} \cdot m} e^{-j\frac{2\pi \ell}{N} \cdot i} \cdot \int_{-\infty}^{\infty} g(t - mT/N) \cdot g(t - iT/N) dt$$

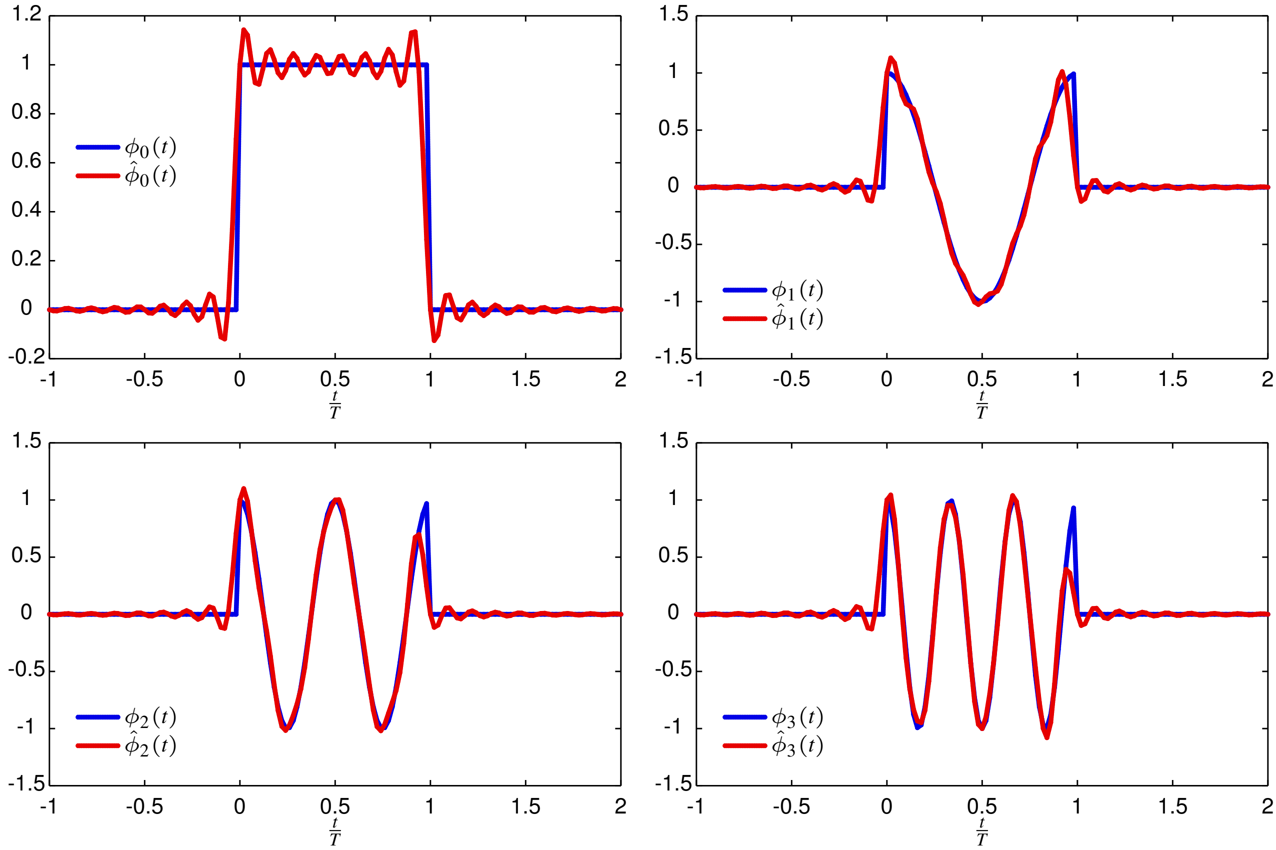
$$\int_{-\infty}^{\infty} g(\tau - mT/N) \cdot g(\tau - iT/N) d\tau = (g(t) * g(-t))|_{t=(m-i)T/N}$$

Since $g(t)$ fulfills Nyquist criterion for ISI

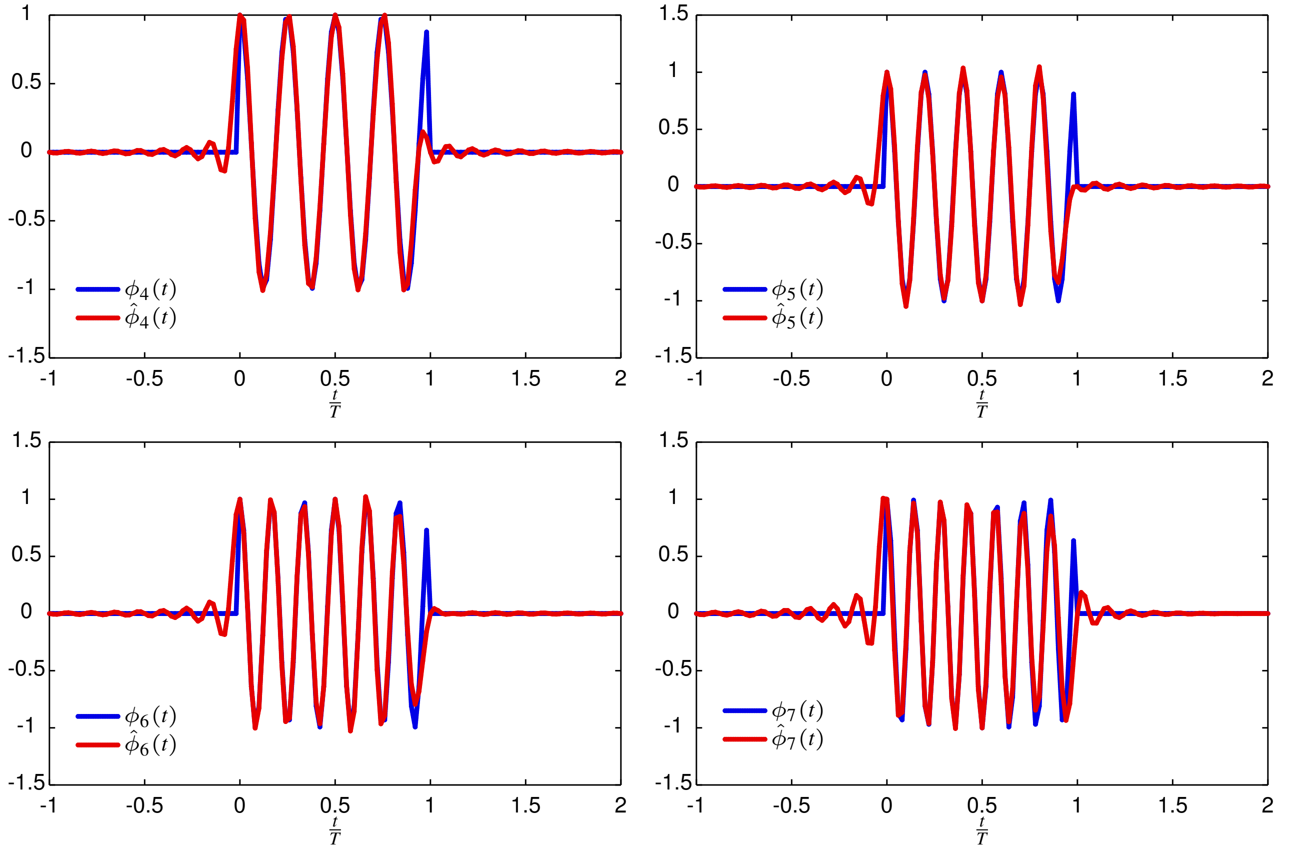
$$\int_{-\infty}^{\infty} g(\tau - mT/N) \cdot g(\tau - iT/N) d\tau = \frac{T}{N} \cdot \delta[m-i]$$

$$\langle \hat{\phi}_k, \hat{\phi}_\ell \rangle = \frac{1}{N} \sum_{m=0}^{N-1} e^{j\frac{2\pi(k-\ell)}{N} \cdot m} = \delta[k-\ell]$$

