DIGITAL COMMUNICATIONS
SOLUTION OF THE EXAM

Question 1

The available range for frequencies can be written as

$$\Delta W = \omega_b - \omega_a = 2\pi \cdot 300 \cdot 10^6 \text{ rad/s, ó } \Delta B = f_b - f_a = 300 \text{ MHz.}$$

a) The main characteristic of MSK modulation is the minimal separation between consecutive frequencies

$$\Delta\omega_i = \omega_i - \omega_{i-1} = \frac{\pi}{T} = \pi \cdot R_s, \text{ or } \Delta f_i = f_i - f_{i-1} = \frac{R_s}{2}.$$

Therefore, maximum rate will be obtained using the whole band, i.e., making $f_0 = f_a$ and $f_3 = f_b$, which means

$$\Delta f_i = \frac{\Delta B}{3} = 100 \text{ MHz,}$$

and in this case the symbol rate is $R_s = 2 \cdot \Delta f_i = 200$ MHz.

The four frequencies are $f_0 = 950$ MHz, $f_1 = 1050$ MHz, $f_2 = 1150$ MHz, $f_3 = 1250$ MHz.

b) A CPFSK modulation has two differences with respect to a MSK modulation. First, minimum frequency separation for a given symbol rate is double

$$\Delta\omega_i = \omega_i - \omega_{i-1} = \frac{2\pi}{T} = 2\pi \cdot R_s, \text{ ó } \Delta f_i = f_i - f_{i-1} = R_s.$$

Second, to transmit at a given rate, frequencies can not take any value, but they have to produce an integer number con cycles in T seconds, i.e., they have to be

$$\omega_i = \frac{2\pi}{T} N_i = 2\pi \cdot R_s \cdot N_i, \text{ or } f_i = \frac{N_i}{T} = R_s \cdot N_i.$$

where N_i is an integer. To minimize bandwidth, integers have to be consecutive ($N_1 = N_0 + 1$, $N_2 = N_0 + 2$, $N_3 = N_0 + 3$). This implies the following constraints. Equation for Δf_i implies a constraint for the symbol rate

$$R_s \leq R_s(max) = \Delta f_i(max) = \frac{\Delta B}{3} = 100 \text{ MHz,}$$

maximum rate that would be obtained if $f_0 = f_a$ and $f_3 = f_b$ would be possible. Expression for ω_i or f_i, along with the available frequency range, implies the following constraints:

$$f_0 = R_s \cdot N_0 \geq f_a, \text{ and } f_3 = R_s \cdot (N_0 + 3) \leq f_b.$$

First constraint can be used to find the minimum value for integer N_0

$$N_0 \geq \frac{f_a}{R_s(max)} = 9.5 \rightarrow N_0(min) = 10.$$

Second constraint implies a high bound for symbol rate

$$R_s \leq \frac{f_b}{N_0 + 3} = \frac{f_b}{13} = 96.15 \text{ Mbauds.}$$

Therefore, maximum rate is obtained taking $f_3 = f_b$, and in this case $\Delta f_i = R_s$, i.e., $f_i = f_{i+1} - R_s$.

Frequencies are, therefore $f_3 = 1250$ MHz, $f_2 = 1153.85$ MHz, $f_1 = 1057.70$ MHz, $f_0 = 961.54$ MHz.

Question 2

a) For bandpass modulations the carrier frequency is the central frequency of the available band

$$\omega_c = \frac{\omega_a + \omega_b}{2} = 2\pi \cdot 875 \text{ Mrad/s}.$$

Using root-raised cosine, with available bandwidth B Hz, maximum transmission rate is

$$R_s = \frac{1}{T} = \frac{B}{1+\alpha} = 120 \text{ Mbauds}.$$

Binary rate is obtained by multiplying symbol rate times the number of bits per symbol, $m = \log_2(M)$, which for a 16-QAM is $m = \log_2(16) = 4$

$$R_b = R_s \cdot m = 480 \text{ Mbits/s}.$$

b) Power spectral density of modulated signal is

$$S_x(j\omega) = \frac{1}{2}\left[S_s(j\omega - j\omega_c) + S_s^*(-j\omega - j\omega_c)\right],$$

