



AUTOMATA THEORY AND FORMAL LANGUAGES

UNIT 7: TURING MACHINE





- Definition of Turing Machine
- Variations of Turing Machines
- Universal Turing Machine
- Additional issues





- **Definition of Turing Machine**
- Variations of Turing Machines
- Universal Turing Machine
- Additional issues





Definition of Turing Machine

Alan Turing

- 23 June 1912 - 7 June 1954
- Contributions:
 - Mathematics,
 - Cryptanalysis,
 - Logic,
 - ...
 - Artificial Intelligence.



<http://www.alanturing.net>

<http://www.mathcomp.leeds.ac.uk/turing2012/>



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Definition of Turing Machine

Alan Turing

- <http://www.alanturing.net>
- http://en.wikipedia.org/wiki/Alan_Turing
- Bibliography:
 - "On Computable Numbers, with an Application to the Entscheidungsproblem" 1936
 - Turing, A.M. Computing machinery and intelligence. Mind, 59, 433-460. 1950
 - "I propose to consider the question, "Can machines think?"
 - http://en.wikipedia.org/wiki/Computing_machinery_and_intelligence
 - <http://blog.santafe.edu/wp-content/uploads/2009/05/turing1950.pdf>
- Charles Petzold. The Annotated Turing: A Guided Tour Through Alan Turing's Historic Paper on Computability and the Turing Machine. Wiley. 2008.
www.theannotatedturing.com





Can machines do everything?

We have studied:

- Simple languages.
- Application of simple languages to solve restricted problems:
 - Analysis.
 - Pattern recognition.
 - Syntax analysis.

In this unit, we want to answer:

- Which languages can be defined by means of any computational device?
- Which problems can be solved/computed?





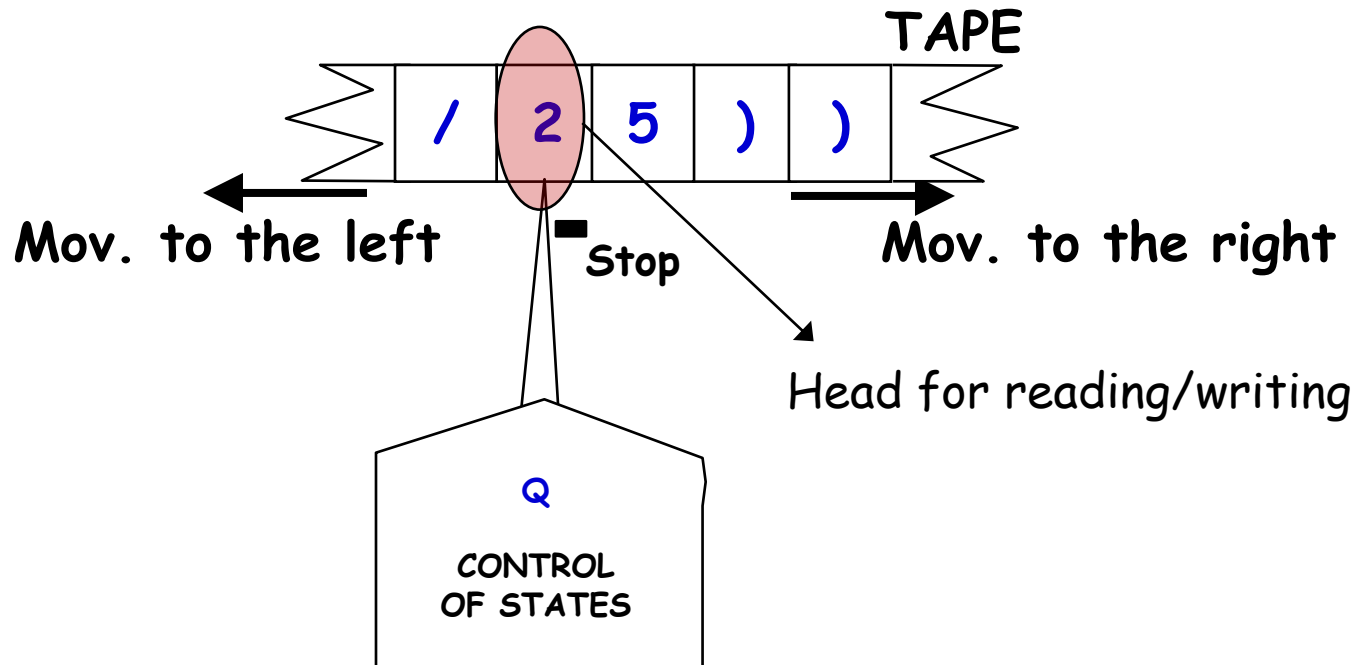
Can machines do everything?

