Exercises for chapter 6

1. The encoding matrix for a linear block code $(4, 8)$ is given next. Compute the coding rate, the minimum distance and the syndrome table.

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

2. A convolutional code has the following generating matrix

$$\mathbf{G}(D) = \left[\begin{array}{ccc} 1 & D & D+1 \\ D^2 & 1 & D^2+D+1 \end{array} \right]$$

   a) Obtain the coding rate of the code.
   b) Plot the schematic representation of the code.
   c) Plor the trellis diagram for the code.
   d) Obtain the minumum Hamming distance, D^H_{min} for the code.

3. A convolutional code with coding rate 1/2 has the following generating matrix:

$$\mathbf{G}(D) = \left[\begin{array}{cc} D^2+1 & D^2+D+1 \end{array} \right].$$

   Transmitter uses a BPSK (or 2-PAM) modulation with normalized levels $A[n] \in \pm 1$. Binary assignment is $B[n] = 0$ for $A[n] = -1$ and $B[n] = 1$ for $A[n] = +1$. The system transmits a cyclic header of 2 zeros between each block of 6 data bits to reset the state of the convolutional. Decode, using the Viterbi algorithm with soft and hard outputs the following received sequence

$$\begin{array}{lcccccccc} q^{(0)}[n]: & +3.06 & -0.70 & -0.58 & -1.37 & -0.82 & -2.63 & -1.37 & -0.85 \\ q^{(1)}[n]: & +1.08 & -1.06 & -2.89 & +0.33 & +1.92 & -1.64 & -0.70 & +2.30 \end{array}$$

   NOTE: For decoding with hard output you need first to get the hard decision on the given sequence.

4. Two channel codes are going to be evaluated in a digital communication system: a linear block code and a convolutional code.

   a) The generating matrix of the linear block code is

$$\mathbf{G} = \left[\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right].$$

      i) Get the code minimum distance.
      ii) Transform $\mathbf{G}$ in a systematic matrix $\mathbf{G}'$ that could be used to obtain a parity check matrix allowing to define a symdrome table for the code.
      iii) Get the parity-check matrix.
      iv) Get the syndrome table.

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b) Next, consider the convolutional code given by the generating matrix

$$\mathbf{G} = \begin{bmatrix} (1+D) & D & 1 & (1+D) \\ D & (1+D) & 1 & 1 \end{bmatrix}.$$

i) Get the shematic representation of the encoder.
ii) Plot the code trellis diagram.
iii) Obtain the code minimum distance, D_{min}.
iv) Get, assuming as starting and ending state the zero state, i.e. $\psi_0 = [0, 0, \cdots, 0]$, the decoded message when the received sequence is

$$\mathbf{r} = [1011000110100110].$$

c) Compare both system performance if the underlying BSC channel has a bit error rate ε.

5. Two channel codes will be analyzed

a) A linear block code has the following generating matrix

$$\boldsymbol{G} = \begin{bmatrix} 0 & 1 & a & 0 & b \\ c & d & 1 & 1 & 1 \end{bmatrix}$$

i) Get a, b, c and d values to obtain the maximum detection and correction capabilities.
ii) Obtain the syndrome table and decode the following received words.

$$\boldsymbol{r}_0 = [10001],\ \boldsymbol{r}_1 = [10011],\ \boldsymbol{r}_2 = [11001]$$

a) A convolutional code has the following generating matrix

$$\boldsymbol{G} = \left[1 + D + D^2,\ 1\right].$$

The information data are transmitted with a 4-QAM modulation with the following binary assignment.

Symbol	$+1+j$	$-1-j$	$+1-j$	$-1+j$
Bits	11	00	10	01

i) Get the schematic representation of the encoder, and the trellis diagram.
ii) Encode the information sequence $B^{(0)}[\ell] = [101100]$ under the assumption that the starting state is the zero state, ψ_0. Plot the path of the output sequence through the trellis diagram.
iii) Get the code performance working both with hard and soft decoding.
iv) Decode the received sequence

$$\boldsymbol{r} = [101001010011],$$

assuming that $B^{(0)}[\ell] = 0$ for $\ell < 0$ and $\ell \geq 4$ (i.e. the initial and final states are ψ_0).