Comparison with continuous time basis functions - Real part



Comparison with continuous time basis functions - Real part



Spectrum of discrete time OFDM

- Power spectral density can be written in terms of the frequency response of equivalent continuous time basis functions

$$S_{sr}(j\omega) = \frac{1}{T} \cdot \sum_{k=0}^{N-1} E_{s,k} \cdot \left|\hat{\Phi}_k(j\omega)\right|^2$$

- Frequency response of discrete time basis functions

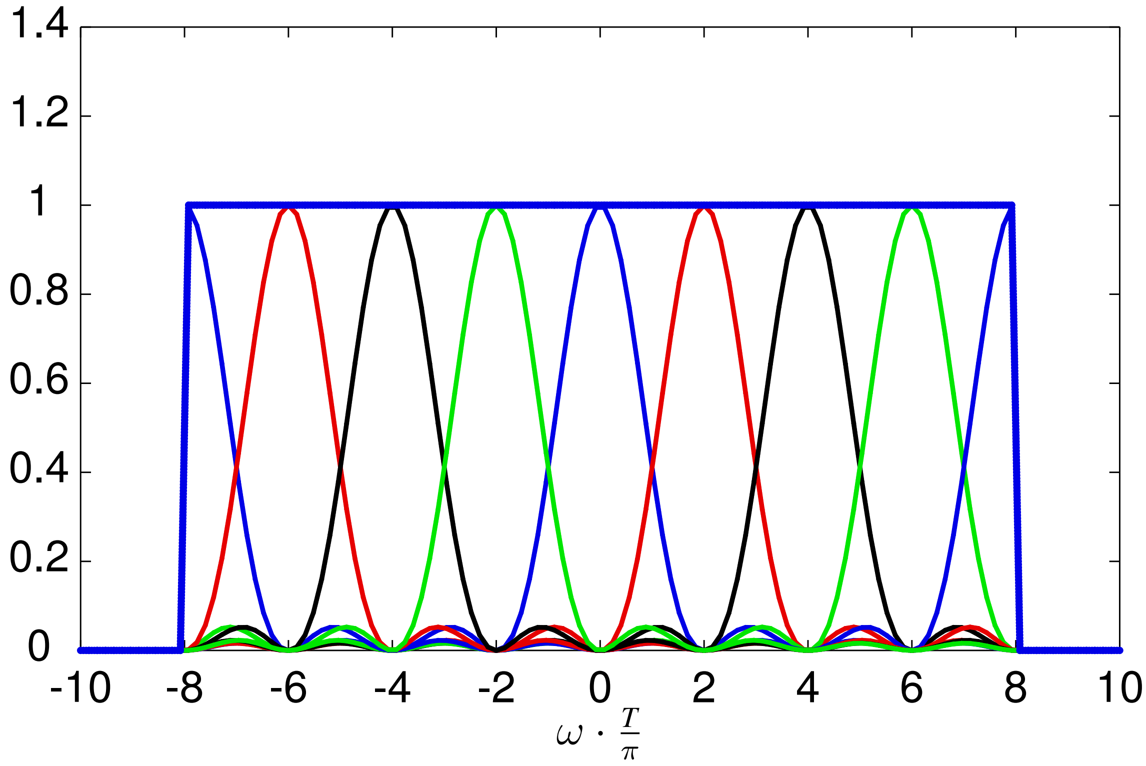
$$\left|\Xi_k\left(e^{j\omega}\right)\right|^2 = \frac{1}{N}\frac{\text{sen}^2[(\omega - 2\pi k/N)N/2]}{\text{sen}^2[(\omega - 2\pi k/N)/2]}$$

- Frequency response of equivalent continuous time basis functions

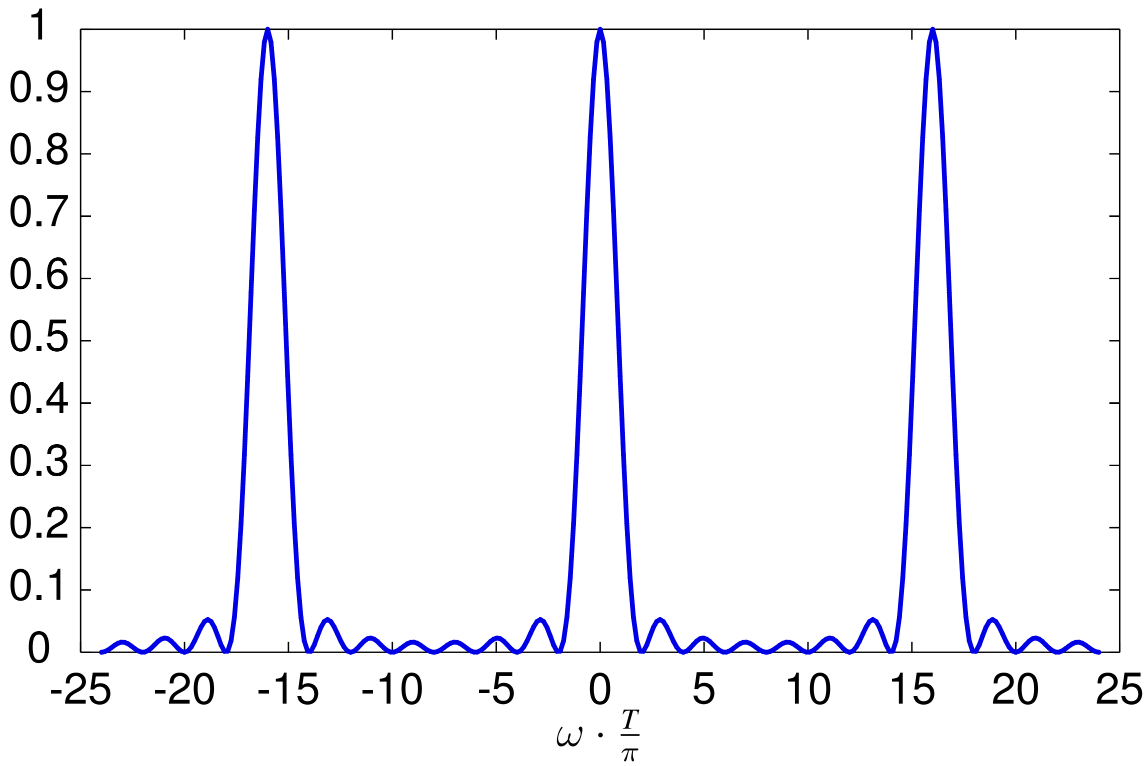
$$\left|\hat{\Phi}_k(j\omega)\right|^2 = \frac{N}{T} \cdot \left|\Xi_k\left(e^{j\omega\frac{T}{N}}\right)\right|^2 \cdot \left(\frac{T}{N}\right)^2 \cdot \Pi\left(\frac{\omega T}{2\pi N}\right)$$

$$\left|\hat{\Phi}_k(j\omega)\right|^2 = \frac{T}{N^2} \cdot \frac{\text{sen}^2[(\omega - 2\pi k/T)T/2]}{\text{sen}^2[(\omega - 2\pi k/T)T/2N]}, \qquad |\omega| < \frac{\pi}{T} \cdot N$$

Spectrum of discrete time OFDM - $N = 8$



Spectrum of discrete time OFDM - Periodicity of $\left|\Xi_k\left(e^{j\omega T}\right)\right|^2$



