

DIGITAL COMMUNICATIONS
SOLUTION OF THE EXAM

Question 1

- a) Mean energy per symbols is obtained by averaging the energy of each symbol, and energy of a symbol is defined as its squared norm (or the addition of the squared coordinates). Therefore

$$E_s = \sum_{i=0}^{M-1} p_A(\mathbf{a}_i) \cdot \mathcal{E}\{\mathbf{a}_i\} = \sum_{i=0}^{M-1} p_A(\mathbf{a}_i) \cdot \underbrace{\sum_{j=0}^{N-1} a_{i,j}^2}_{\|\mathbf{a}_i\|^2} = (1-p) \cdot 0 + p \cdot A^2 = p \cdot A^2.$$

- b) The baseband power spectral density of a PAM signal is

$$S_s(j\omega) = \frac{1}{T} S_A(e^{j\omega T}) |G(j\omega)|^2 = \frac{E_s}{T} |G(j\omega)|^2.$$

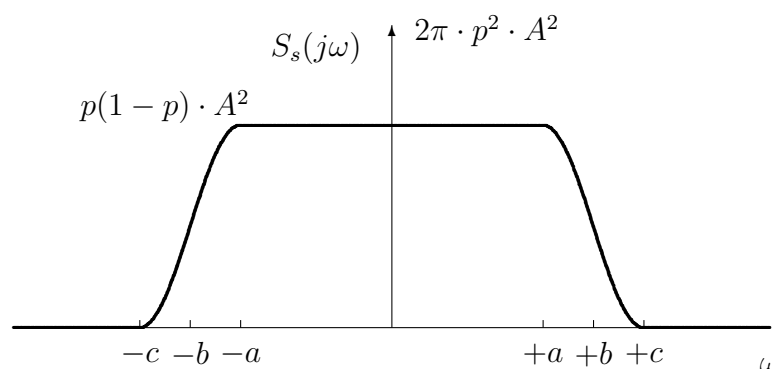
As $G(j\omega) = H_{RRC}^{\alpha,T}(j\omega)$ is a squared root raised cosine, $|G(j\omega)|^2 = \left| H_{RRC}^{\alpha,T}(j\omega) \right|^2 = H_{RC}^{\alpha,T}(j\omega)$ is a raised cosine, which is normalized, and therefore has amplitude T in the flat bandpass.

Power spectral density of the data sequence is obtained through the Fourier transform of the autocorrelation function

$$S_A(e^{j\omega}) = TF\{R_A[k]\} = \sum_k R_A[k] \cdot e^{-j\omega k} = p(1-p) \cdot A^2 + p^2 \cdot A^2 \cdot 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k),$$

i.e., a constant value $p(1-p) \cdot A^2$ plus a delta train in $\omega = 0$ and in ω being integer multiples of 2π with amplitudes $p^2 \cdot A^2 \cdot 2\pi$. In the expression for $S_s(j\omega)$, power spectral density $S_A(e^{j\omega})$ is evaluated at ωT , which means that now delta functions are at zero and at integer multiples of $\frac{2\pi}{T}$. The only delta function being in the pass band of the raised cosine, $|G(j\omega)|^2$ (goes up to $\frac{\pi}{T} \cdot (1 + \alpha)$ rad/s) is the one at zero.

Therefore, $S_s(j\omega)$ is as in the following picture



with values

$$a = \frac{\pi}{T} \cdot (1 - \alpha), \quad b = \frac{\pi}{T}, \quad c = \frac{\pi}{T} \cdot (1 + \alpha),$$

Bandwidth is therefore

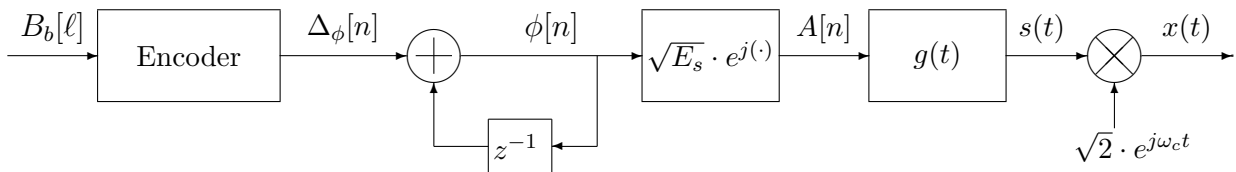
$$W = \frac{\pi}{T} \cdot (1 + \alpha) = R_s \cdot \pi \cdot (1 + \alpha) = \pi \times 1.25 \text{ krad/s},$$

or

$$B = \frac{R_s}{2} \cdot (1 + \alpha) = 625 \text{ Hz}.$$

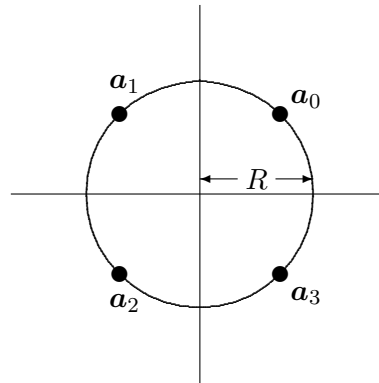
Question 2

a) Block diagram for the transmitter is



This picture represents up to the generation of the bandpass signal, $x(t)$, although in the exam only the system up to the complex baseband signal $s(t)$ was demanded.

b) With this initial reference phase, constellation is a 4-QAM, i.e., symbol phases are $\phi[n] \in \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$ radians, as plotted in the picture



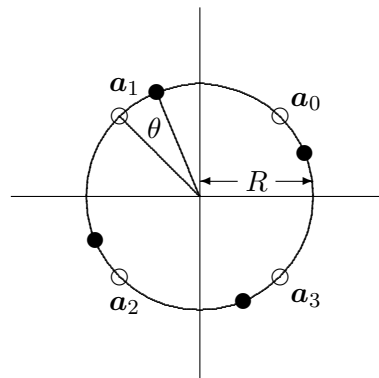
The energy of all symbols is the same one (all symbols have the same norm), and therefore mean energy per symbol is

$$E_s = \sum_{i=0}^{M-1} p_A(\mathbf{a}_i) \cdot \mathcal{E}\{\mathbf{a}_i\} = \sum_{i=0}^{M-1} p_A(\mathbf{a}_i) \cdot \|\mathbf{a}_i\|^2 = R^2.$$