6. Two linear block codes are given by the following generating matrices:

$$\mathbf{G}_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{G}_2 = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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a) Are the codes systematic?

b) Get the error detecting and correcting capabilities.

c) Choose the best code from previous section, get its syndrome table and decode the following received words:

$$\boldsymbol{r}_a = [01101] \text{ and } \boldsymbol{r}_b = [11111].$$

7. A convolutional code has the following generating matrix:

$$\boldsymbol{G}(D) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & D & D+1 \end{bmatrix}$$

a) Obtain the code rate and get the schematic representation of the encoder.

b) Obtain the state diagram, the trellis diagram and get the minimum distance of the code.

c) Assuming the starting state the zero state, encode the following input sequence

$$B^{(0)}[\ell] = 11011000.$$

d) Determine if the following sequence 110010111000 is a possible codeword. Assume any possible starting state.

e) Consider now a simplified version of the previous code with generating matrix[1]:

$$\boldsymbol{G}(D) = \begin{bmatrix} D & D+1 \end{bmatrix}$$

Assuming the starting and ending state the zero state, find the decoded message when the received message is

$$\boldsymbol{r} = [10001111011011].$$

8. For a communication system, three different codes $\mathcal{C}_1$, $\mathcal{C}_2$ y $\mathcal{C}_3$ are defined. The corresponding codewords are the following

$$\begin{aligned} \mathcal{C}_1 &= \{01, 10\}, \\ \mathcal{C}_2 &= \{00000, 01010\}, \\ \mathcal{C}_3 &= \{00000, 10100, 01111, 11011\}. \end{aligned}$$

a) Obtain for each code the parameters k, n, the coding rate and the minimum Hamming distance.

b) Determine which of this codes are linear and for those get the generating matrix.

c) Find the systematic codes among the three different encoders.

d) Is it possible to improve the performance of $\mathcal{C}_2$ without modifying the parameters k, n? explain and discuss which would be the way.

e) If the received sequence is $\boldsymbol{r} = [11111]$, get the word witn maximum likelihood between the set of possible transmitted codeword. Explain the procedure used to obtain the transmitted codeword with highest likelihood.

9. We want to design a communication system with a channel code with rate 1/2. There are two possibilities:

[1]Note that the new code is a simplification of the previous one, obtained removing the first input and the first output.

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- A linear block code with the following generating matrix:

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

- A convolutional code with the following generating matrix:

$$\boldsymbol{G}(D) = [D,\ 1 + D + D^2].$$

In both cases, after the encoder, a 2-PAM (o BPSK) modulation is used, with a distance between symbols in the constellation d_{min}^{BPSK}.

a) Obtain the set of codewords of the block codes, the parity-check matrix and the minimum distance.

b) Compute the probability of error with the block encoder using a "hard" detector. Provide an expression for this probability as function of the minimum distance of the code and the distance in the constellation d_{min}^{BPSK}.

c) Plot the trellis diagram of the convolutional code.

d) Calculate the probability of error with the convolution encoder using both "soft" and "hard" observations, in this case assuming $d_{min}^{BPSK} = 1$.

e) Find the transmitted sequence with the block encoder for the message $\boldsymbol{b} = [100110]$. Then, assuming that the channel produces errors in the first, sixth and ninth bits, obtain the recovered sequence at the output of the decoder.

f) Find the transmitted sequence with the convolutional encoder for the message $\boldsymbol{b} = [100]$. Assume that the starting and ending states are the zero state, and the channel produces an error in the first bit. Obtain the sequence at the output of the decoder using a "hard" decision criterion.

10. A convolutional code has the following generating matrix

$$\boldsymbol{G}(D) = [1,\ 1 + D,\ 1 + D + D^2].$$

a) Obtain the encoder sketch and the Trellis diagram.

b) Discuss if the code is systematic or not.

c) Assume that the received sequence is

$$\boldsymbol{r} = [111\ 111\ 110\ 011\ 001],$$

and assume also the starting and ending states are the zero state, which is forzed by means of the transmission of the appropriate number of zeros. Determine the most likelihood transmitted sequence and the corresponding message.

11. In a digital communication system we decide to improve the error correction capability of a repetition block code with generating matrix $\mathbf{G} = [1\ 1\ 1]$. For that we concatenate the output of the repetition code to the input of a convolutional code with generating matrix

$$\boldsymbol{G} = \begin{bmatrix} 1 & D & 0 & 0 \\ D & 1 & 0 & 0 \\ 0 & 0 & 1 + D & D \end{bmatrix}$$

For the block and convolutional encoders separately answer the following questions:

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a) Get the codewords of the block code and obtain its minimum distance.

b) Get the block diagram of the convolutional code and its trellis diagram. For the trellis diagram it is not necessary to label all the transitions, only label those that emerge from the state $\psi_0 = [000]$ and those that arrive to the same state $\psi_0 = [000]$. What would be the minimum distance of that code?