Receiver for discrete time OFDM

- Analytic expression for the output of samplers

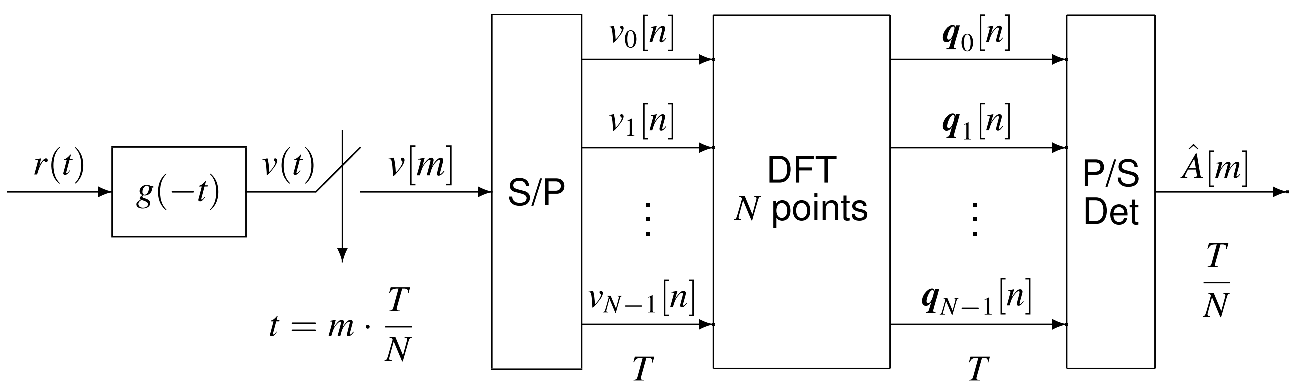
$$q_k[n] = \left(v(t) * \hat{\phi}_k^*(-t)\right)\Big|_{t=nT}$$
$$= \sqrt{\frac{N}{T}} \cdot \sum_m \xi_k^*[m] \cdot \left(v(t) * g\left(-t - m\frac{T}{N}\right)\right)\Big|_{t=nT}$$
$$= \sqrt{\frac{1}{T}} \cdot \sum_{m=0}^{N-1} e^{-j\frac{2\pi k}{N}\cdot m} \left(v(t) * g(-t)\right)\big|_{t=nT+m\frac{T}{N}}$$

- By definig sequence $v[m] = v(t) * g(-t)\big|_{t=m\frac{T}{N}}$, now $q_k[n]$ is

$$q_k[n] = \frac{1}{\sqrt{T}} \sum_{m=0}^{N-1} e^{-j\frac{2\pi k}{N}\cdot m} \cdot v[nN+m]$$

- For each discrete instant n, the N observations $q_k[n]$, $k = 0, 1, \cdots, N-1$ are given by the DFT of a block of N samples of signal $v[m]$

Receiver for discrete time OFDM (II)



Noise at the receiver

- Receiver filter for carrier of index k: $\sqrt{2} \cdot f_k(t)$
- Spectral power density of noise sequence at that carrier

$$S_{z,k}(e^{j\omega}) = \frac{2}{T} \cdot \sum_i S_n\left(j\frac{\omega}{T} - j\frac{\omega_c}{T} - j\frac{2\pi i}{T}\right) \cdot \left|F_k\left(j\frac{\omega}{T} - j\frac{2\pi i}{T}\right)\right|^2$$

- $f_k(t) = \phi_k^*(-t)$
- $n(t)$: white, Gaussian, stationary $S_n(j\omega) = N_0/2$
  - $z_k[n]$ white, circularly symmetric

$$\sigma_{z,k}^2 = N_0, \; k = 0, \cdots, N-1$$

  - A consequence of orthogonality for pulses of each subchannel

$$E\{z_i[n] \cdot z_k^*[n]\} = 0, \text{ if } i \neq k$$

Equivalent baseband discrete channel

- Output of the matched filter (before sampling)

$$q_k(t) = \sum_{i=0}^{N-1} \sum_{\ell} A_i[\ell] \cdot p_{k,i}(t - \ell T) + z_k(t), \qquad k = 0, \cdots, N-1$$

- Joint response of i-th transmitter, k-th receiver and equivalent baseband channel

$$p_{k,i}(t) = \phi_i(t) * h_{eq}(t) * f_k(t)$$

- Sampled output

$$q_k[n] = \sum_{i=0}^{N-1} \sum_{\ell} A_i[\ell] \cdot p_{k,i}[n - \ell] + z_k[n], \qquad k = 0, \cdots, N-1$$

$$q_k[n] = \sum_{i=0}^{N-1} A_i[n] * p_{k,i}[n] + z_k[n], \qquad k = 0, \cdots, N-1$$

  - N^2 equivalent discrete channels are defined

$$p_{k,i}[n], \; i \in \{0, 1, \cdots, N-1\}, \; k \in \{0, 1, \cdots, N-1\}$$

connecting all N inputs (index i) with all N outputs (index k)

Generalization of Nyquist ISI criterion

- Condition for avoiding intersymbol interference (ISI)

$$p_{i,i}[n] = K \cdot \delta[n]$$

- Condition for avoiding intercarrier interference (ICI)

$$p_{k,i}[n] = 0, \text{ for } k \neq i, \ \forall n$$

- Generalization of Nyquist ISI criterion in frequency domain

$$\boldsymbol{P}(e^{j\omega}) = \boldsymbol{I}_{N\times N}$$

$P_{k,i}(e^{j\omega})$: Fourier transform of $p_{k,i}[n]$
$\boldsymbol{P}(e^{j\omega})$: matrix with elements $P_{k,i}(e^{j\omega})$ (row k, column i)

  - Difficult to fulfill all constraints: N^2 constraints, N degrees of freedom

Extension to discrete time OFDM

- Joint input-channel-output responses

$$p_{k,i}(t) = \frac{N}{T}\sum_m \sum_\ell \xi_i[m] \cdot \xi_k^*[\ell] \cdot \left(g(t - mT/N) * h_{eq}(t) * g(-t - \ell T/N)\right)$$

- Equivalent discrete channels are

$$p_{k,i}[n] = \frac{1}{T}\sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} e^{j\frac{2\pi i}{N}\cdot m} \cdot e^{-j\frac{2\pi k}{N}\cdot \ell} \cdot d[nN + \ell - m]$$

$d[m]$: samples of joint response of reconstruction filter, baseband equivalent channel and receiver (matched) filter at $\frac{T}{N}$

$$d[m] = \left(g(t) * h_{eq}(t) * g(-t)\right)\big|_{t=m\frac{T}{N}}$$

REMARK: with this definition $v[m] = s[m] * d[m] + z[m]$

- Conditions of generalized Nyquist ISI criterion are fulfilled if

$$d[m] = K \cdot \delta[m]$$

Modifications to eliminate ISI and ICI

- Assumption: response $d[m]$ is causal and with finite length $K_d + 1$
  - Channel $d[m]$ has a memory of K_d lags
- New discrete time basis functions (length is extended C samples)

$$\tilde{\xi}_k[m] = \frac{1}{\sqrt{N}} \cdot e^{j\frac{2\pi k}{N}\cdot m} \cdot w_{N+C}[m+C], \qquad k = 0, \cdots, N-1$$

  - Non null values for $m \in [-C, N-1]$ (instead of $m \in [0, N-1]$)
  - Constraint to fully eliminate ISI and ICI:

$$C \geq K_d$$

- Samples of the signal to be generated are now given by

$$\tilde{s}[m] = \sqrt{\frac{N}{T}} \cdot \sum_n \sum_{k=0}^{N-1} A_k[n] \cdot \tilde{\xi}_k[m - n(N+C)]$$

- Signal at the demodulator are obtaned as follows

$$q_k[n] = \frac{1}{\sqrt{T}} \cdot \sum_{m=0}^{N-1} \xi_k^*[m] \cdot v[n(N+C)+m]$$