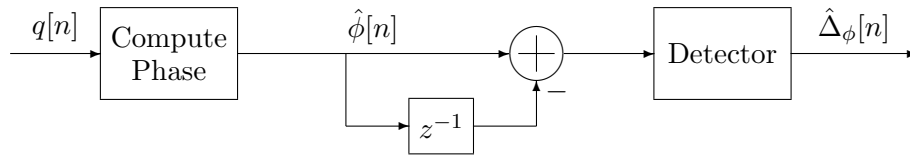
c) A Gray encoding has to be performed over the incremental phase (instead of the absolute phase). One example could be

$\Delta_\phi[n]$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
bits	00	10	11	01

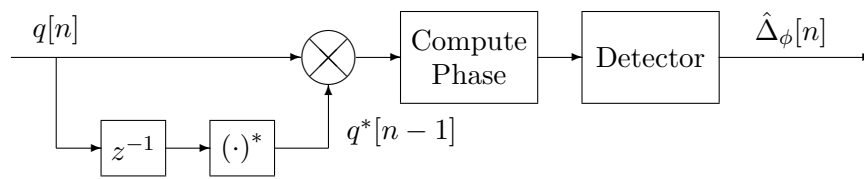
d) The effect is a rotation of $-\theta$ radians, since the equivalent discrete channel is $p[n] = e^{-j\theta} \delta[n]$. In the picture, the situatio for a value $\theta = \frac{\pi}{8}$ rad. is plotted



- e) At the receiver, basically the difference between the phases of current symbols and previous symbol has to be estimated. This can be done directly, evaluating the phase of each observation. Another possibility is to take the product of current observation times the previous observation complex conjugated, because this product produces a complex number whose phase is the difference between phases of both symbols plus noise.



Receiver DPSK (a)



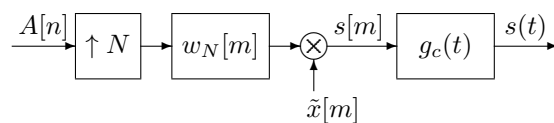
Receiver DPSK (b)

Question 3

- a) Samples of the direct sequence spread spectrum with spreading factor N and spreading sequence $x[n]$ are given by the expression

$$s[m] = \tilde{x}[m] \cdot \sum_n A[n] \cdot w_N[m - nN],$$

where $\tilde{x}[m]$ is the cyclic extension of the spreading sequence and $w_N[m]$ is a discrete time causal window of length N samples. This expression indicates that samples are generated in blocks of N , each block associated with a symbol of sequence $A[n]$. From a symbol $A[n]$, N samples are generated by replicating N times the symbol value and by multiplying each replica by one of the values of the spreading sequence, as it can be seen in the block diagram



Therefore, from the first 3 symbols of sequence $A[n]$, 9 samples of signal $s(t)$ are generated, samples at chip rate T_c

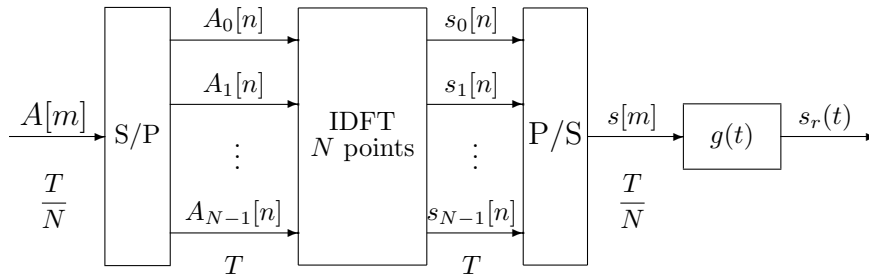
$$\begin{aligned} A[0] &\Rightarrow s[0] = A[0] \times x[0] = +1, \quad s[1] = A[0] \times x[1] = -1, \quad s[2] = A[0] \times x[2] = +1 \\ A[1] &\Rightarrow s[3] = A[1] \times x[0] = -3, \quad s[4] = A[1] \times x[1] = +3, \quad s[5] = A[1] \times x[2] = -3 \\ A[2] &\Rightarrow s[6] = A[2] \times x[0] = +1, \quad s[7] = A[2] \times x[1] = -1, \quad s[8] = A[2] \times x[2] = +1. \end{aligned}$$

If filters of the raised cosine family are used, the required bandwidth for the bandpass modulation is

$$B = R_s \cdot (1 + \alpha) \times N = 7.5 \text{ kHz},$$

because in this case $R_s = \frac{R_b}{2} = 2$ kbauds (a constellation of 4 symbols is used, and therefore this has $m = 2$ bits per symbol).

- b) In the case of an OFDM modulation, samples are evaluated through the inverse DFTs of size N of N symbols, each one transmitted in one of the carriers. First, a serial to parallel conversion is performed, as seen in the block diagram (without cyclic prefix in this case)



In this conversion, first data block is

$$n = 0 \Rightarrow A_0[0] = A[0], A_1[0] = A[1], A_2[0] = A[2], A_3[0] = A[3],$$

and second block is

$$n = 1 \Rightarrow A_0[1] = A[4], A_1[1] = A[5], A_2[1] = A[6], A_3[1] = A[7],$$

It is necessary to remember the scaling factor $\frac{1}{\sqrt{T}}$ with respect to the definition of the IDFT of order N .

Therefore, first block of samples is obtained from the IDFT of symbols of the first block, i.e., $A_k[0]$

$$s[m] = \frac{1}{\sqrt{T}} \sum_{k=0}^{N-1} A_k[0] \cdot e^{j \frac{2\pi k}{N} m}, \quad m = 0, 1, 2, 3$$

In this particular case

$$s[0] = \frac{1}{\sqrt{T}} \left(\underbrace{A_0[0]}_{+1} \cdot \underbrace{e^{j \frac{2\pi \cdot 0}{4} \cdot 0}}_{e^{j0}=1} + \underbrace{A_1[0]}_{-3} \cdot \underbrace{e^{j \frac{2\pi \cdot 1}{4} \cdot 0}}_{e^{j0}=1} + \underbrace{A_2[0]}_{+1} \cdot \underbrace{e^{j \frac{2\pi \cdot 2}{4} \cdot 0}}_{e^{j0}=1} + \underbrace{A_3[0]}_{+3} \cdot \underbrace{e^{j \frac{2\pi \cdot 3}{4} \cdot 0}}_{e^{j0}=1} \right) = \frac{2}{\sqrt{T}}.$$