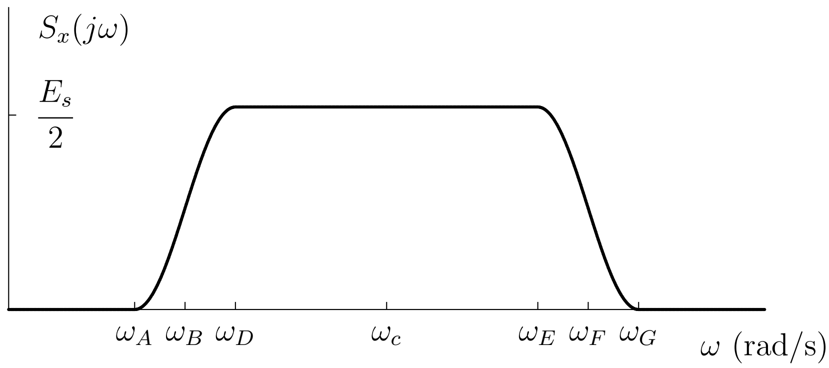
where $S_s(j\omega)$ is the power spectral density of the complex baseband signal, which for white sequences is

$$S_s(j\omega) = \frac{E_s}{T}\left|G(j\omega)\right|^2.$$

In this case, if transmitter filter is a root-raised cosine, the squared response $|G(j\omega)|^2$ is a raised cosine (whose amplitude is T if normalized). The mean energy per symbol for a 16-QAM modulation is

$$E_s = E[|A[n]|^2] = \frac{4}{16}2 + \frac{8}{16}10 + \frac{4}{16}18 = 10.$$

Therefore, power specral densitiy $S_x(j\omega)$ will have the following shape



with

$$\omega_A = \omega_c - \frac{\pi}{T}(1+\alpha) = 2\pi \cdot 800\text{Mrad/s},\ \omega_B = \omega_c - \frac{\pi}{T} = 2\pi \cdot 815\text{Mrad/s},$$

$$\omega_D = \omega_c - \frac{\pi}{T}(1-\alpha) = 2\pi \cdot 830\text{Mrad/s},\ \omega_E = \omega_c + \frac{\pi}{T}(1-\alpha) = 2\pi \cdot 920\text{Mrad/s},$$

$$\omega_F = \omega_c + \frac{\pi}{T} = 2\pi \cdot 935\text{Mrad/s},\ \omega_G = \omega_c + \frac{\pi}{T}(1+\alpha) = 2\pi \cdot 950\text{Mrad/s}.$$

c) First of all, conditions for transmission without ISI and for $z[n]$ being white will be presented. Given the channel behavior in the usable band, we are facing transmission through Gaussian channel (the only distortion produced during transmission, if signal is inside the usable band, is the addition of white and Gaussian noise). Therefore, condition to avoid ISI is that the joint response between transmitter and receiver, $p(t) = g(t) * f(t)$ (or $P(j\omega) = G(j\omega)F(j\omega)$), has to fulfill Nyquist criterion:

$$p(t)\big|_{t=nT} = C \cdot \delta[n] \text{ ro } \frac{1}{T}\sum_k P\left(j\omega - j\frac{2\pi k}{T}\right) = C.$$

Condition for $z[n]$ being white is that the ambiguity function of receiver filter, ($r_f(t) = f(t) * f(-t)$, ó $R_f(j\omega) = |F(j\omega)|^2$) has to fulfill Nyquist criterion

$$r_f(t)\big|_{t=nT} = C \cdot \delta[n] \text{ or } \frac{1}{T}\sum_k \left|F\left(j\omega - j\frac{2\pi k}{T}\right)\right|^2 = C.$$

Now, each case will be analyzed.