New equivalent discrete channels

- With the introduced modification, channels are now

$$\begin{aligned}
p_{k,i}[n] =& \frac{1}{T} \cdot \sum_{m=-C}^{N-1} \sum_{\ell=0}^{N-1} e^{j\frac{2\pi i}{N}\cdot m} \cdot e^{-j\frac{2\pi k}{N}\cdot \ell} \cdot d[n(N+C)+\ell-m] \\
=& \frac{1}{T} \cdot \sum_{\ell=0}^{N-1} \sum_{u=\ell-N+1}^{\ell+C} e^{-j\frac{2\pi i}{N}\cdot u} \cdot e^{j\frac{2\pi(i-k)}{N}\cdot \ell} \cdot d[n(N+C)+u] \\
=& \frac{1}{T} \cdot \sum_{u=0}^{K_d} e^{-j\frac{2\pi i}{N}\cdot u} \cdot d[u] \cdot \delta[n] \cdot \sum_{\ell=0}^{N-1} e^{j\frac{2\pi(i-k)}{N}\cdot \ell} \\
=& \frac{N}{T} \cdot \delta[n] \cdot \delta[k-i] \cdot \underbrace{\sum_{u=0}^{K_d} e^{-j\frac{2\pi i}{N}\cdot u} \cdot d[u]}_{\text{DFT of } d[m]} = \frac{N}{T} \cdot \delta[n] \cdot \delta[k-i] \cdot D[i]
\end{aligned}$$

$D[k]$: coefficient of index k of the N points DFT for $d[m]$

- ISI and ICI are fully eliminated

Equivalent channels with cyclic extension

- With cyclic extension such that $C \geq K_d$ channels are now

$$p_{k,i}[n] = \frac{N}{T} \cdot \delta[n] \cdot \delta[k-i] \cdot D[k]$$

- ISI and ICI are fully eliminated
- Observation for carrier of index k, $q_k[n]$, are now given by

$$q_k[n] = \frac{N}{T} \cdot A_k[n] \cdot D[k] + z_k[n]$$

$D[k]$: coefficient of index k of the N points DFT for $d[n]$

  - Different signal to noise ratio for each carrier (gain factor $D[k]$)

OFDM seem as a process in blocks

- Samples are processed in blocks
- Samples for n-th block

$s^{(n)}[m]$ for $m = 0, \cdots, N-1$ are given by the N values of $\text{IDFT}_N\left(\{A_k[n]\}_{k=0}^{N-1}\right)$

- Cyclic extension of samples at each block

$$\tilde{s}^{(n)}[m] = \begin{cases} s^{(n)}[m+N] & m = -C, \cdots, -1 \\ s^{(n)}[m] & m = 0, \cdots, N-1 \end{cases}$$

- Transmission through $d[m]$

$$\tilde{v}^{(n)}[m] = \tilde{s}^{(n)}[m] * d[m] + \tilde{z}^{(n)}[m]$$

- Elimination of the cyclic extension

$$v^{(n)}[m] = \tilde{v}^{(n)}[m] \cdot w_N[m]$$

- Demodulation

$q_k[n]$ for $k = 0, \cdots, N-1$ are given by the N values of $\text{DFT}_N\left(\left\{v^{(n)}[m]\right\}_{m=0}^{N-1}\right)$

OFDM seem as a process in blocks (II)

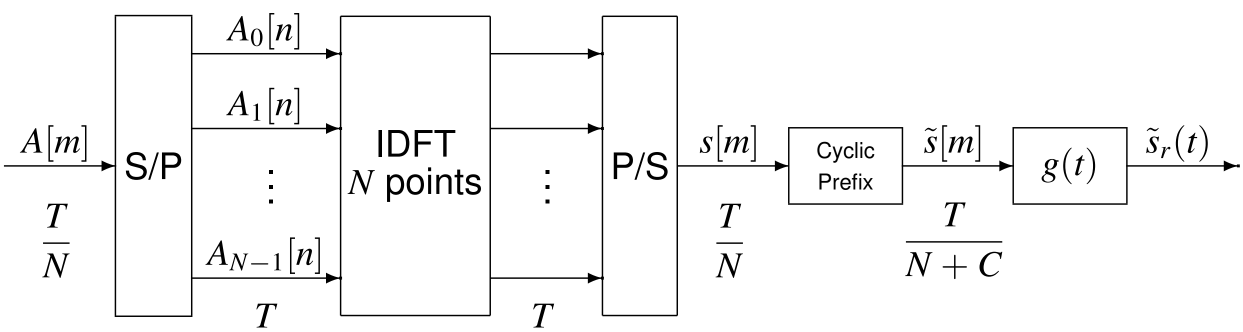
- $q_k[n]$: is obtained from the DFT of N points of $v^{(n)}[m]$
- Cyclic extension is introduced to simulate a circular convolution
- The linear convolution of $\tilde{s}^{(n)}[m]$ with $d[m]$ is equivalent to the circular convolution of $s^{(n)}[m]$ and $d[m]$
- This is usefull because of the property of DFT of being multiplicative under circular convolution

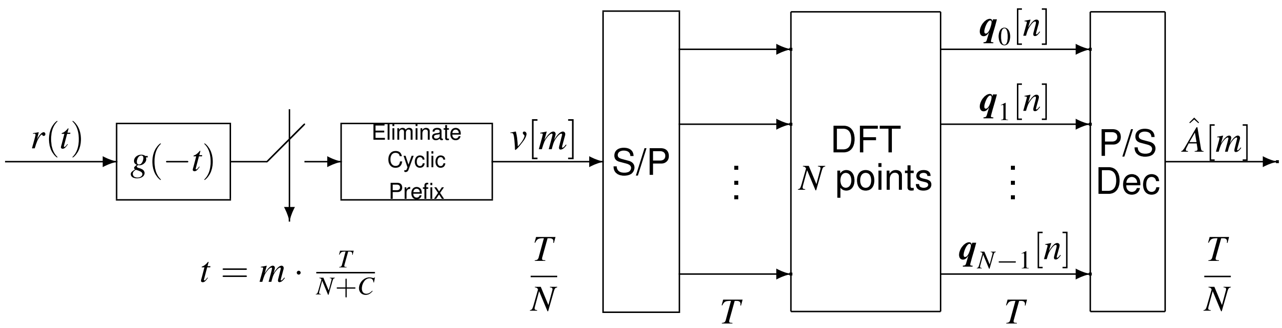
$$\text{If } z[n] = x[n] \circledast y[n] \text{ then } \text{DFT}_N(z[n]) = \text{DFT}_N(x[n]) \times \text{DFT}_N(y[n])$$

- Taking this into account, whitout noise, and abusing of notation

$$\begin{aligned} q_k[n] =& \text{DFT}_N(\tilde{s}^{(n)}[m] * d[m]) \\ =& \text{DFT}_N(s^{(n)}[m] \circledast d[m]) \\ =& \text{DFT}_N(s^{(n)}[m]) \times \text{DFT}_N(d[m]) \\ =& \text{DFT}_N(\text{IDFT}_N(A_k[n])) \times \text{DFT}_N(d[m]) \\ =& A_k[n] \times D[k] \end{aligned}$$

Modulator/demodulator for OFDM with cyclic prefix





Spectral efficiency of OFDM with cyclic prefix

- OFDM signal is constructed from samples with interpolation filter $g(t)$

$$s_r(t) = \sum_m s[m] \cdot g(t - mT_s), \text{ with } g(t) = \text{sinc}\left(\frac{N}{T_s} t\right)$$

T_s: sampling period associated to samples $s[m]$
  - Bandwidth of corresponding bandpass modulated signal $x(t)$ is

$$W = \frac{2\pi}{T_s} \text{ rad/s}, \; B = \frac{1}{T_s} \text{ Hz}$$

- OFDM without cyclic prefix
  - In this case samples are interpolated at $T_s = \frac{T}{N}$

$$W = \frac{2\pi}{T} \times N \text{ rad/s}, \; B = R_s \times N \text{ Hz}$$

- OFDM with cyclic prefix
  - In this case samples are interpolated at $T_s = \frac{T}{N+C}$