- Which is the solution?
 - He says that she always say the truth.
 - She says that he always lies (he never says the truth).
- Can a computer answer to any possible question in a dialog system?
- Is a grammar ambiguous?
- Is there a solution for $X^N+Y^N=Z^N$, with $N>2$?



What is a Turing Machine?

- Mechanical device:
 - Infinite tape divided into cells with a head for reading/writing.
 - This head can be moved from to left or right or stay in the same cell.





Turing Machine: Operation

Operations that a TM carries out:

- Being in a state p and reading a symbol in the cell on which the R/W head is, it carries out three actions:
 - Transits to a new state.
 - Writes a new symbol in the tape, in the same cell where the current symbol has been read. This symbol replaces the previously read (unless it is the same).
 - Moves the R/W head to the left, to the right or stops in the same position.



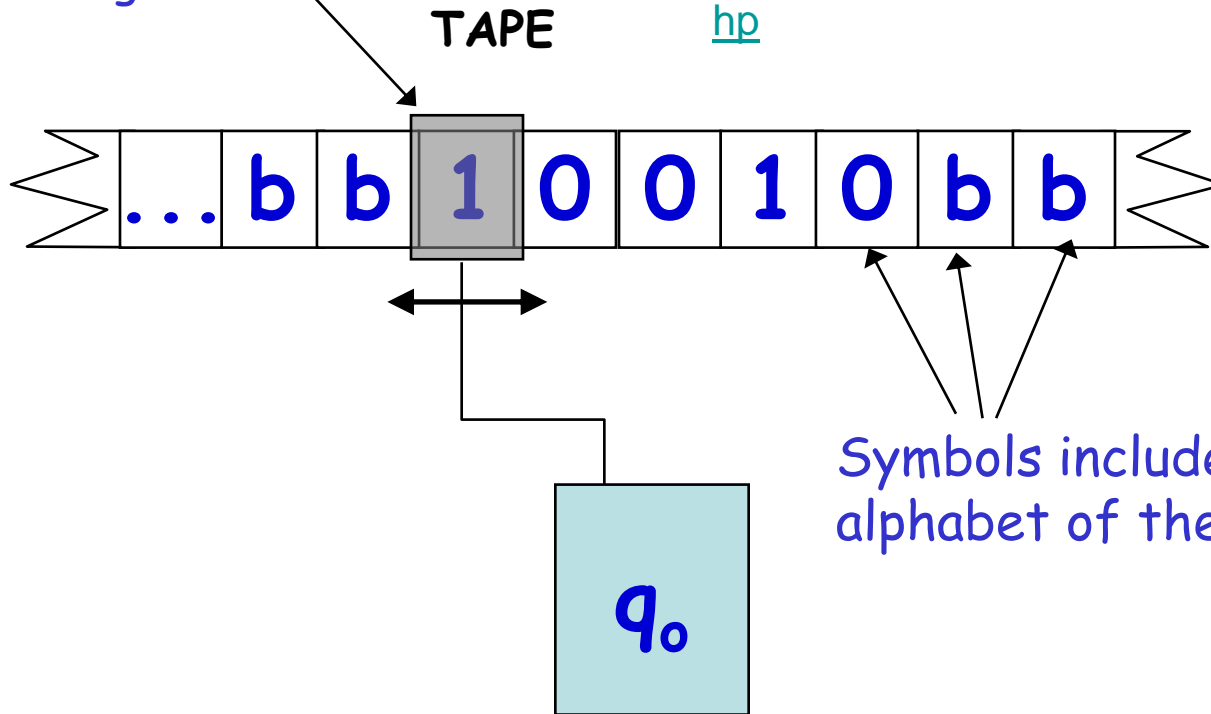
Variations of Turing Machines

Turing Machine: Initial Situation

Initial position of
the reading head

Example:

<http://www.aturingmachine.com/index.php>



Symbols included in the
alphabet of the tape, Γ

Initial State



Turing Machine: Operation

Characteristics of the Tape:

- Infinite tape.
- It can contain a character in each cell.
- It can be read.
- It can be written.
- Initially it is considered with infinite blank symbols to the right and left of the word.
- It can be moved to the left or right (a cell each time) or not move.