$$s[1] = \frac{1}{\sqrt{T}} \left(\underbrace{A_0[0]}_{+1} \cdot \underbrace{e^{j \frac{2\pi \cdot 0}{4} \cdot 1}}_{e^{j0}=+1} + \underbrace{A_1[0]}_{-3} \cdot \underbrace{e^{j \frac{2\pi \cdot 1}{4} \cdot 1}}_{e^{j\pi/2}=+j} + \underbrace{A_2[0]}_{+1} \cdot \underbrace{e^{j \frac{2\pi \cdot 2}{4} \cdot 1}}_{e^{j\pi}=-1} + \underbrace{A_3[0]}_{+3} \cdot \underbrace{e^{j \frac{2\pi \cdot 3}{4} \cdot 1}}_{e^{j3\pi/2}=-j} \right) = \frac{-j6}{\sqrt{T}}.$$

$$s[2] = \frac{1}{\sqrt{T}} \left(\underbrace{A_0[0]}_{+1} \cdot \underbrace{e^{j \frac{2\pi \cdot 0}{4} \cdot 2}}_{e^{j0}=+1} + \underbrace{A_1[0]}_{-3} \cdot \underbrace{e^{j \frac{2\pi \cdot 1}{4} \cdot 2}}_{e^{j\pi}=-1} + \underbrace{A_2[0]}_{+1} \cdot \underbrace{e^{j \frac{2\pi \cdot 2}{4} \cdot 2}}_{e^{j2\pi}=+1} + \underbrace{A_3[0]}_{+3} \cdot \underbrace{e^{j \frac{2\pi \cdot 3}{4} \cdot 2}}_{e^{j3\pi}=-1} \right) = \frac{2}{\sqrt{T}}.$$

$$s[3] = \frac{1}{\sqrt{T}} \left(\underbrace{A_0[0]}_{+1} \cdot \underbrace{e^{j \frac{2\pi \cdot 0}{4} \cdot 3}}_{e^{j0}=+1} + \underbrace{A_1[0]}_{-3} \cdot \underbrace{e^{j \frac{2\pi \cdot 1}{4} \cdot 3}}_{e^{j3\pi/2}=-j} + \underbrace{A_2[0]}_{+1} \cdot \underbrace{e^{j \frac{2\pi \cdot 2}{4} \cdot 3}}_{e^{j3\pi}=-1} + \underbrace{A_3[0]}_{+3} \cdot \underbrace{e^{j \frac{2\pi \cdot 3}{4} \cdot 3}}_{e^{j9\pi/2}=+j} \right) = \frac{+j6}{\sqrt{T}}.$$

For the second data block now symbols correspond to $A_k[1]$

$$s[m] = \frac{1}{\sqrt{T}} \sum_{k=0}^{N-1} A_k[1] \cdot e^{j \frac{2\pi k}{N} (m-N)}, \quad m = 4, 5, 6, 7$$

For values in this particular case we have

$$s[4] = \frac{1}{\sqrt{T}} \left(\underbrace{A_0[1]}_{+1} \cdot \underbrace{e^{j \frac{2\pi \cdot 0}{4} \cdot 0}}_{e^{j0}=1} + \underbrace{A_1[1]}_{-1} \cdot \underbrace{e^{j \frac{2\pi \cdot 1}{4} \cdot 0}}_{e^{j0}=1} + \underbrace{A_2[1]}_{+1} \cdot \underbrace{e^{j \frac{2\pi \cdot 2}{4} \cdot 0}}_{e^{j0}=1} + \underbrace{A_3[1]}_{-1} \cdot \underbrace{e^{j \frac{2\pi \cdot 3}{4} \cdot 0}}_{e^{j0}=1} \right) = 0.$$

$$s[5] = \frac{1}{\sqrt{T}} \left(\underbrace{A_0[1]}_{+1} \cdot \underbrace{e^{j\frac{2\pi \cdot 0 \cdot 1}{4}}}_{e^{j0=+1}} + \underbrace{A_1[1]}_{-1} \cdot \underbrace{e^{j\frac{2\pi \cdot 1 \cdot 1}{4}}}_{e^{j\pi/2=+j}} + \underbrace{A_2[1]}_{+1} \cdot \underbrace{e^{j\frac{2\pi \cdot 2 \cdot 1}{4}}}_{e^{j\pi=-1}} + \underbrace{A_3[1]}_{-1} \cdot \underbrace{e^{j\frac{2\pi \cdot 3 \cdot 1}{4}}}_{e^{j3\pi/2=-j}} \right) = 0.$$

$$s[6] = \frac{1}{\sqrt{T}} \left(\underbrace{A_0[1]}_{+1} \cdot \underbrace{e^{j\frac{2\pi \cdot 0 \cdot 2}{4}}}_{e^{j0=+1}} + \underbrace{A_1[1]}_{-1} \cdot \underbrace{e^{j\frac{2\pi \cdot 1 \cdot 2}{4}}}_{e^{j\pi=-1}} + \underbrace{A_2[1]}_{+1} \cdot \underbrace{e^{j\frac{2\pi \cdot 2 \cdot 2}{4}}}_{e^{j2\pi=+1}} + \underbrace{A_3[1]}_{-1} \cdot \underbrace{e^{j\frac{2\pi \cdot 3 \cdot 2}{4}}}_{e^{j3\pi=-1}} \right) = \frac{4}{\sqrt{T}}.$$

$$s[7] = \frac{1}{\sqrt{T}} \left(\underbrace{A_0[1]}_{+1} \cdot \underbrace{e^{j\frac{2\pi \cdot 0 \cdot 3}{4}}}_{e^{j0=+1}} + \underbrace{A_1[1]}_{-1} \cdot \underbrace{e^{j\frac{2\pi \cdot 1 \cdot 3}{4}}}_{e^{j3\pi/2=-j}} + \underbrace{A_2[1]}_{+1} \cdot \underbrace{e^{j\frac{2\pi \cdot 2 \cdot 3}{4}}}_{e^{j3\pi=-1}} + \underbrace{A_3[1]}_{-1} \cdot \underbrace{e^{j\frac{2\pi \cdot 3 \cdot 3}{4}}}_{e^{j9\pi/2=+j}} \right) = 0.$$

Another important point is to take into account that during the serial to parallel conversion, there is a rate conversion, and now OFDM symbol rate (the rate for each sequence transmitted in one of the N carriers) is

$$R_s^{OFDM} = \frac{R_s}{N} = 500 \text{ bauds.}$$

In these modulations, the OFDM symbol duration is defined as $T = \frac{1}{R_s^{OFDM}}$. This value is important to obtain the bandwidth.