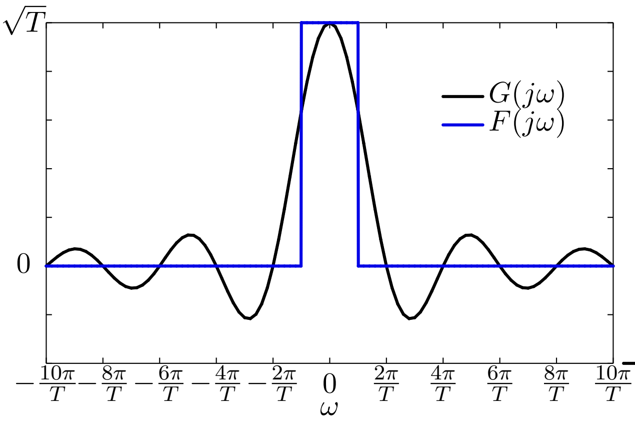
i) In this case both conditions are fulfilled, becasuse if $f(t) = f_a(t) = g(-t)$, then we have that $p(t) = r_g(t) = r_f(t)$ is a raised cosine function, which fulfills Nyquist criterion. Therefore, there is NO ISI, and noise $z[n]$ IS white.

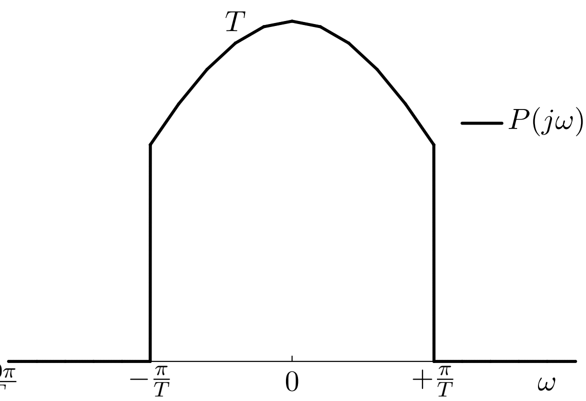
ii) In this case, ISI will be present. It ca be easily seen in frequency domain. In this case

$$G(j\omega) = \sqrt{T} \cdot \Pi\left(\frac{\omega T}{2\pi}\right), \text{ and } F(j\omega) = \sqrt{T} \cdot \text{sinc}\left(\frac{\omega T}{2\pi}\right)$$

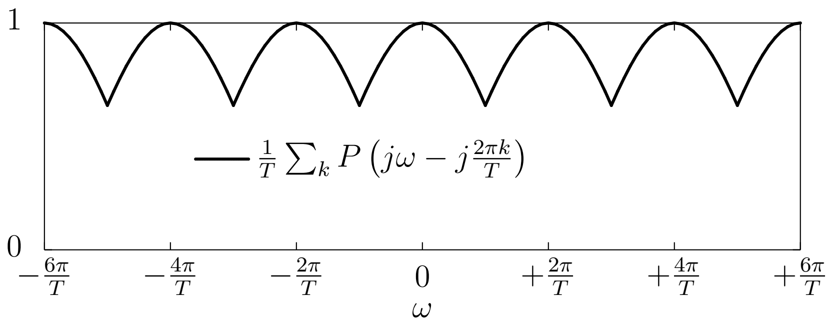
$$P(j\omega) = G(j\omega)F(j\omega) = \begin{cases} T \cdot \text{sinc}\left(\frac{\omega T}{2\pi}\right), & \text{if } |\omega| \leq \frac{\pi}{T} \\ 0, & \text{if } |\omega| > \frac{\pi}{T}. \end{cases}$$

Next picture plots $G(j\omega)$ and $F(j\omega)$, along with its product, $P(j\omega)$





Obviously, addition of replicas of this function each $\frac{2\pi}{T}$ is not constant, as can be seen in next figure. Therefore, THERE IS ISI.



Noise $z[n]$ IS white, because the ambiguity function of the receiver $f(t) = f_b(t)$ is a triangle beteen $-T$ y T, which fulfills Nyquist criterion.