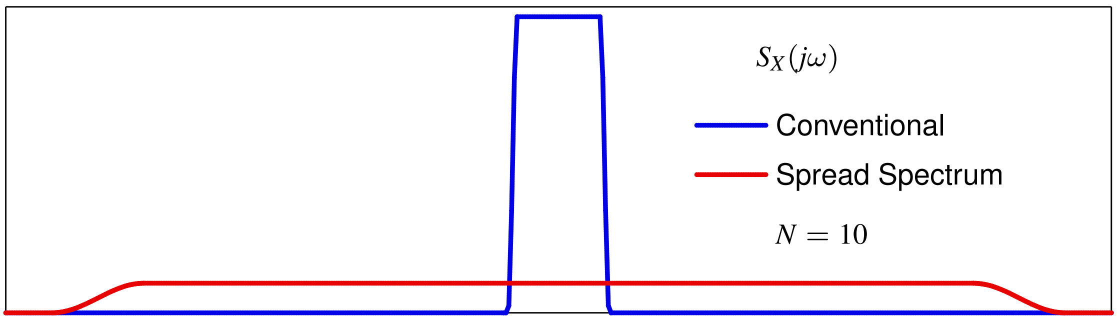
$$W = \frac{2\pi}{T} \times (N+C) \text{ rad/s}, \; B = R_s \times (N+C) \text{ Hz}$$

- Efficiency of OFDM using cyclic prefix of lenght C

$$\eta = \frac{N}{N+C}$$

Spread spectrum modulations

- Myth: spread spectrum modulation increases capacity
  - Reality:
    - ★ Provides low sensibility to channel distortion (including jammers)
    - ★ Allows secure communications
- Bandwidth is deliberately much higher than in conventional modulations
  - Bandwidth is increased by a factor N
    - ★ This allows some degree of inmunity to narrow band interferences / fading



- Origin: to combat intentional interference (jamming) in military systems
  - Current days applications
    - ★ Multiple access
      - CDMA: Code division multiple access
    - ★ Applications requiring robustness against local (in frequency) fadings
    - ★ To limit power flux density in satellite downlinks

Increasing the bandwith of a digital communication signal

- Time and frequency expressions for PAM modulated signals transmitting at symbol rate $R_s = \frac{1}{T}$ bauds

$$s(t) = \sum_n A[n] \cdot g(t - nT), \quad S_s(j\omega) = \frac{1}{T} \cdot S_A(e^{j\omega T}) \cdot |G(j\omega)|^2$$

  - Bandwith using root-raised cosine filters at T with roll-off α

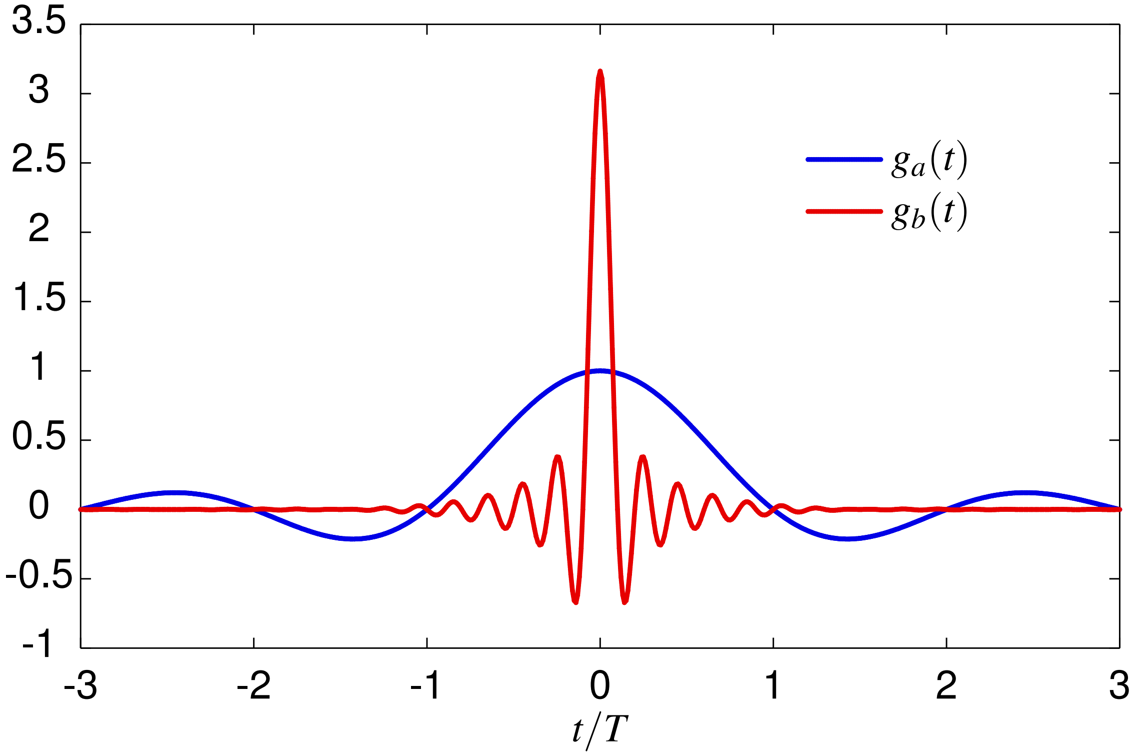
baseband (BB): $W = \frac{\pi}{T}(1+\alpha)$, bandpass (BP): $W = \frac{2\pi}{T}(1+\alpha)$

- Goal: to increase bandwidth by an expansion factor N

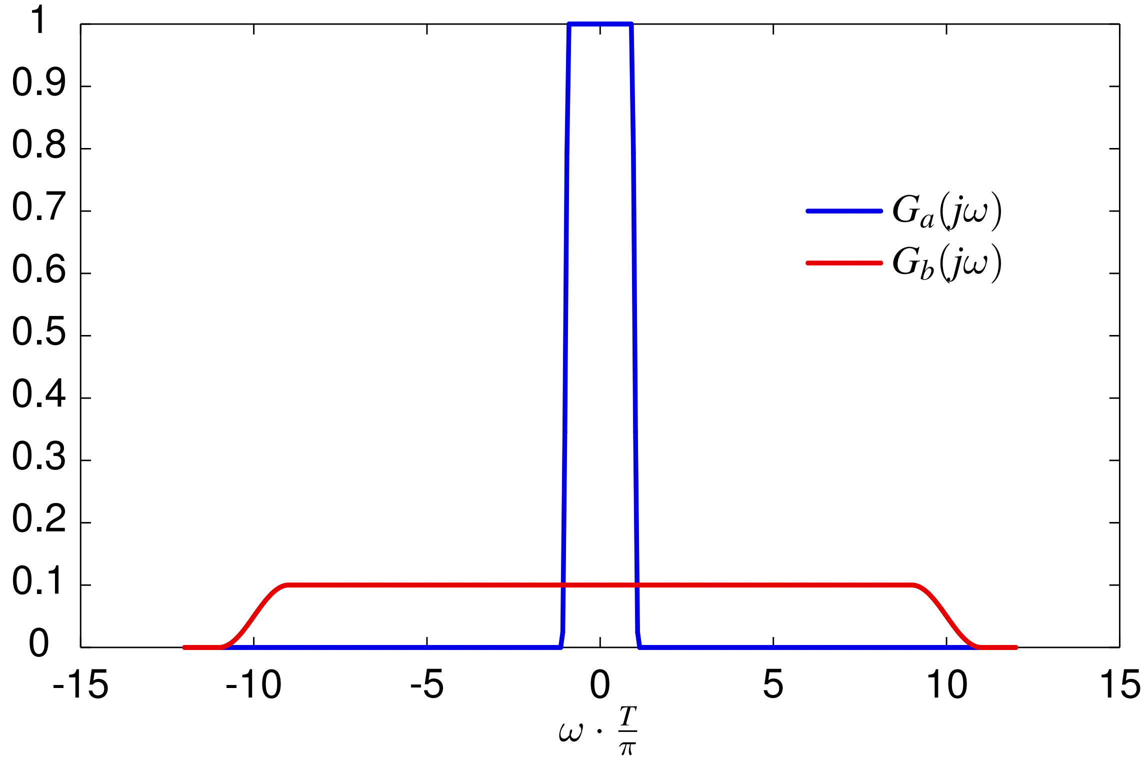
baseband (BB): $W = N \times \frac{\pi}{T}(1+\alpha)$, bandpass (BP): $W = N \times \frac{2\pi}{T}(1+\alpha)$