- i) In this case, if the minimum possible bandwidth is required, the OFDM will have to be used without cyclic prefix, and therefore the sequence of samples of the transmitted signal is

$$\underbrace{s[0], s[1], s[2], s[3]}_{\text{Block 1}}, \underbrace{s[4], s[5], s[6], s[7]}_{\text{Block 2}}.$$

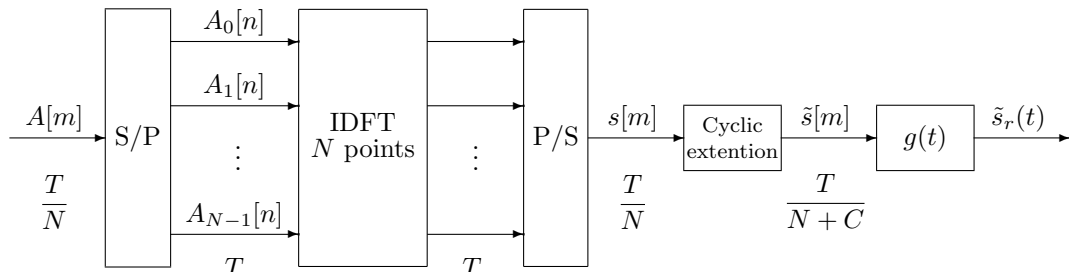
Without cyclic prefix, bandwidth of modulated signal is

$$B = R_s^{OFDM} \times N = R_s = 2 \text{ kHz.}$$

- ii) Now, to remove ISI and ICI a cyclic prefix has to be used, with length at least the channel memory for channel $d[m]$, K_d . In this case $K_d = 2$. Because the bandwidth using a prefix of C samples is

$$B = R_s^{OFDM} \times (N + C) = R_s \cdot \frac{N + C}{N},$$

minimum bandwidth allowing to remove ISI and ICI is obtained using the lowest value of C ($C \geq K_d$), in this case $C = K_d = 2$. This implies to have $B = 3 \text{ kHz}$.



At it is now showed in the picture, samples of the signal are obtained by introducing a cyclic extension of the prefix, producing samples $\tilde{s}[m]$ having now size $N + C = 6$ samples per block

$$\tilde{s}[-2] = s[2], \tilde{s}[-1] = s[3], \tilde{s}[0] = s[0], \tilde{s}[1] = s[1], \tilde{s}[2] = s[2], \tilde{s}[3] = s[3],$$

for the first block, and

$$\tilde{s}[4] = s[6], \tilde{s}[5] = s[7], \tilde{s}[6] = s[4], \tilde{s}[7] = s[5], \tilde{s}[8] = s[6], \tilde{s}[9] = s[7],$$

for the second one, being in this case the sequence of samples

$$\underbrace{\tilde{s}[-2], \tilde{s}[-1], \tilde{s}[0], \tilde{s}[1], \tilde{s}[2], \tilde{s}[3]}_{\text{Block 1}}, \underbrace{\tilde{s}[4], \tilde{s}[5], \tilde{s}[6], \tilde{s}[7], \tilde{s}[8], \tilde{s}[9]}_{\text{Block 2}},$$

or,

$$\underbrace{s[2], s[3], s[0], s[1], s[2], s[3]}_{\text{Block 1}}, \underbrace{s[6], s[7], s[4], s[5], s[6], s[7]}_{\text{Block 2}}.$$

Exercise 1

- a) Taking into account the values of the two equivalent discrete channels, the noiseless output of the system is

$$o[n] = A_1[n] * p_1[n] + A_2[n] * p_2[n] = 0.9 \cdot A_1[n] - 0.1 \cdot A_1[n - 1] + 0.8 \cdot A_2[n] - 0.2 \cdot A_2[n - 1].$$

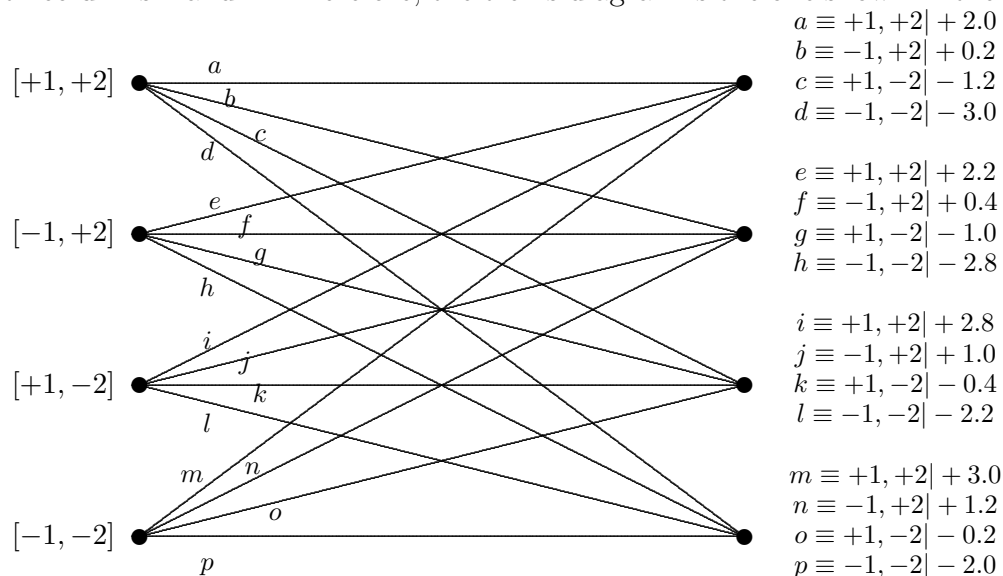
Noiseless output at a given discrete instant depends on values of the two sequences at the same instant and at the previous instant. Therefore, it can take the set of values shown in the following table

$A_1[n]$	$A_2[n]$	$A_1[n - 1]$	$A_2[n - 1]$	$o[n]$	$A_1[n]$	$A_2[n]$	$A_1[n - 1]$	$A_2[n - 1]$	$o[n]$
+1	+2	+1	+2	$a \equiv +2.0$	+1	+2	+1	-2	$i \equiv +2.8$
-1	+2	+1	+2	$b \equiv +0.2$	-1	+2	+1	-2	$j \equiv +1.0$
+1	-2	+1	+2	$c \equiv -1.2$	+1	-2	+1	-2	$k \equiv -0.4$
-1	-2	+1	+2	$d \equiv -3.0$	-1	-2	+1	-2	$l \equiv -2.2$
+1	+2	-1	+2	$e \equiv +2.2$	+1	+2	-1	-2	$m \equiv +3.0$
-1	+2	-1	+2	$f \equiv +0.4$	-1	+2	-1	-2	$n \equiv +1.2$
+1	-2	-1	+2	$g \equiv -1.0$	+1	-2	-1	-2	$o \equiv -0.2$
-1	-2	-1	+2	$h \equiv -2.8$	-1	-2	-1	-2	$p \equiv -2.0$