Turing Machine: Operation

- There are many variants of Turing Machines...
- ... but they all are equivalent.
- The tape is one-dimensional and **infinite** by both sides.
- Initially:
 - The tape contains the word, and the rest of elements of the tape (left and right of the word) are the blank symbol (b or \square).
 - At the beginning, the R/W head is located on the left-most element of the word.





Turing Machine: Formal Definition

- Septuple: $(\Sigma, \Gamma, b, Q, q_0, f, F)$, where:
 - Γ : alphabet of symbols in the tape
 - $\Sigma \subset \Gamma$: alphabet of input symbols
 - $b \in \Gamma, b \notin \Sigma$: is the blank symbol (the only symbol allowed to be in the tape infinitely at any step during the computation). It indicates empty cell (\square).
 - Q : set of states (finite).
 - $q_0 \in Q$: initial state.
 - $F \subset Q$: set of final states.
 - f : transition function

$$Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

L or -: left
R or +: right
S or =: stay



Turing Machine: Transition Function

f: Transition function, table with double input

$$Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

L: mov. to the left
R: mov. to the right
S: stop

$\downarrow Q / \Gamma \rightarrow$	Symbol	Symbol
Input	State, symbol, movement	...
Input	State, symbol, movement	
...	...	

Empty cells in the table:

- Transitions that are NOT possible \rightarrow The machine stops.



Turing Machine: Example

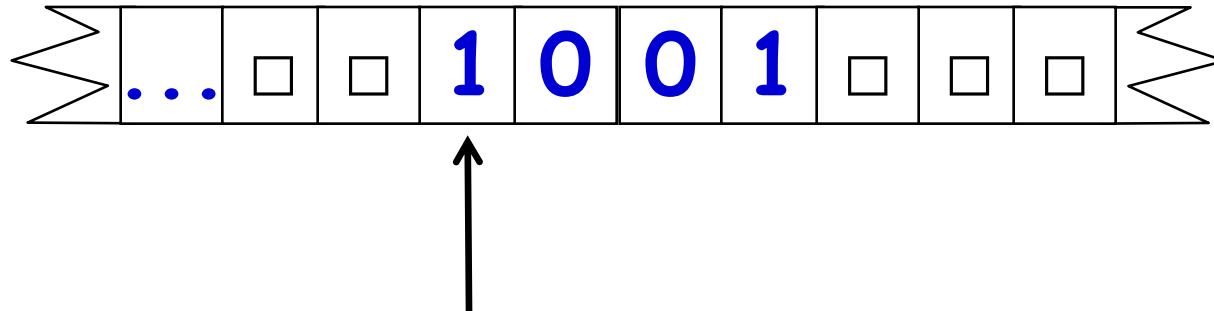
$\mathcal{M} = (\Sigma = \{0, 1\}, \Gamma = \{0, 1, \square\}, \square, Q = \{q_0, q_1, q_F\}, q_0, F = \{q_F\}, f)$

f	0	1	\square
-> q_0	$(q_0, 0, +)$	$(q_1, 1, +)$	$(q_F, 0, =)$
q_1	$(q_1, 0, +)$	$(q_0, 1, +)$	$(q_F, 1, =)$
* q_F			