- Transmission without ISI: root-raised cosine filters
- A possible option: pulses fulfilling Nyquist ISI criterion at T/N
  - If Nyquist criterion is fulfilled at T/N it is also fulfilled at T
  - Bandwidth increases by a factor N
  - Problem: ambiguity function of pulses is localized in time
    $\Rightarrow$ Power of the signal is localized in time

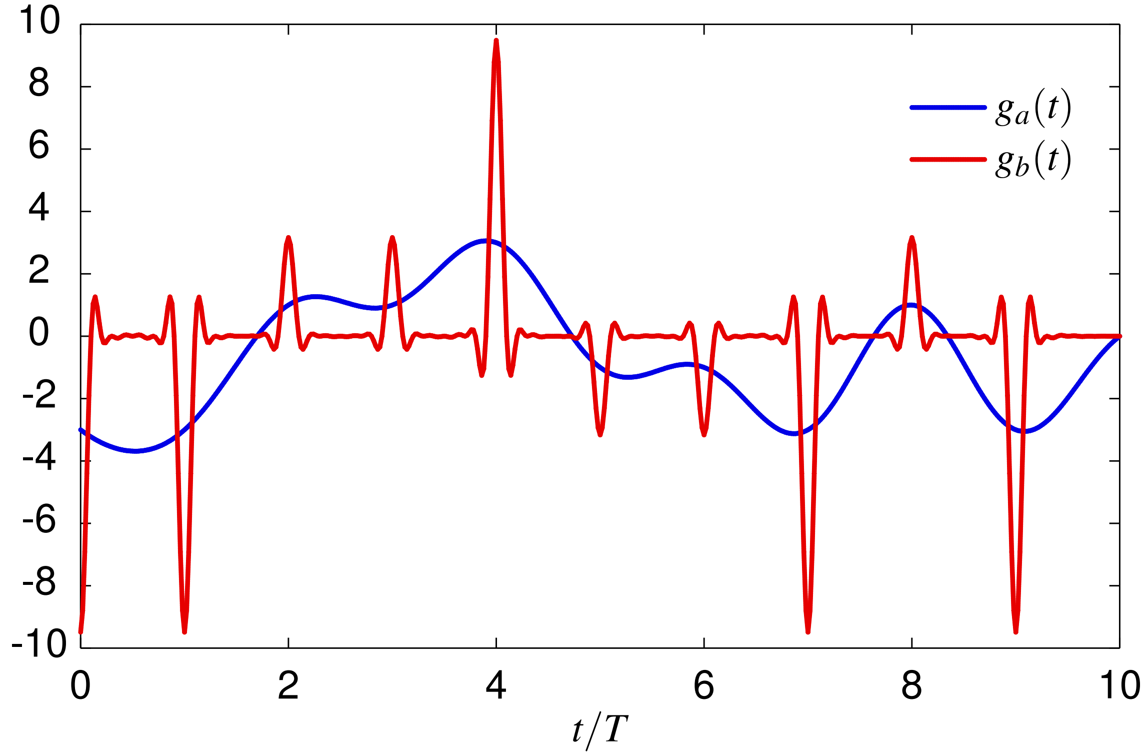
Raised cosine pulses: $g_a(t)$ at T and $g_b(t)$ at $\frac{T}{N}$ ($N = 10, \alpha = 0{,}1$)



Frequency response of pulses at T and $\frac{T}{N}$ ($N = 10, \alpha = 0{,}1$)



Example of waveforms: 4-PAM, $N = 10$, $\alpha = 0{,}5$



Direct sequence spread spectrum (DS-SS)

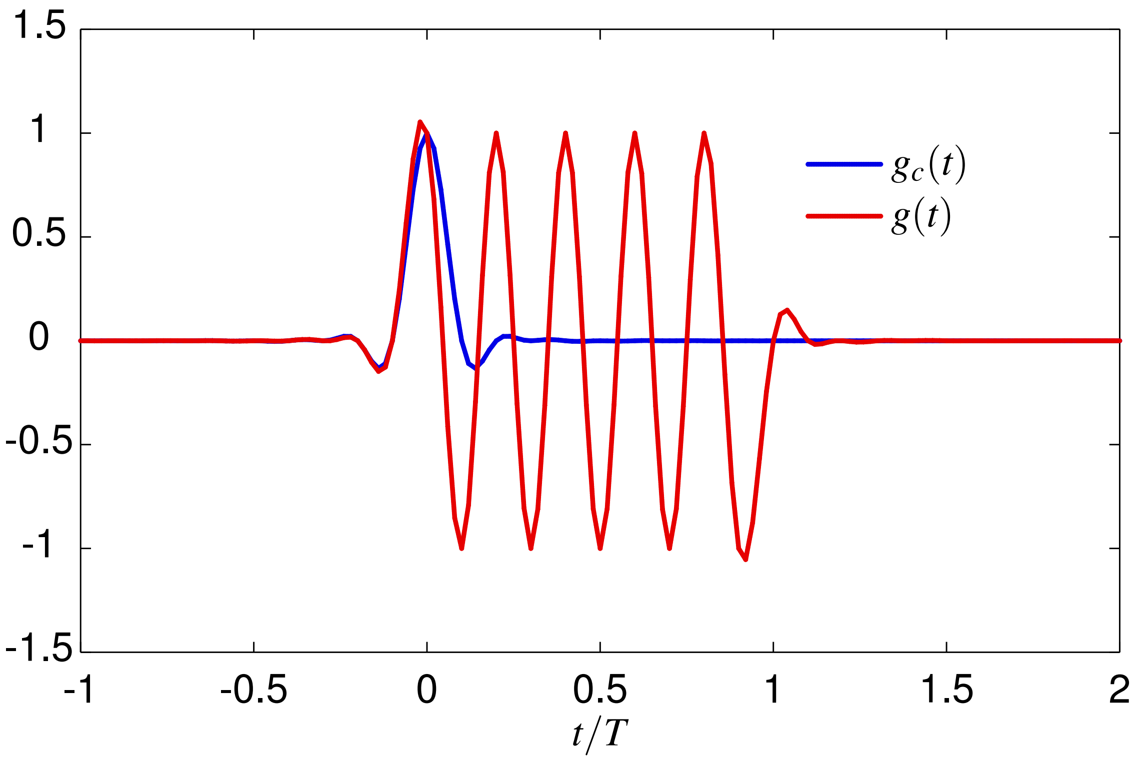
- This method is an alternative that avoids to localize power in time
- Family of pulses