Question 3

a) For OFDM modulation, discrete time implementation allows to reduce the hardware complexity for implemantation of both transmitter and receiver, because the only necessary hardware is a reconstruction filter to move $s[m]$ to continuous time, $s_r(t)$. Therefore, an additional advantage is the generation of a bandlimited signal. For spread spectrum modulation, again the advantage is hardware complexity, because the necessary hardware is a trasmitter filter fulfilling Nyquist at chip period, instead of a linear combination of N filters at chip period T_c to implement the response of the transmission filter at symbol period T.

b) In OFDM, by means of the introduction of a cyclic prefix of C samples, being C at least equal to K_d, where K_d is the memory of equivalent discrete channel at $\frac{T}{N+M}$, $d[m]$.For direct sequence spread spectrum, choice of an appropriate spreading sequence will mitigate ISI. Theoretically, ISI is completely removed if

$$\sum_m x[m] \cdot x^*[m+k] = C \cdot \delta[k], \text{ para todo } k.$$

c) If receiver at symbol rate is matched to the joint response between $g(t)$ and $h_{eq}(t)$, which will be denoted as $g_h(t) = h_1 g(t) + h_2 g(t - \tau_2)$, expression for the output of the receiver filter $q(t)$ has to be developed

$$q(t) = v(t) * f(t).$$

Taking into account that $g(t) = \sum_{m=0}^{N-1} x[m] \cdot g_c(t - mT_c)$, and that $q[n] = q(t)\big|_{t=nT}$

$$q[n] = \sum_{m=0}^{N-1} x^*[m] h_1^* \big[v(t) * g_c(-t - mT_c)\big]_{t=nT=nNT_c} + \sum_{m=0}^{N-1} x^*[m] h_2^* \big[v(t) * g_c(-t - mT_c - CT_c)\big]_{t=nT=nNT_c}.$$

By defining

$$v[m] = v(t) * g_c(-t)\big|_{t=mT_c}$$

$q[n]$ can be written as

$$q[n] = \sum_{m=0}^{N-1} h_1^* \cdot x^*[m] \cdot v[nN+m] + \sum_{m=0}^{N-1} h_2^* \cdot x^*[m] \cdot v[nN+m+C].$$

Problem 1

For codes 1 y 2, assignment between uncoded words $\boldsymbol{b}_i$ of k bits, and coded words of n bits, $\boldsymbol{c}_i$ is given in the following tables

$\boldsymbol{b}_i$	$\boldsymbol{c}_i$		
0	0	0	0
1	1	1	1

$\boldsymbol{b}_i$			$\boldsymbol{c}_i$			
0	0	0	0	0	0	0
0	0	1	0	0	1	1
0	1	0	0	1	0	1
0	1	1	0	1	1	0
1	0	0	1	0	0	1
1	0	1	1	0	1	0
1	1	0	1	1	0	0
1	1	1	1	1	1	1

a) Taking into account this assignment for codes 1 and 2 and that $\boldsymbol{c}_i = \boldsymbol{b}_i \cdot \boldsymbol{G}$, and the shape of $\boldsymbol{H}_3$, generating matrices are

$$\boldsymbol{G}_1 = \left[\begin{array}{c|cc} 1 & 1 & 1 \end{array}\right],\ \boldsymbol{G}_2 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right],\ \boldsymbol{G}_3 = \left[\begin{array}{ccc|cc} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

b) Dictionnary for code 3 is given in the following table

i	$\boldsymbol{b}_i$		$\boldsymbol{c}_i$				
0	0	0	0	0	0	0	0
1	0	1	1	0	1	0	1
2	1	0	0	1	1	1	0
3	1	1	1	1	0	1	1

Detection and correction capabilities of a linear block code is given by minimum distance of the code, d_{min}. In particular, a linear block code is able to detect up to a maximum of d errors and correct up to a maximum of t errors, with

$$d = d_{min} - 1,\ t = \left\lfloor \frac{d_{min} - 1}{2} \right\rfloor.$$

Minimum distance can be obtained by looking the minimal number of ones in the encoded words without considering the all zero word. In this case