If we name the codewords at the output of the block encoder at discrete instant ℓ as $\mathbf{c}[\ell] = [c_0[\ell]\ c_1[\ell]\ c_2[\ell]]$ and $\mathbf{B}[\ell] = [B^{(0)}[\ell]\ B^{(1)}[\ell]\ B^{(2)}[\ell]]$ as the input words to the convolutional encoder at discrete instant ℓ, the concatenation is made by assigning $B^{(i)}[\ell] = c_i[\ell]$ for $i = 0 \ldots 2$. Answer the following questions related to the concatenated code.

c) Get the code rate of the concatenated code. Obtain the trellis diagram arising from the concatenation of both codes and its minimum distance.

d) Compare the error correction capabilities of the concatenated code with each of the component encoders separately. Discuss if the concatenated encoder needs more or less bandwidth than the block code to maintain the bit rate. Discuss similarly for the convolutional code.

e) From the trellis diagram obtained for the concatenated code get a block diagram and the generating matrix in D polynomials of a convolutional encoder that behaves exactly as the concatenated code.

12. Two block codes are given with the following assignments between uncoded block bits, $\mathbf{b}_i$, and codewords $\mathbf{c}_i$.

i	$\mathbf{b}_i$	$\mathbf{c}_i$
0	0 0	0 0 0 0 0
1	0 1	1 1 0 1 0
2	1 0	1 0 1 1 1
3	1 1	0 1 1 0 1

Código A

i	$\mathbf{b}_i$	$\mathbf{c}_i$
0	0 0	0 0 0 0 0
1	0 1	1 0 1 0 1
2	1 0	1 1 0 1 0
3	1 1	1 1 1 1 1

Código B

a) For each of the codes:

   i) Explain if the code is linear or not and if it is systematic. Answer should be clearly reasoned.

   ii) Get the number of errors that each code is able to correct.

b) For the linear code (if both are, consider the one most appropriate, clearly justifying your election), get the generating matrix and the parity check matrix.

c) For the same code, get the syndrome table and decode, providing the estimated uncoded bits $\hat{\mathbf{b}}_i$, the following received words.

$$\mathbf{r_a} = 1\ 1\ 1\ 0\ 1,\ \mathbf{r_b} = 1\ 0\ 0\ 1\ 1.$$

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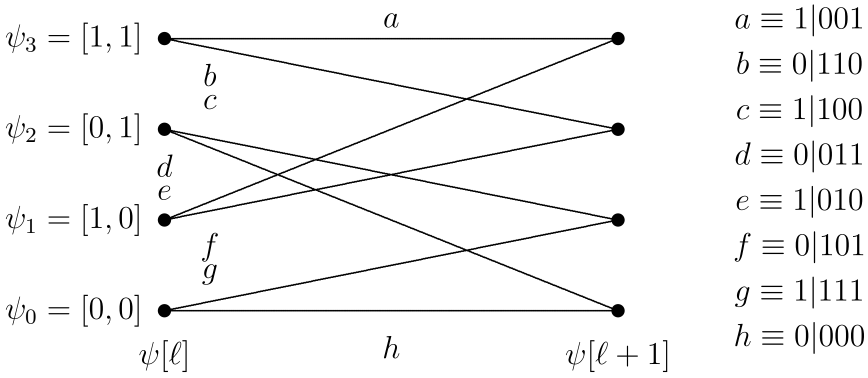
13. A convolutional code, has the following trellis diagram



a) Get the generating matrix and the encoder schematic representation.

b) Assuming that all previously transmitted bits are zero, encode the following symbol sequence

$$B^{(0)}[0] = 0,\ B^{(0)}[1] = 1,\ B^{(0)}[2] = 1,\ B^{(0)}[3] = 0,\ B^{(0)}[4] = 1,$$

and get the approximate error probability if hard decoding is used.

c) Decode the first three symbols, $\hat{B}^{(0)}[\ell]$, $\ell \in \{0,1,2\}$, for the following received sequence, assuming that $B^{(0)}[\ell] = 0$ for $n < 0$ and for $n \geq 3$

$$\mathbf{r} = 010\ 111\ 000\ 111\ 011$$

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