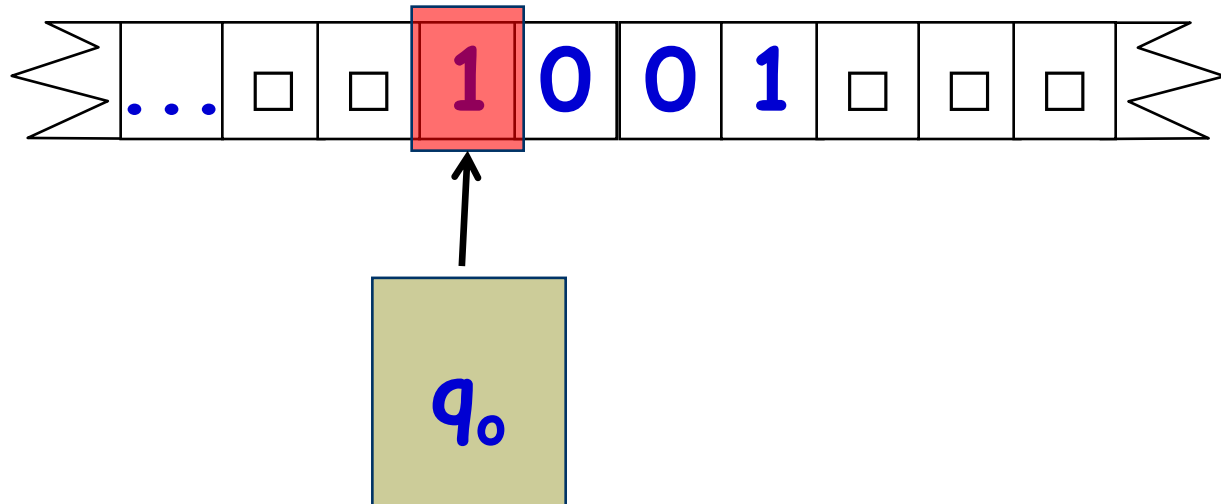
Turing Machine: Example

f	0	1	□
-> q0	(q0, 0, +)	(q1, 1, +)	(qF, 0, =)
q1	(q1, 0, +)	(q0, 1, +)	(qF, 1, =)
* qF			