To decode both sequences simultaneously, there are two inputs in the system, $A_1[n]$ y $A_2[n]$ (columns 1 and 2 in the table), and a single output, $o[n]$, being the system state given by

$$\psi[n] = [A_1[n - 1], A_2[n - 1]]$$

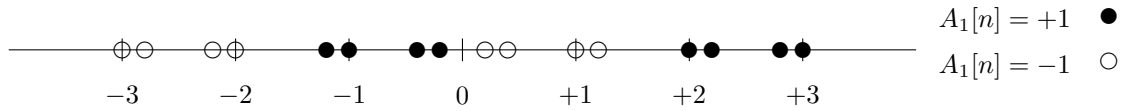
which means that $\psi[n]$ corresponds with columns 3 and 4 of the table, and that estate $\psi[n + 1]$ corresponds with columns 1 and 2. Therefore, the trellis diagram is the one shown in the figure



- b) To compute the probability of error for user 1, with a delay $d = 0$, the conditional probabilities of error have to be averaged

$$P_{e1} = \frac{1}{2} P_{e|A_1[n]=+1} + \frac{1}{2} P_{e|A_1[n]=-1}.$$

To compute these conditional probabilities, it is necessary to know the points of the extended constellation at $o[n]$ that are generated when $A_1[n] = +1$ and when $A_1[n] = -1$. These points (can be obtained from previous table) are plotted in next figure



Conditional probability of error is

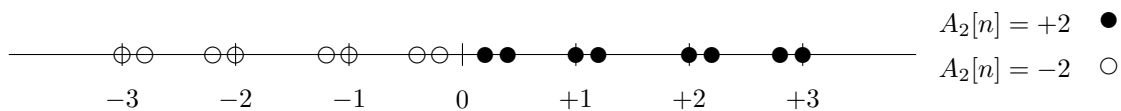
$$P_{e|A_1[n]=+1} = \frac{1}{8} \left[(1 - Q(0.2)) + (1 - Q(0.4)) + (1 - Q(1)) + (1 - Q(1.2)) + Q(2) + Q(2.2) + Q(2.8) + Q(3) \right]$$

In this case $P_{e|A_1[n]=-1} = P_{e|A_1[n]=+1}$, and therefore $P_{e1} = P_{e|A_1[n]=+1}$.

To compute the probability of error for user 2, with a delay $d = 0$, the conditional probabilities of error have to be averaged

$$P_{e2} = \frac{1}{2} P_{e|A_2[n]=+2} + \frac{1}{2} P_{e|A_2[n]=-2}$$

To compute these conditional probabilities, it is necessary to know the points of the extended constellation at $o[n]$ that are generated when $A_2[n] = +2$ and when $A_2[n] = -2$. This points (can be obtained from previous table) are plotted in next figure



Conditional probability of error for $A_2[n] = +2$ is

$$P_{e|A_2[n]=+2} = \frac{1}{8} \left[Q(0.2) + Q(0.4) + Q(1) + Q(1.2) + Q(2) + Q(2.2) + Q(2.8) + Q(3) \right]$$

By symmetry $P_{e|A_2[n]=-2} = P_{e|A_2[n]=+2}$, and of course $P_{e2} = P_{e|A_2[n]=+2}$.

Clearly, $P_{e1} > P_{e2}$, because the energy of constellation is higher for user 2 (sequence $A_2[n]$), which makes its contribution to the noiseless output being greater.

- c) For a two coefficients ZF equalizer designed considering only user 1, solution is given by the product of pseudo-inverse of channel matrix for user 1 and desired joint channel-equalizer response

$$\mathbf{w} = \underbrace{(\mathbf{P}_1^H \mathbf{P}_1)^{-1} \mathbf{P}_1^H}_{\mathbf{P}_1^\#} \cdot \mathbf{c}_{d1},$$

where

$$\mathbf{P}_1 = \begin{bmatrix} p_1[0] & 0 \\ p_1[1] & p_1[0] \\ 0 & p_1[1] \end{bmatrix} = \begin{bmatrix} +0.9 & 0 \\ -0.1 & +0.9 \\ 0 & -0.1 \end{bmatrix}, \quad \mathbf{c}_{d1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

After calculations, the solution is

$$\mathbf{w} = \begin{bmatrix} +1.1109 \\ +0.1219 \end{bmatrix}.$$

Now, joint response of channel $p_1[n]$ and equalizer is

$$c[n] = p_1[n] * w[n] = \sum_{k=0}^{K_w} w[k] \cdot p_1[n - k].$$

In matrix notation

$$\mathbf{c}_1 = \mathbf{P}_1 \cdot \mathbf{w} = \begin{bmatrix} +1 \\ -0.0014 \\ -0.0122 \end{bmatrix}.$$

It can be seen that residual ISI is relatively low.

Taking into account system for user 2

$$\mathbf{P}_2 = \begin{bmatrix} p_2[0] & 0 \\ p_2[1] & p_2[0] \\ 0 & p_2[1] \end{bmatrix} = \begin{bmatrix} +0.8 & 0 \\ -0.2 & +0.8 \\ 0 & -0.2 \end{bmatrix},$$

joint response of channel $p_2[n]$ and equalizer is

$$\mathbf{c}_2 = \mathbf{P}_2 \cdot \mathbf{w} = \begin{bmatrix} +0.8888 \\ -0.1246 \\ -0.0244 \end{bmatrix}.$$

To compute the probability of error, when there is a single user the equalizer output is

$$u[n] = \underbrace{c[d]}_{\text{gain}} \cdot A[n-d] + \underbrace{\sum_{\substack{k=0 \\ k \neq d}}^{K+K_w} c[k] \cdot A[n-k]}_{\text{residual ISI}} + \underbrace{\sum_{k=0}^{K_w} w[k] \cdot z[n-k]}_{\text{filtered noise } z'[n]},$$

The usual approximation for the probability of error consists in assuming that ISI and noise are independent, and therefore their energies are additive, and that both noise and ISI have Gaussian distribution. With this assumption, and taking into account that the gain factor scalates the received constellation

$$P_e \approx k \cdot Q \left(\frac{d_{min} \cdot |c[d]|}{2\sqrt{\sigma_{z'}^2 + \sigma_{ISI}^2}} \right)$$

where

$$\sigma_{z'}^2 = \sigma_z^2 \cdot \sum_{k=0}^{K_w} |w[k]|^2$$

and

$$\sigma_{ISI}^2 = E_s \cdot \sum_{\substack{k=0 \\ k \neq d}}^{K+K_w} |c[k]|^2.$$

