Exercises for chapter 4

1. A digital communication system has the following equivalent discrete channel:

$$p[n] = \delta[n] - \delta[n-1].$$

The power spectral density of noise in the equivalent discrete model is $\sigma_z^2 = \frac{N_o}{2} = 0.01$ and the transmitted modulation is a 2-PAM, $A[n] \in \{\pm 1\}$.

a) Draw the received constellation (without noise) and obtain the probability of error obtained with a symbol-by-symbol memoryless detector.
b) Draw the basic trellis diagram for the ML sequence detector.
c) Obtain the probability of error of the ML sequence detector.
d) Design the linear equalizer based on the ZF criterion and obtain the probability of error of such equalizer.
e) Design the linear equalizer based on the MMSE criterion and obtain the probability of error of such equalizer.
f) Design the linear equalizer based on the MMSE criterion with 3 coefficients and delay $d = 1$, and obtain the probability of error of such equalizer.
g) Compare the performances of the three different designed equalizers.

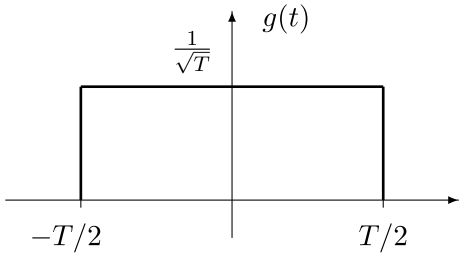
2. A 4-PSK constellation, with equiprobable symbols $A[n] \in \{+1, -1, +j, -j\}$ and $E_s = 1$, is transmitted through the following equivalent discrete channel

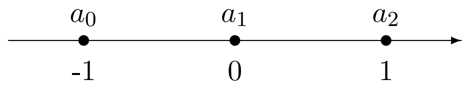
$$p[n] = \delta[n] + j0.8\delta[n-1],$$

with additive Gaussian white noise. The goal is to evaluate the performance of the system with different receivers.

a) If in the receiver there is a memoryless symbol by symbol detector
   i) Obtain the received constellation without noise in the channel.
   ii) Calculate the probability of error P_e.
b) If the receiver is composed by a linear equalizer using the ZF criterion (without constrains in its complexity) and then a symbol by symbol detector
   i) Obtain the transfer function of the equalizer.
   ii) Estimate P_e.
c) If the receiver is a ML (maximum likelihood) detector of sequences:
   i) Obtain the trellis diagram and the minimum distance to an erroneous event.
   ii) Obtain P_e and compare with the probabilities of error previously obtained.

3. A PAM based communication system PAM uses in the transmitter as shaping filter the pulse $g(t) = \frac{1}{\sqrt{T}}\Pi\left(\frac{t}{T}\right)$ shown in the figure The system transmits the three symbols of the constellation shown in the figure with the same probability. Assume a Gaussian channel with power spectral density $N_0/2$ Watts/Hz.





a) Obtain the equivalent discrete channel when the channel is $h(t) = \delta(t) - 0.5 \cdot \delta(t - \frac{T}{2})$, and the receiver uses a matched filter to the transmitter.

b) Assume (also for the remaining parts of this problem) the equivalent discrete channel

$$p[n] = 0.75 \cdot \delta[n] - 0.25 \cdot \delta[n-1].$$

First of all, obtain the optimum memoryless symbol by symbol detector and calculate the probability of error.

c) Draw the trellis diagram associated to the previous equivalent discrete channel and calculate the probability of error using a ML detector of sequences. Compare its performance with the previous receiver.

d) Decode the following sequence

$$\boldsymbol{q} = [0.2 \quad -0.35 \quad -0.35 \quad -0.2].$$

using a a ML detector. Assume that $A[n] = 0$ for $n < 0$ and $n \geq 3$.

NOTE: The Fourier transform of $g(t)$ is $G(j\omega) = \sqrt{T}\text{sinc}\left(\frac{\omega T}{2\pi}\right)$.