$$g(t) = \sum_{m=0}^{N-1} x[m] \cdot g_c(t - mT_c)$$

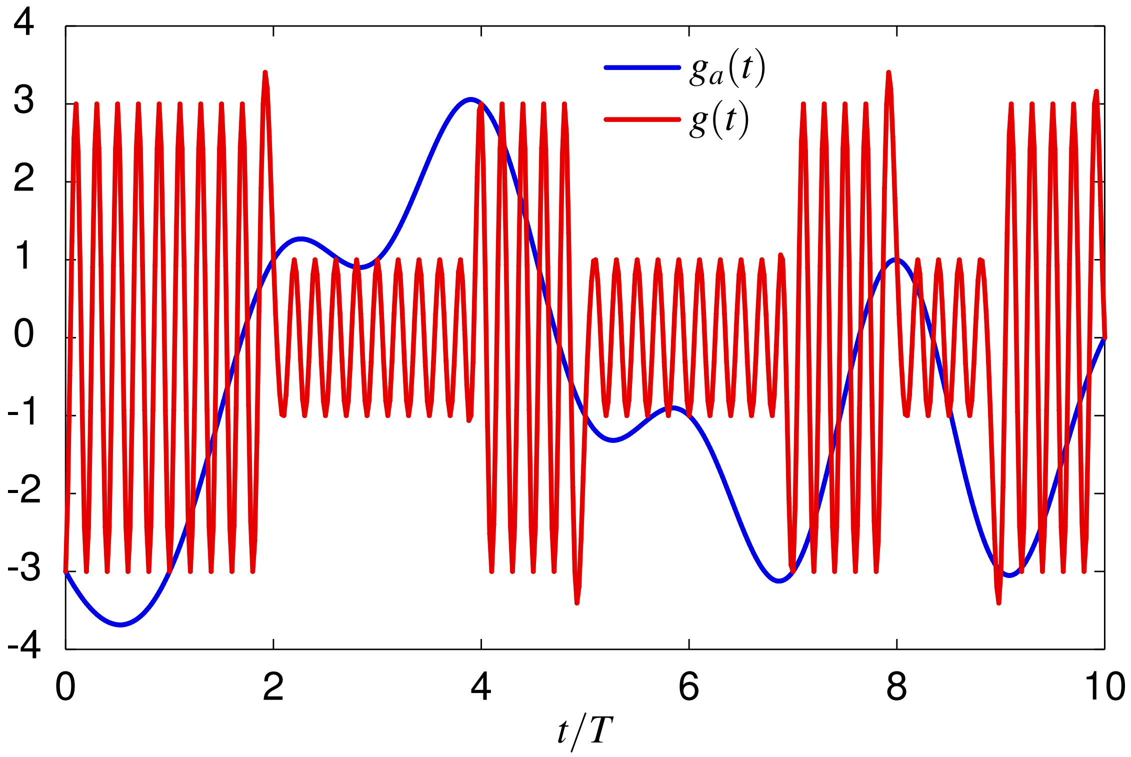
  - $x[m]$: spreading sequence (*chip* sequence)
  - T_c: *chip* period $T_c = \frac{T}{N}$
  - $g_c(t)$: pulse with anbiguity function fulfilling Nyquist at T_c
- The analytic expression for the modulated signal is

$$s(t) = \sum_{n} A[n] \cdot \sum_{\ell=0}^{N-1} x[\ell] \cdot g_c(t - \ell T_c - nT)$$

Example using raised cosine pulses $N = 10$, $\alpha = 0{,}5$



Example of waveform using raised cosine pulses $N = 10$, $\alpha = 0{,}5$



Direct sequence spread spectrum - Alternative notation

$$s(t) = \sum_n A[n] \sum_{m=nN}^{nN+N-1} x[m-nN] \cdot g_c(t-mT_c)$$
$$= \sum_n A[n] \sum_m \tilde{x}[m] \cdot w_N[m-nN] \cdot g_c(t-mT_c)$$

- Periodic sequence $\tilde{x}[m]$ is defined from the spreading sequence
$$\tilde{x}[m] = \sum_k x[m-kN]$$
- Signal $s(t)$ can be generated modulating with $g_c(t)$ the discrete sequence
$$s[m] = \tilde{x}[m] \cdot \sum_n A[n] \cdot w_N[m-nN]$$

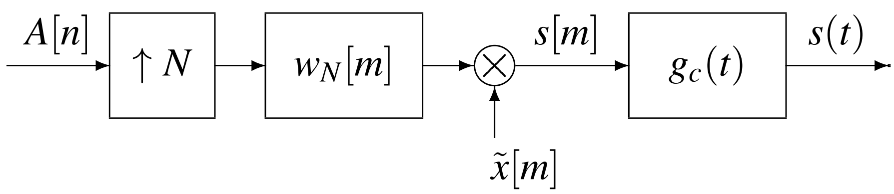
DS-SS transmitter - Block diagram

DS-SS: *Direct Sequence Spread Spectrum*

- Samples to be modulated

$$s[m] = \tilde{x}[m] \cdot \sum_n A[n] \cdot w_N[m - nN]$$

- Block diagram for the transmitter



Spectrum of a DS-SS signal

- Power spectral density of baseband signal $s(t)$

$$S_s(j\omega) = \frac{1}{T} \cdot S_A\left(e^{j\omega T}\right) \cdot |G(j\omega)|^2$$

- Frequency response of pulse $g(t)$

$$G(j\omega) = G_c(j\omega) \cdot \sum_{m=0}^{N-1} x[m] \cdot e^{-j\omega m T_c}$$

- Power spectral density of the DS-SS signal

$$S_s(j\omega) = \frac{1}{T} \cdot S_A\left(e^{j\omega T}\right) \cdot \left|X\left(e^{j\omega T_c}\right)\right|^2 \cdot |G_c(j\omega)|^2$$

Baseband receiver

- Assuming that the receiver filter is $f(t) = g^*(-t)$, and being $v(t)$ the baseband received signal

$$\begin{aligned} q[n] =& (v(t) * g^*(-t))|_{t=nT} \\ =& \sum_{m=0}^{N-1} x^*[m] \cdot (v(t) * g_c^*(-t - mT_c))|_{t=nT} \\ =& \sum_{m=0}^{N-1} x^*[m] \cdot (v(t) * g_c^*(-t))|_{t=nT+mT_c} \end{aligned}$$

- Definition of sequence $v[m]$: sampling $v(t) * g_c^*(-t)$ at chip period

$$v[m] = v(t) * g_c^*(-t)\big|_{t=mT_c=m\frac{T}{N}}$$

- The demodulator output can be written as follows

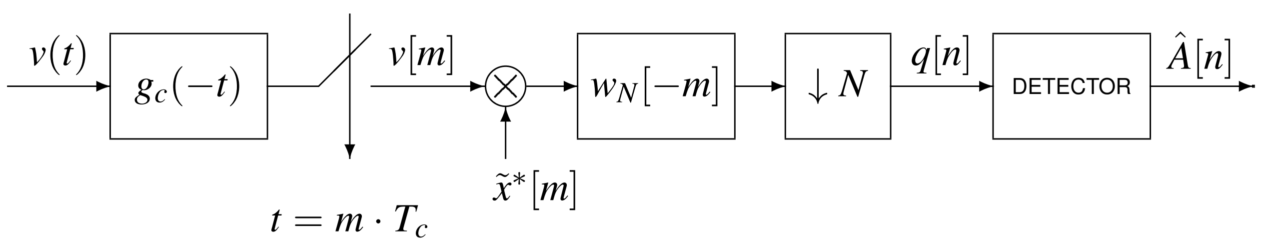
$$q[n] = \sum_{m=0}^{N-1} x^*[m] \cdot v[nN + m]$$

Baseband receiver - Block diagram

- Expression for demodulator output was

$$\begin{aligned} q[n] =& \sum_{m=0}^{N-1} x^*[m] \cdot v[nN + m] \\ =& (v[m] \cdot \tilde{x}^*[m]) * w_N[-m - nN] \end{aligned}$$

- Block diagram for the receiver



Characteristic of noise at the receiver

- Channel introduces additive Gaussian noise $n(t)$ with PSD $S_n(j\omega)$
- If the following conditions happen
  - $f(t) = g^*(-t)$
  - $n(t)$ is white and $S_n(j\omega) = N_0/2$ W/Hz

  $z[n]$ is white, circularly symmetric and with variance

$$\sigma_z^2 = N_0$$

Equivalent discrete channel

- The equivalent discrete channel is

$$p[n] = (g(t) * h_{eq}(t) * f(t))\big|_{t=nT}$$

- Receiver filter is $f(t) = g^*(-t)$

$$\begin{aligned} p[n] &= \sum_{\ell=0}^{N-1} \sum_{m=0}^{N-1} x[\ell] \cdot x^*[m] \cdot (g_c(t-\ell T_c) * h_{eq}(t) * g_c(-t-mT_c))_{t=nT} \\ &= \sum_{\ell=0}^{N-1} \sum_{m=0}^{N-1} x[\ell] \cdot x^*[m] \cdot d[nN+m-\ell] \end{aligned}$$