Now, we have to take into account that there are two users and that contribution of user 2 can be seen as an ICI for user 1, and the output of the equalizer is

$$u[n] = \underbrace{c_1[d]}_{\text{gain}} \cdot A_1[n-d] + \underbrace{\sum_{\substack{k=0 \\ k \neq d}}^{K+K_w} c_1[k] \cdot A_1[n-k]}_{\text{residual ISI}} + \underbrace{\sum_{k=0}^{K+K_w} c_2[k] \cdot A_2[n-k]}_{\text{ICI user 2}} + \underbrace{\sum_{k=0}^{K_w} w[k] \cdot z[n-k]}_{\text{filtered noise } z'[n]}.$$

Therefore, assuming also that ICI term is independent of ISI and noise, and that it also has a Gaussian distribution, probability of error can be approximated as follows

$$P_{e1} \approx k \cdot Q \left(\frac{d_{min1} \cdot |c_1[d]|}{2\sqrt{\sigma_{z'}^2 + \sigma_{ISI}^2 + \sigma_{ICI}^2}} \right)$$

where

$$\sigma_{z'}^2 = \sigma_z^2 \cdot \sum_{k=0}^{K_w} |w[k]|^2,$$

$$\sigma_{ISI}^2 = E_{s1} \cdot \sum_{\substack{k=0 \\ k \neq d}}^{K+K_w} |c_1[k]|^2,$$

and

$$\sigma_{ICI}^2 = E_{s2} \cdot \sum_{k=0}^{K+K_w} |c_2[k]|^2.$$

Using the values for this exercise, $d_{min1} = 2$, $E_{s1} = 1$, $E_{s2} = 4$, $\sigma_{z'}^2 = 1.2491$, $\sigma_{ISI}^2 = 1.5 \times 10^{-4}$, $\sigma_{ICI}^2 = 3.2241$, finally we get

$$P_{e1} \approx Q(0.4727) = 0.3182.$$

- d) In this case, when the presence of user 2 is known, equations corresponding to the interference introduced by this user have to be introduced in the system. We have to remember equations corresponding to user 1

$$\mathbf{c}_1 = \mathbf{P}_1 \cdot \mathbf{w},$$

which defined the objective joint response $c_1[n] = \delta[n - d]$. But know it is also necessary to consider the presence of user 2, whose joint response is given as

$$\mathbf{c}_2 = \mathbf{P}_2 \cdot \mathbf{w}$$

and for which the desired response is $c_2[n] = 0$ for all n .

The whole system has now 6 equations, 3 for the joint response of each user, which in matrix notation becomes

$$\mathbf{c}_{dT} = \mathbf{P}_T \cdot \mathbf{w}, \text{ with } \mathbf{P}_T = \begin{bmatrix} p_1[0] & 0 \\ p_1[1] & p_1[0] \\ 0 & p_1[1] \\ p_2[0] & 0 \\ p_2[1] & p_2[0] \\ 0 & p_2[1] \end{bmatrix} = \begin{bmatrix} +0.9 & 0 \\ -0.1 & +0.9 \\ 0 & -0.1 \\ +0.8 & 0 \\ -0.2 & +0.8 \\ 0 & -0.2 \end{bmatrix}, \mathbf{c}_{dT} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solution is now given by the pseudo-inverse matrix of the new channel matrix

$$\mathbf{w} = \underbrace{(\mathbf{P}_T^H \mathbf{P}_T)^{-1} \mathbf{P}_T^H}_{\mathbf{P}_T^\#} \cdot \mathbf{c}_{dT}.$$

Operating, the solution is

$$\mathbf{w} = \begin{bmatrix} +0.6171 \\ +0.1029 \end{bmatrix}.$$

Joint response of channel $p_1[n]$ and equalizer is

$$\mathbf{c}_1 = \mathbf{P}_1 \cdot \mathbf{w} = \begin{bmatrix} +0.5554 \\ +0.0309 \\ -0.0103 \end{bmatrix},$$

and joint response of channel $p_2[n]$ and equalizer is

$$\mathbf{c}_2 = \mathbf{P}_2 \cdot \mathbf{w} = \begin{bmatrix} +0.4937 \\ -0.0411 \\ -0.0206 \end{bmatrix}.$$

Using values, $d_{min1} = 2$, $E_{s1} = 1$, $E_{s2} = 4$, $\sigma_{z'}^2 = 0.3914$, $\sigma_{ISI}^2 = 0.0011$, $\sigma_{ICI}^2 = 0.9835$, probability of error is approximated as

$$P_{e1} \approx Q(0.4735) = 0.3179.$$

Exercise 2

a) First we consider the linear block code

i) A linear code fulfill several properties. The ones that are relevant here are the following ones

- The null codeword with n zeros belongs to the code, and it is associated to the null message word with k zeros. This implies that

$$\mathbf{c}_0 = 0\ 0\ 0\ 0\ 0\ 0$$

- Any linear combination of codewords is another codeword. This means that $\mathbf{c}_2 + \mathbf{c}_4$, has to be a codeword, and this is not in the table, which means that it has to be \mathbf{c}_6

$$\mathbf{c}_6 = \mathbf{c}_2 + \mathbf{c}_4 = 1\ 1\ 0\ 0\ 1\ 1$$

This word can also be obtained from the generator matrix (obtained in next section). The code is not systematic, because neither the first three bits and last three bits of the encoded word reply the 3 associated message bits.