4. A communication system is described by the following equivalent discrete channel

$$p[n] = \frac{1}{2}\delta[n] - \delta[n-1] + \frac{1}{2}\delta[n-2].$$

Assume to use a 2-PAM modulation $A[n] \in [\pm 1]$, and that the noise is white, Gaussian with spectral density $N_0/2$ and $N_0 = 2 \cdot 10^{-2}$.

a) Obtain the linear ZF (zero forcing) equalizer without constrains in the complexity and the probability of error.

b) Obtain the linear ZF (zero forcing) equalizer with 2 coefficients and delay $d = 1$.

5. A communication system transmits a 2-PAM modulation, $A[n] = \pm 1$, through the following equivalent discrete channel

$$p[n] = -0.3\delta[n] + 0.8\delta[n-1] - 0.2\delta[n-2]$$

and discrete noise $z[n]$ white, Gaussian and with spectral density $N_0/2$.

a) Assume to use a symbol by symbol detector.

i) Draw the constellation at the output of the channel without noise.

ii) Write the probability density function of $q[n]$ given $A[n] = 1$.

iii) Write the probability density function of $q[n]$ given $A[n] = -1$.

iv) Obtain the probability of error P_e in two different scenarios: first assuming to take decision over $\hat{A}[n]$; second assuming to take decision over $\hat{A}[n-1]$. What is the reason of the different performances in the two scenarios?

b) Using a linear equalizer

i) Obtain the linear ZF equalizer with 3 coefficients with $d=0$ and $d=1$. In order to design the ZF equalizer, first obtain the channel matrix P and then the impulsive response c_d, in both cases ($d=0$ and $d=1$).

ii) If we have a filter with coefficients $\mathbf{w}_{ZF} = [1.15,\ 0.22,\ 0.02]^T$ obtained for $d=1$, find the power of the noise and of the ISI at the output of the equalizer [1].

6. The equivalent discrete channel for the communication system is

$$p[n] = \frac{1}{4}\delta[n] + \delta[n-1] - \frac{1}{4}\delta[n-2].$$

The system uses a 2-PAM constellation $A[n] \in \{\pm 1\}$. The variance of the discrete noise $z[n]$ is $\sigma_z^2 = 0.1$.

a) If we use memoryless symbol by symbol detector, choose the optimum delay d for the decision and calculate the probability of error.

b) Design a linear ZF equalizer without constrains, and calculate the corresponding probability of error.

c) Design the linear ZF and MMSE equalizers with 3 coefficients and delay $d=2$ (write the equation system that you have to solve defining the involved terms, but it is not necessary to obtain the coefficients of the corresponding equalizer).

7. To compare the performances of two different channels, the following sequence is transmitted.

n	-2	-1	0	1	2	3	4	5	6	7
$A[n]$	$+1$	$+1$	$+1$	-1	$+1$	-1	-1	-1	$+1$	$+1$

At the output of each channel without noise, we obtain the following sequences:

n	0	1	2	3	4	5	6	7
$o_1[n]$	$+0.1$	-0.3	$+0.3$	-0.3	-0.1	-0.1	$+0.3$	$+0.1$

n	0	1	2	3	4	5	6	7
$o_2[n]$	$+0.1$	-0.3	$+0.7$	-0.7	$+0.3$	-0.1	$+0.3$	-0.3

Assume also that the first channel $p[n]$ that provides $o_1[n]$ is formed by two samples (i.e., $N=2$) and the second channel $p[n]$ that provides $o_2[n]$ is formed by three samples (i.e., $N=3$).

a) Obtain for both channels the received constellations and determine, for each received symbols, the set of transmitted symbols (i.e., the constellation that really has been transmitted).

b) Using in both channels a memoryless detector with delay $d=0$, determine the channel with worst performance calculating the corresponding probability of error.

c) Obtain the equivalent discrete channels $p_1[n]$ y $p_2[n]$ that have generated the received sequence.