- Code 1: $d_{min} = 3$, $d = 2$, $t = 1$
- Code 2: $d_{min} = 2$, $d = 1$, $t = 0$
- Code 3: $d_{min} = 3$, $d = 2$, $t = 1$

c) Concatenating codes 1 and 2, the equivalent code will have size $k = 1$, $n = 4$, with the following encoded words

k	C1	C1+C2
0	000	0000
1	111	1111

Now minimum distance is $d_{min} = 4$, which allows to increment by 1 the detection capability of code 1.

d) If now codes 2 and 3 are concatenated, the code size is $k = 3$ and $n = 4$, and dictionary of the code is

k	C2	C2+C3
$\boldsymbol{b}_0 = 000$	0000	$\boldsymbol{c}_0 = 00000\ 00000$
$\boldsymbol{b}_1 = 001$	0011	$\boldsymbol{c}_1 = 00000\ 11011$
$\boldsymbol{b}_2 = 010$	0101	$\boldsymbol{c}_2 = 10101\ 10101$
$\boldsymbol{b}_3 = 011$	0110	$\boldsymbol{c}_3 = 10101\ 01110$
$\boldsymbol{b}_4 = 100$	1001	$\boldsymbol{c}_4 = 01110\ 10101$
$\boldsymbol{b}_5 = 101$	1010	$\boldsymbol{c}_5 = 01110\ 10101$
$\boldsymbol{b}_6 = 110$	1100	$\boldsymbol{c}_6 = 11011\ 00000$
$\boldsymbol{b}_7 = 111$	1111	$\boldsymbol{c}_7 = 11011\ 11011$

Minimum distance is now $d_{min} = 4$.

Generating matrix can easily be obtained, because it contains the 3 encoded words corresponding to uncoded words of 3 bits having a single one, i.e.

$$\boldsymbol{G}_{2+3} = \begin{bmatrix} \boldsymbol{c}_4 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

e) In this case, we only have to take into account that syndrome is obtained from an error pattern ε as $\boldsymbol{s} = \boldsymbol{e} \cdot \boldsymbol{H}_3^T$. Syndrome table is now

$\boldsymbol{e}$	$\boldsymbol{s}$
00000	000
10000	100
01000	010
00100	001
00010	011
00001	101
11000	110
01001	111

Assignment for last two error patterns, which have to be error patterns of two errors producing the symdromes that are not produced for the 5 patterns of a single error, is not unique. There exist other error patterns of two error producing these syndromes.

Decoding will follow the following steps:

1.- Syndrome is obtained: $\boldsymbol{s} = \boldsymbol{r}\boldsymbol{H}^T = 001$

2.- Error pattern is identified (from table): $\boldsymbol{s} = 001 \rightarrow \boldsymbol{e} = 00100$

3.- Error pattern is corrected: $\hat{\boldsymbol{c}} = \boldsymbol{r} + \boldsymbol{e} = 11011 = \boldsymbol{c}_3$

4.- Decoding is performed (looking at the dictionnary for the code): $\hat{\boldsymbol{c}} = \boldsymbol{c}_3 \rightarrow \hat{\boldsymbol{b}} = \boldsymbol{b}_3 = 11$

Problem 2

a) Optimal delay is given by the position in the equivalent discrete channel where this channel has the maximum absolute value. In this case, the maximum value of the channel is at position $n = 1$. Therefore, optimal delay is $\boxed{d = 1}$.

To obtain the probability of error, the noiseless outputs of $p[n]$ (extended constellation), have to be obtained. Noiseless output is given as $o[n] = A[n] * p[n]$,

$$o[n] = A[n] * p[n] = \frac{3}{4}A[n] + \frac{5}{4}A[n-1]$$

and all posible values are in the table

$A[n]$	$A[n-1]$	$o[n]$
$+1$	$+1$	$+2$
-1	$+1$	$+\frac{1}{2}$
$+1$	-1	$-\frac{1}{2}$
-1	-1	-2