To obtain performance of the code, we need to know the minimum distance of the code, that can be obtained through the minimum number of ones appearing in a codeword different of the null codeword. In this case, the minimum number of ones is 3 (for instance coded words $\mathbf{c}_1, \mathbf{c}_3, \mathbf{c}_5, \mathbf{c}_7$ have 3 ones). Therefore

$$d_{min} = 3.$$

Detection capability for a linear block code is

$$d = d_{min} - 1 = 2 \text{ errors.}$$

Correction capability is

$$t = \left\lfloor \frac{d_{min} - 1}{2} \right\rfloor = 1 \text{ error.}$$

ii) To obtain the generator matrix, the relationship between message (uncoded) words of length k bits and coded words of length n bits has to be taken into account

$$\mathbf{c}_i = \mathbf{b}_i \mathbf{G},$$

which means that the elements of the generating matrix are $\mathbf{c}_4, \mathbf{c}_2$ and \mathbf{c}_1 , i.e., the ones related with uncoded words with a single one

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_4 \\ \mathbf{c}_2 \\ \mathbf{c}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

To obtain the parity check matrix, as the code is not systematic, Gauss elimination has to be used to obtain an associated matrix (expanding the same coded words, with a different message-codewords assignment) with systematic structure. In general, two options are possible, considering a systematic matrix at the beginning (\mathbf{G}'_a) or at the end (\mathbf{G}'_b), and then the relationship between generating and parity check matrices is

$$\mathbf{G}'_a = [\mathbf{I}_k | \mathbf{P}] \rightarrow \mathbf{H}_a = [\mathbf{P}^T | \mathbf{I}_{n-k}]$$

$$\mathbf{G}'_b = [\mathbf{P} | \mathbf{I}_k] \rightarrow \mathbf{H}_b = [\mathbf{I}_{n-k} | \mathbf{P}^T]$$

In this particular code, it is only possible to obtain a systematic structure at the beginning

$$\mathbf{G}'_a = \left[\begin{array}{ccc|ccc} \mathbf{g}_1 + \mathbf{g}_2 & & & & & \\ \mathbf{g}_0 + \mathbf{g}_2 & & & & & \\ \mathbf{g}_0 + \mathbf{g}_1 + \mathbf{g}_2 & & & & & \end{array} \right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right]$$

Therefore, the parity check matrix is

$$\mathbf{H} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right]$$

iii) Syndrome table is obtained from relationship

$$\mathbf{s} = \mathbf{e} \cdot \mathbf{H}^T.$$

Syndrome table for this code is:

e	s
0 0 0 0 0 0	0 0 0
1 0 0 0 0 0	1 1 0
0 1 0 0 0 0	1 0 1
0 0 1 0 0 0	0 1 1
0 0 0 1 0 0	1 0 0
0 0 0 0 1 0	0 1 0
0 0 0 0 0 1	0 0 1
0 0 1 1 0 0	1 1 1

Table starts with the pattern error with all zeros (pattern of no errors) Then patterns with a single error are introduced (vectors \mathbf{e} with a single one), and for them the associated syndromes are given by the columns of parity check matrix \mathbf{H} . This is not a perfect code, and after including all these error patterns, there is still a syndrome value that is not included in the table (1 1 1, in this case). To complete the table, erroneous patterns or two erroneous bits producing this syndrome have to be found. There are several possible erroneous patterns (the one in the table, but also 1 0 0 0 0 1, or 0 1 0 0 1 0).

Now, to decode the received words, the following procedure is used:

1.- Syndrome for received word is computed

$$\mathbf{s} = \mathbf{r} \mathbf{H}^T = 0 1 1.$$

2.- The erroneous pattern associated to this syndrome is identified in the table

$$\mathbf{e} = 0 0 1 0 0 0.$$

3.- The error detected in the received word is corrected

$$\hat{\mathbf{c}} = \mathbf{r} + \mathbf{e} = 1\ 1\ 0\ 0\ 1\ 1 = \mathbf{c}_6.$$

4.- The uncoded message word is obtained from the dictionary of the code

$$\hat{\mathbf{c}} = \mathbf{c}_6 \rightarrow \hat{\mathbf{b}} = \mathbf{b}_6 = 1\ 1\ 0.$$

b) Now we consider the convolutional code

i) Some properties of the convolutional encoder can be obtained from trellis diagram:

- Number of input ($k = 2$) and number of outputs $n = 4$ are obtained from the number of bits appearing in the first and second part, respectively, of the branch label.
- The total number of memories ($M_t = 2$) is obtained from the number of states.

It is necessary to know, on the one hand the number of memories associated to each input, and the connections with the outputs.

The number of memories associated to each input is obtained looking at the possible transitions between states. In this case, the 4 states are fully connected, which means that there is a memory for each input ($M^{(0)} = 1$ and $M^{(1)} = 1$).

In general, the value of each output in a given moment ℓ depends on the values of bits at the input of the encoder $B^{(0)}[\ell]$ and $B^{(1)}[\ell]$, that can be grouped in a vector

$$\mathbf{B}[\ell] = [B^{(0)}[\ell], B^{(1)}[\ell]]$$

and previous bits in each input, which are in the delay units defining the state

$$\psi[\ell] = [B^{(0)}[\ell - 1], B^{(1)}[\ell - 1]].$$

To know which outputs are connected to a bit in the first input $B^{(0)}[\ell]$, the simplest method is to look for the value of outputs in the following situation

$$B^{(0)}[\ell] = 1, B^{(1)}[\ell] = 0, B^{(0)}[\ell - 1] = 0, B^{(1)}[\ell - 1] = 0 \rightarrow \mathbf{B}[\ell] = [1, 0], \psi[\ell] = [0, 0]$$

which corresponds with branch o of the trellis diagram. In this case, the outputs that are connected are first and third outputs.

To know which outputs are connected to a bit in the first input $B^{(1)}[\ell]$, the simplest method is to look for the value of outputs in the following situation

$$B^{(0)}[\ell] = 0, B^{(1)}[\ell] = 1, B^{(0)}[\ell - 1] = 0, B^{(1)}[\ell - 1] = 0 \rightarrow \mathbf{B}[\ell] = [0, 1], \psi[\ell] = [0, 0]$$

which corresponds with branch n of the trellis diagram. In this case, second and fourth outputs are connected.

To know the outputs that are connected to a bit in the first memory $B^{(0)}[\ell - 1]$, the simplest method is to look for the value of outputs in the following situation

$$B^{(0)}[\ell] = 0, B^{(1)}[\ell] = 0, B^{(0)}[\ell - 1] = 1, B^{(1)}[\ell - 1] = 0 \rightarrow \mathbf{B}[\ell] = [0, 0], \psi[\ell] = [1, 0]$$

which corresponds with branch l of the trellis diagram. The only output that is connected is second output.