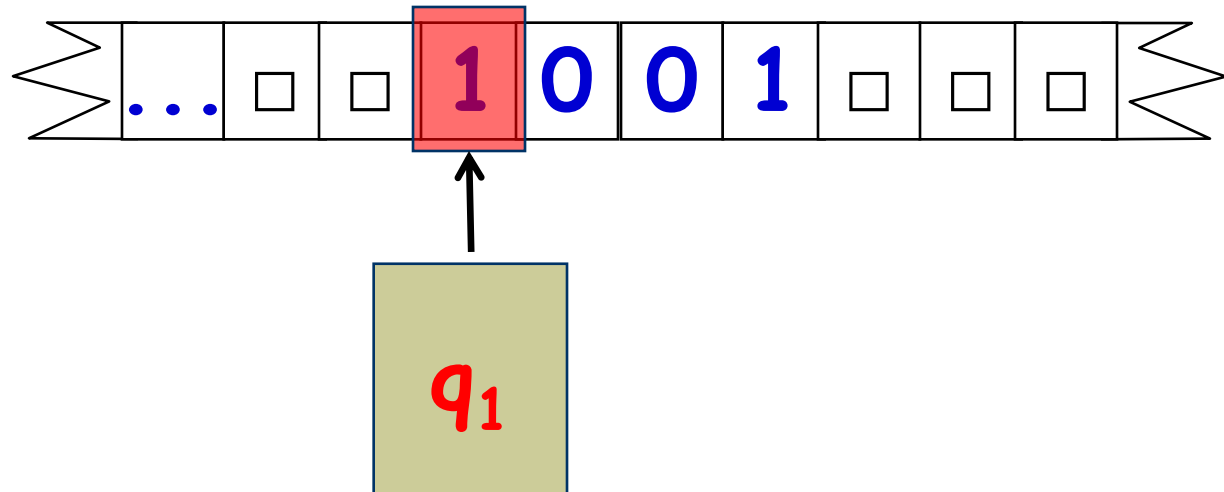
Turing Machine: Example

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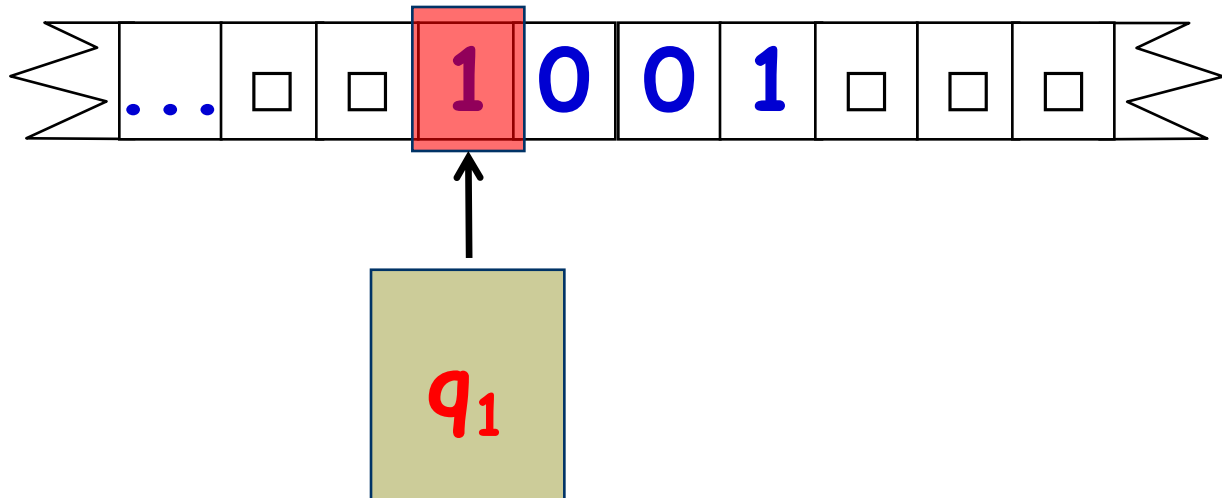
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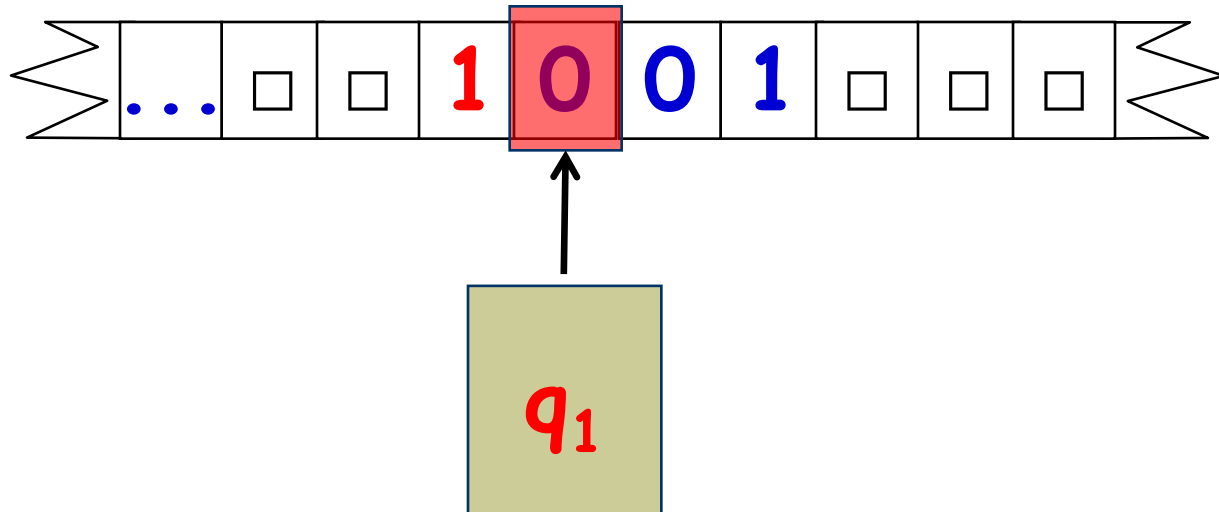