- Baseband equivalent discrete channel at chip period is

$$d[m] = (g_c(t) * h_{eq}(t) * g_c(-t)))\,\big|_{t=mT_c}$$

Conditions to eliminate ISI

- From previous expression, ISI is avoided if

$$\sum_{m=0}^{N-1} x[m+k] \cdot x^*[m] = \delta[k]$$

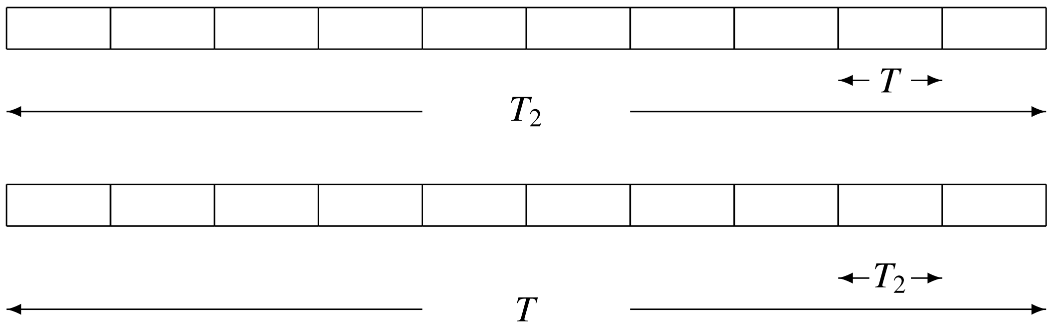
which is fulfilled if

  - the ambiguity function of $x[m]$ is a delta function
  - $\left|X(e^{j\omega T_c})\right|^2$ is constant

- Examples of sequence with (almost) flat spectrum
  - $x[m] = e^{j\theta} \cdot \delta[m-k]$ (problem: time localization)
  - Pseudo-noise sequences

Frequency hopping spread spectrum (FH-SS)

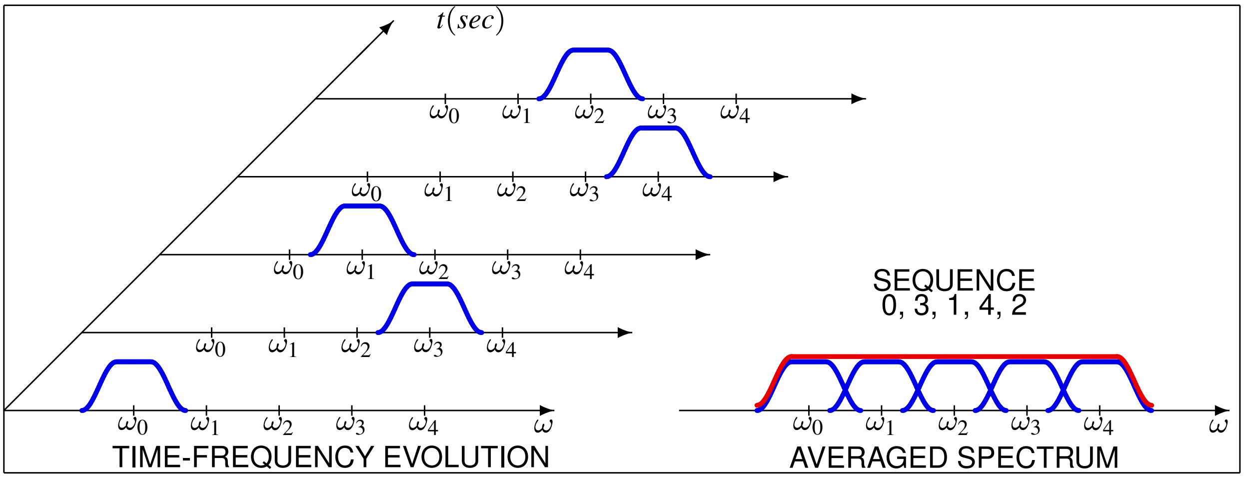
- Designed to work in channels with attenuation "valleys"
  - Idea: to alternate good and bad portions of the spectrum
    - ★ Carrier frequency changes periodically
    - ★ Period for "hopping" in carrier frequencies: T_2
- Clasification
  - Slow frequency hopping: $T_2/T = N \in \mathbb{Z} > 1$
  - Fast frequency hopping: $T/T_2 = N \in \mathbb{Z} > 1$



- It can be implemented using a variety of basic modulations
  - An example: Continuoud phase FSK (CPFSK)

FH-SS spectrum - cyclic time evolution

- Frequency hops are guided by the spreading sequence
  - Pseudorandom sequence defining the order of carriers in the hopping
  - Has to be known by both transmitter and receiver
- Example using 5 carriers



Expressions using CPFSK modulations

- M-ary CPFSK signal: $I[n] \in \{\pm 1, \pm 3, \cdots, \pm(M-1)\}$

$$x(t) = \sqrt{\frac{2E_s}{T}} \cdot \sum_n \text{sen}\left(\omega_c t + I[n]\frac{\pi}{T}t\right) \cdot w_T(t-nT)$$

- Slow frequency hopping signal

$$x(t) = \sqrt{\frac{2E_s}{T}} \cdot \sum_m \sum_{n=0}^{T_2/T-1} \text{sen}\left(\omega_c t + x[m]\frac{\pi}{T}t + I[n+mN]\frac{\pi}{T}t\right) \cdot w_T(t-nT-mT_2)$$

- Fast frequency hopping signal

$$x(t) = \sqrt{\frac{2E_s}{T}} \cdot \sum_n \sum_{m=0}^{T/T_2-1} \text{sen}\left(\omega_c t + x[m+nN]\frac{\pi}{T_2}t + I[n]\frac{\pi}{T_2}t\right) \cdot w_{T_2}(t-nT-mT_2)$$

- $x[m]$: deterministic sequence organizing the changes in frequency ($2kM$)

Multiple medium access - CDMA

- One of the applications of spread spectrum is multiple medium access
  - CDMA: Code Division Medium Access
- Each user uses a different spreading sequence
- Conditions to select sequences for different uses are particular for each kind of spread spectrum modulation

CDMA - DS-SS

- Basic modulation parameter are identical for all users
  - $g_c(t)$, T, T_c
- Multiuser signals in CDMA: L users
  - Each user has a different spreading sequence $x_i[m]$
  - The pulses of users at symbol time are given by

$$g_i(t) = \sum_{m=0}^{N-1} x_i[m] \cdot g_c(t - m \cdot T_c)$$

  - Complex baseband signal

$$s(t) = \sum_{i=0}^{L-1} s_i(t)$$

$$s_i(t) = \sum_n A_i[n] \cdot g_i(t - nT) = \sum_n \sum_{m=0}^{N-1} A_i[n] \cdot x_i[m] \cdot g_c(t - mT_c - nT)$$

Condition for orthogonality of the pulses

- Inner product of two different pulses at symbol period is

$$\langle g_i(t), g_j(t)\rangle = \int_{-\infty}^{\infty} g_i(t)\cdot g_j^*(t)\, dt$$
$$= \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} x_i[m]\cdot x_j^*[\ell]\cdot \int_{-\infty}^{\infty} g_c(t-mT_c)\cdot g_c^*(t-\ell T_c)\, dt$$
$$= \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} x_i[m]\cdot x_j^*[\ell]\cdot \left(g_c(t-mT_c) * g_c^*(-t-\ell T_c)\right)_{t=0}$$
$$= \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} x_i[m]\cdot x_j^*[\ell]$$

- Pulses are orthogonal if spreading sequences fulfill the condition

$$\sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} x_i[m]\cdot x_j^*[\ell] = \delta[i-j], \quad i,j\in\{0,1,\cdots,L-1\}$$

- Several kind of sequences are used in practical systems
  - Gold sequences (1967), Kasami code, Welch sequences,...

CDMA - FH-SS

- Different users employ different spreading sequences
  - Sequences can not produce spectral overlapping at any moment
- A simple example with 5 carriers and 2 users