Probability of error is given by

$$P_e = \frac{1}{2}P_{e|A[n-1]=+1} + \frac{1}{2}P_{e|A[n-1]=-1}.$$

In this case $P_{e|A[n-1]=+1} = P_{e|A[n-1]=-1}$, and

$$P_e = P_{e|A[n-1]=+1} = P_{e|A[n-1]=-1} = \frac{1}{2}Q\left(\frac{1/2}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{2}{\sqrt{N_0/2}}\right).$$

b) First, the joint channel-equalizer response is obtained

$$c[n] = p[n] * w[n] = \sum_{k=0}^{K_w} w[k] \cdot p[n-k],$$

being $K_w = 1$ in this case. Therefore

$$c[0] = -\frac{12}{25} \cdot \frac{3}{4} = -\frac{36}{100} = -0.36$$
$$c[1] = -\frac{12}{25} \cdot \frac{5}{4} + -\frac{5}{4} \cdot \frac{3}{4} = 0$$
$$c[2] = \frac{4}{5} \cdot \frac{5}{4} = 1$$

Now, the equalizer outpur can be written as

$$u[n] = A[n] * c[n] + z[n] * w[n] = o_u[n] + z'[n],$$

where $o_u[n]$ is the noiseless output of the equalizer and $z'[n]$ is the filtered noise, whose variance (power) is

$$\sigma_{z'}^2 = \sigma_z^2 \cdot \sum_k |w[k]|^2 = 0.8704 \cdot \sigma_z^2 = 0.8704 \cdot \frac{N_0}{2}.$$

Optimal delay is now the position where $c[n]$ has the maximum absolute value, which in this case is $\boxed{d=2}$.

Table for noiseless outputs, which allows to compute the probability of error is

$A[n]$	$A[n-1]$	$A[n-2]$	$o_u[n]$
$+1$	$+1$	$+1$	$+0.64$
-1	$+1$	$+1$	$+1.36$
$+1$	-1	$+1$	$+0.64$
-1	-1	$+1$	$+1.36$
$+1$	$+1$	-1	-1.36
-1	$+1$	-1	-0.64
$+1$	-1	-1	-1.36
-1	-1	-1	-0.64

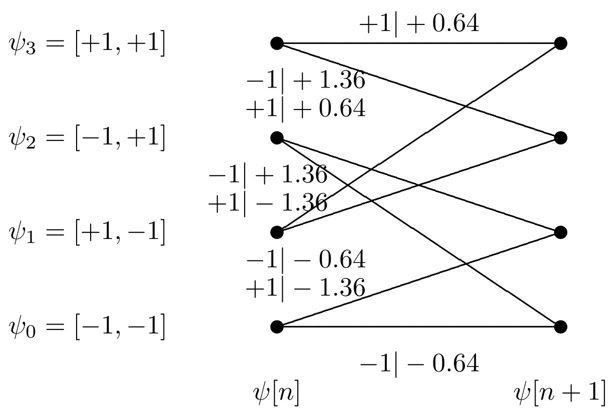
Probability of error is

$$P_e = \frac{1}{2} P_{e|A[n-1]=+1} + \frac{1}{2} P_{e|A[n-1]=-1}.$$

In this case $P_{e|A[n-1]=+1} = P_{e|A[n-1]=-1}$, and therefore

$$P_e = P_{e|A[n-1]=+1} = P_{e|A[n-1]=-1} = \frac{1}{2} Q\left(\frac{0.64}{\sigma_{z'}}\right) + \frac{1}{2} Q\left(\frac{1.36}{\sigma_{z'}}\right).$$

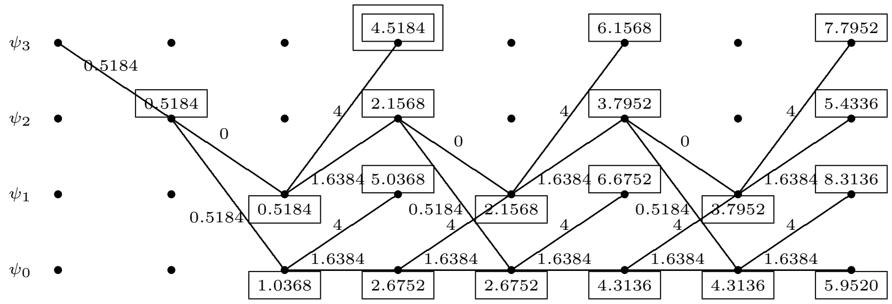
c) Trellis diagram can be easily obtained from table for $o_u[n]$, and is plotted in next figure