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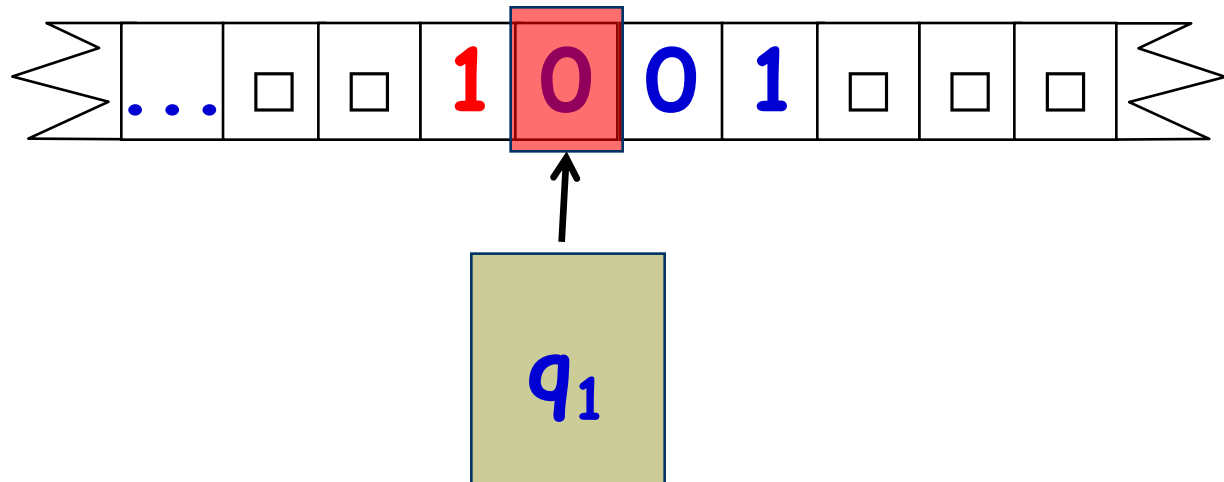
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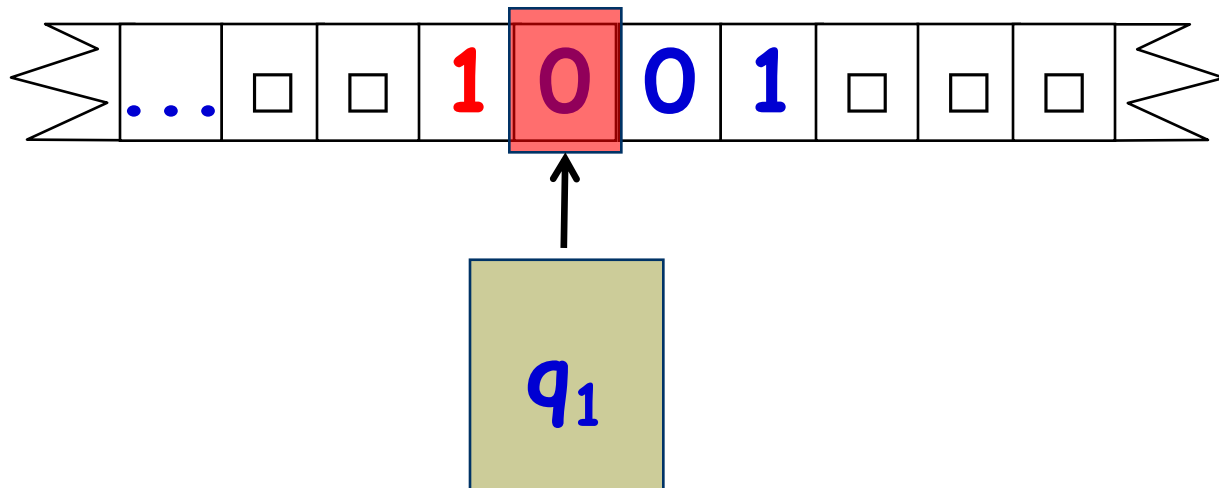
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-> q0	(q0, 0, +)	(q1, 1, +)	(qF, 0, =)
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