Finally, to know the outputs that are connected to a bit in the second memory $B^{(1)}[\ell - 1]$, the simplest method is to look for the value of outputs in the following situation

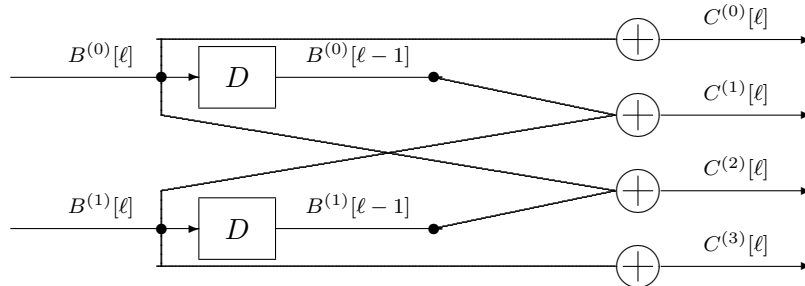
$$B^{(0)}[\ell] = 0, B^{(1)}[\ell] = 0, B^{(0)}[\ell - 1] = 0, B^{(1)}[\ell - 1] = 1 \rightarrow \mathbf{B}[\ell] = [0, 0], \psi[\ell] = [0, 1]$$

which corresponds with branch l of the trellis diagram. The only output that is connected is third output.

Combining this information, generating matrix is

$$\mathbf{G}(D) = \begin{bmatrix} 1 & D & 1 & 0 \\ 0 & 1 & D & 1 \end{bmatrix}.$$

Schematic representation of the code is given in the following picture



- ii) A simple method to obtain the encoded sequence using the trellis diagram is to represent the transmitted data as a path through the trellis, and obtain the encoded bits from the second part of the branch labels of branches associated to this path. Initial state is $[0,0]$, and the 4 blocks of 2 bits are 10, 11, 01 and 00. This means that the path is given by the following branches

$$o(00) - i(11) - b(01) - h(00),$$

and therefore the encoded sequence is

$$\mathbf{C} = 1010 \ 1011 \ 0011 \ 0010 .$$

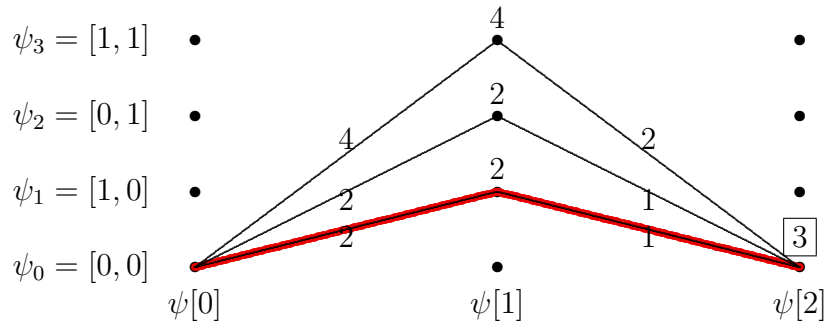
To calculate the probability of error, the minimum Hamming distance between two different encoded sequences, D_{min}^H , the number of transitions associated to this erroneous event, z , have to be obtained, and then

$$Pe \approx c \cdot \sum_{i=t}^{n \cdot z} \binom{n \cdot z}{i} \cdot \varepsilon^i \cdot (1 - \varepsilon)^{n \cdot z - i}$$

- D_{min}^H : minimum Hamming distance between two different encoded sequences.
- z : length of erroneous event with minimum distance D_{min}^H
- $t = \left\lfloor \frac{D_{min}^H - 1}{2} \right\rfloor$ (correction capability over $n \times z$ bits)
- ε : bit error rate during transmission (BER)

Therefore, it is necessary to compute D_{min}^H and z , and this will be done through the trellis comparing with the all zeros encoded sequence, making use of the linearity of the convolutional encoder.

Figure shows the branch metric for this computation, highlighting the path that is associated with the erroneous event of minimum distance (in fact, there are two, path going through state ψ_2 accumulates the same metric).



Therefore minimum distance is $D_{min}^H = 3$, and moreover $t = 1$, $z = 2$, $n = 4$.

iii) Decoding is done by applying Viterbi algorithm.

This picture shows the branch metric for application of Viterbi algorithm, highlighting survival paths (broader lines) and the final solution (double line).

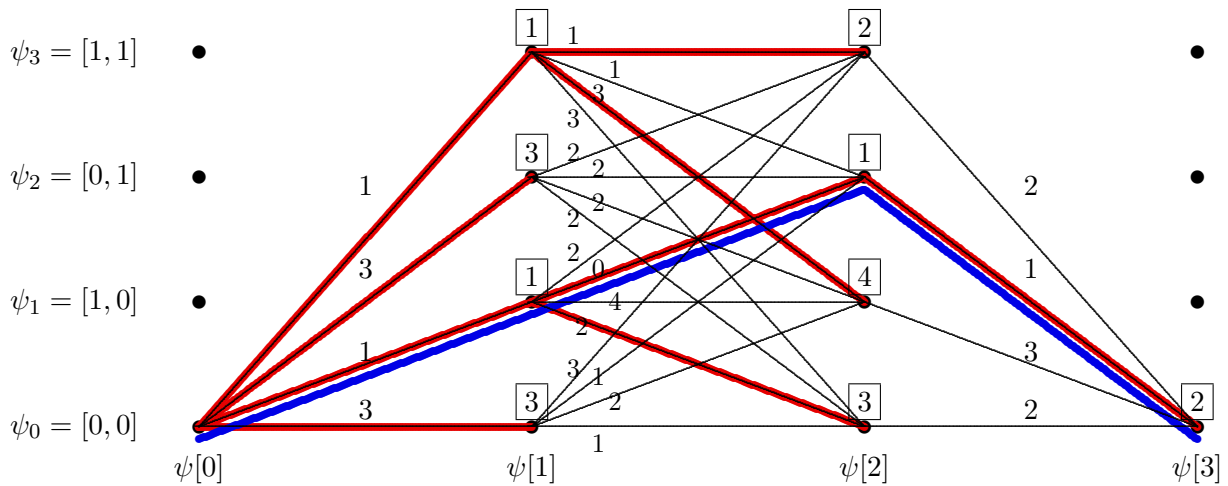


Table contains the accumulated metrics for each state, highlighting the one associated to survival paths. Decoded bits are also shown

	$\ell = 1$	$\ell = 2$	$\ell = 3$	
ψ_3	$\boxed{1}$	$\boxed{2}/5/3/6$		$\hat{B}^{(0)}[0] = 1$
ψ_2	$\boxed{3}$	$2/5/\boxed{1}/4$		$\hat{B}^{(1)}[0] = 0$
ψ_1	$\boxed{1}$	$\boxed{4}/5/5/5$		$\hat{B}^{(0)}[1] = 0$
ψ_0	$\boxed{3}$	$4/5/\boxed{3}/4$	$4/\boxed{2}/7/5$	$\hat{B}^{(1)}[1] = 1$

Solution is

$$\hat{B}[m] = 1\ 0\ 0\ 1$$