Probability of error for maximum likelihood sequence detectors can be approximated as

$$P_e \approx k_0 \cdot Q\left(\frac{D_{min}}{\sqrt{N_0/2}}\right),$$

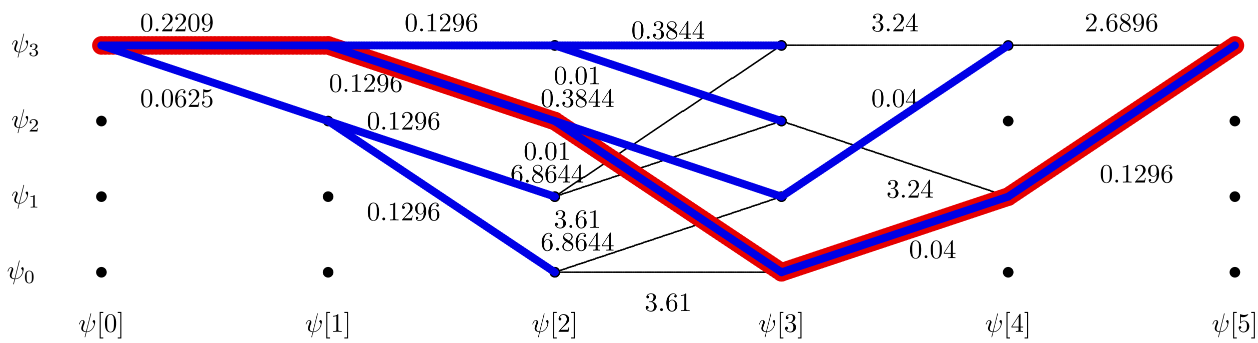
where D_{min} is the minimum euclidean distance between the noiseless outputs corresponding to two different transmitted sequences. Becasuse we know that the sequence of all ones has another sequence whose output is at minimum euclidean distance, D_{min} can be obtained from the trellis taking the sequence of all ones as a reference, and considering that the noiseless output generated by this sequences is $o[n] = +0.64$ for all n. Development of calculation is in next figure



$$D_{min} = \sqrt{4.5184}$$

REMARK: Indeed, calculations could be finished after processing the third transition, taking into account that the only two accumulated metrics lower than 4.5184, which are 2.1568 and 2.6752, and that the only path to come back to ψ_3 is from ψ_1, and that the metric for this branch is 4. This means, that is not possible to come back to fuse with the reference sequence ($A[n]+1$, $\forall n$, or what is the same, $\psi[n] = \psi_3$, $\forall n$) with accumulated metric lower than 6.1568 and 6.6752, respectively.

d) To decode the sequence with observations, Viterbi algorithm has to be applied through the trellis diagram for the system with equalizer. Next picture plots the branch metric, the survival paths obtained in the application of the algorithm (double width) and the path corresponding to the maximum likelihood sequence (thicker width). Accumulated metrics for each one of the states are given in the attached table. For each state, fist metric corresponds to the branch coming higher, and the second one with the branch coming lower to the node. Boldface is used to remark the metric of the survival path (which corresponds to the metric of this state).



	$\psi[1]$	$\psi[2]$	$\psi[3]$	$\psi[4]$	$\psi[5]$
ψ_3	**0.2209**	**0.3505**	**0.7349**/7.0565	3.9749/**0.7749**	3.4645/**0.5301**
ψ_2	**0.0625**	**0.3525**	**0.3605**/3.9605		
ψ_1		**0.1921**	**0.7349**/7.0565	3.6005/**0.4005**	
ψ_0		**0.1921**	**0.3605**/3.9605		

Therefore, the solution (maximum likelihood sequence) is the following one:

$$\hat{A}[0] = +1,\ \hat{A}[1] = -1,\ \hat{A}[2] = -1.$$