[1] The symbol ^{T} denotes the transposition of a vector or matrix.

d) Obtain the probability of error or both channels using a ML sequence detector.[2] Without constrains for the receptor, which channel has the best performances?

e) Obtain the sequence that has been transmitted with the highest probability if in the channel $p_2[n]$ we received:

n	0	1	2	3	4	5	6	7
$q_2[n]$	-0.5	-0.2	-0.3	-0.7	-0.5	-0.2	$+0.5$	$+0.1$

when $A[n] = +1$ for $n < 0$ and for $n > 5$.

8. A baseband communication system uses as transmitter filter a causal normalized rectangular of duration T sec, and a 2-PAM constellation is transmitted through a linear Gaussian channel with spectral density $N_0/2$, with $N_0 = 0.02$ and the following impulse response

$$h(t) = \delta(t) - 4 \cdot \delta\left(t - \frac{3T}{2}\right) + \frac{5}{2} \cdot \delta\left(t - 2T\right).$$

a) Calculate the equivalent discrete channel $p[n]$, if the receiver uses a matched filter to the transmitter filter.

b) Then, consider that the equivalent discrete channel is

$$p[n] = \delta[n] - 2 \cdot \delta[n-1] + \frac{1}{2} \cdot \delta[n-2].$$

Calculate the probability of error when we use the best memoryless symbol by symbol detector.

c) Design the linear ZF equalizer with 3 coefficients and delay in the decision $d = 1$[3].

d) Design the linear ZF equalizer with 3 coefficients and delay $d = 3$ with the MMSE criterion.

e) If the coefficients of the equalizer are

$$w[0] = -0.2,\ w[1] = -0.6,\ w[2] = -0.1,$$

estimate the optimum delay for the decision and calculate the corresponding probability of error.

9. When we transmit two 2-PAM (also known as BPSK) sequences, partially unknown with length $L = 4$

$$\mathbf{A}_1 = \{1, A_1[1], A_1[2], A_1[3]\} \qquad \mathbf{A}_2 = \{-1, A_2[1], A_2[2], -1\}$$

through a channel of which we know that has length 3

$$p[n] = p[0]\delta[n] + p[1]\delta[n-1] + p[2]\delta[n-2]$$

we get two different sequences in the receiver $o_1[n]$ and $o_2[n]$ when there is no noise. We also assume that $A_i[n] = +1$ for $n < 0$ and $n \geq 4$ and for $i \in \{1, 2\}$.

n	0	1	2	3	4	5
$o_1[n]$	-0.1	-1.1	$+1.3$	-0.3	-0.1	-0.1

n	0	1	2	3	4	5
$o_2[n]$	-1.1	$+0.3$	$+0.1$	$+0.1$	$+1.1$	-0.3

[2]For the channel $p_2[n]$ it is possible to assume that $A[n] = +1$, $\forall n$, is associated to an erroneous event with minimum Euclidean distance.

[3]It is not necessary to solve the equation system, but you have to write the numeric values of all the terms involved in the system.

a) Get the coefficients of the channel that the two sequences went through and the trellis diagram.

b) Get the unknown values of each of the sequences.

c) If you would decide to use a memoryless detector, get the error probability. Assume that $d = 0$ in this case and that the power spectral density of the noise is given by $N_0/2$. [4]

d) Get the ZF equalizer with two coefficients for $d = 0$ and $d = 1$.

e) Get the sequence obtained at the output of the equalizer for $d = 1$, $u[n]$, for $n = 1$ and $n = 2$, when at the input of the equalizer we have $o_1[n]$. Did you fully recover the original sequence $\mathbf{A}_1$? Explain your answer.

SOME INTERESTING RELATIONSHIPS

Inverse of a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \qquad \text{con } D = a \cdot d - b \cdot c$$

Some integrals, for $|a| \geq |b|$ and n being an integer number:

$$\int_{-\pi}^{\pi} \frac{1}{a + b \cdot \cos(n \cdot \omega)} \, d\omega = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

$$\int_{-\pi}^{\pi} \frac{1}{(a + b \cdot \cos(n \cdot \omega))^2} \, d\omega = \frac{2\pi a}{\sqrt{(a^2 - b^2)^3}}$$

[4] If you were unable to solve a), for this item and for following solve depending on $p[0]$, $p[1]$ and $p[2]$.