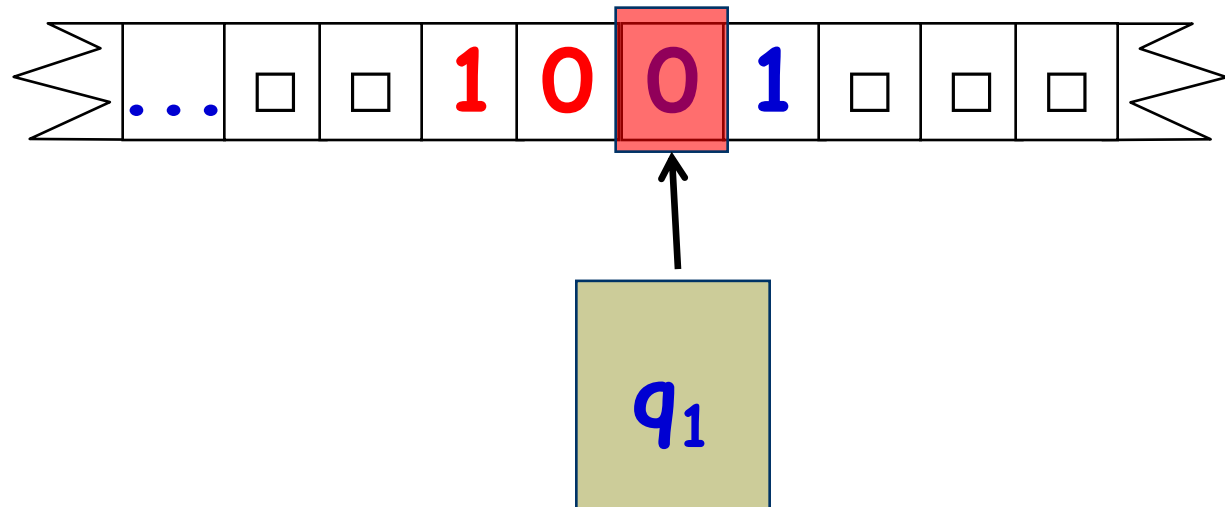
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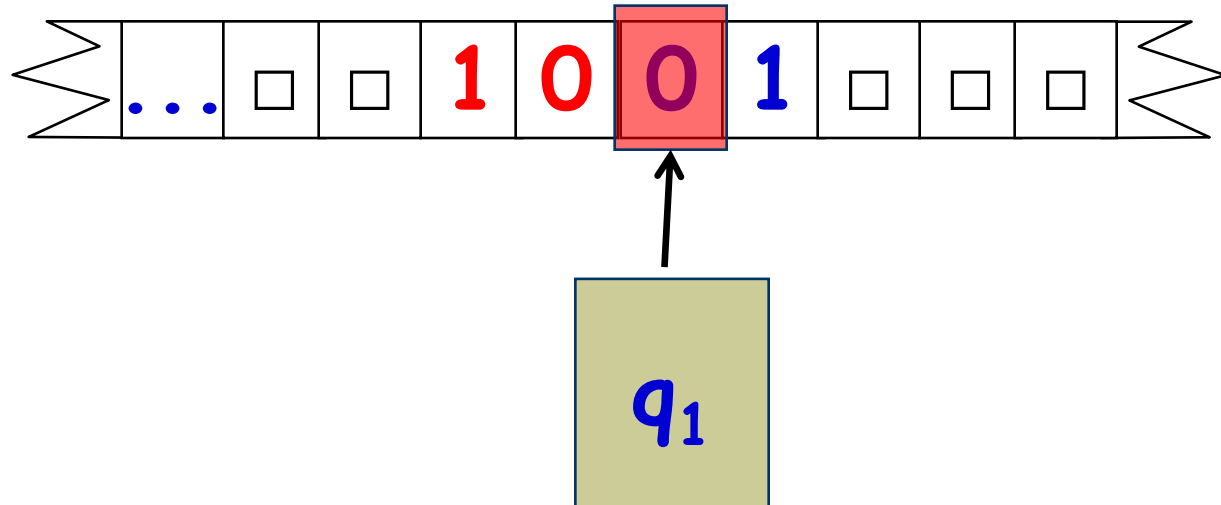
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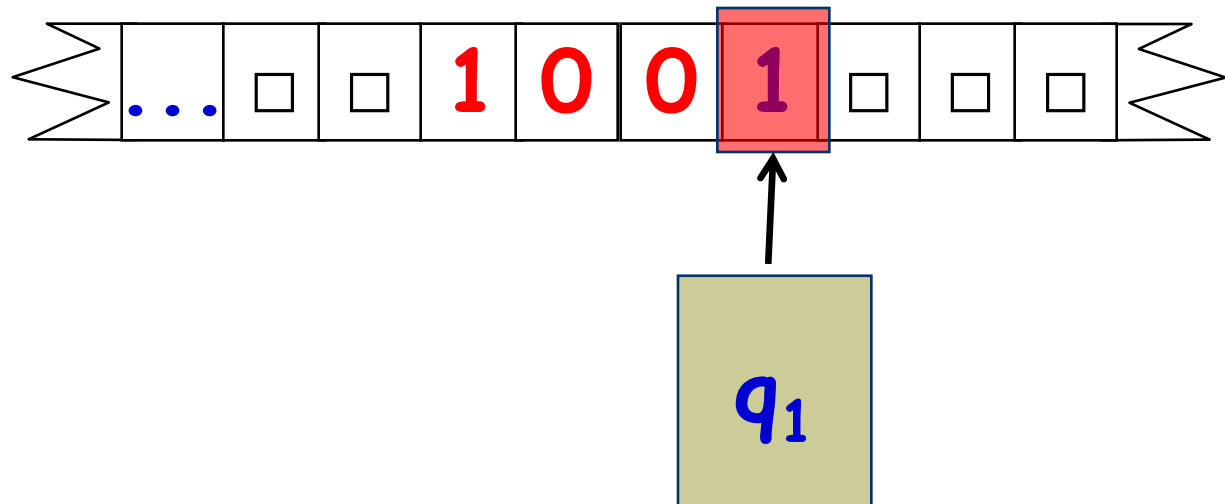
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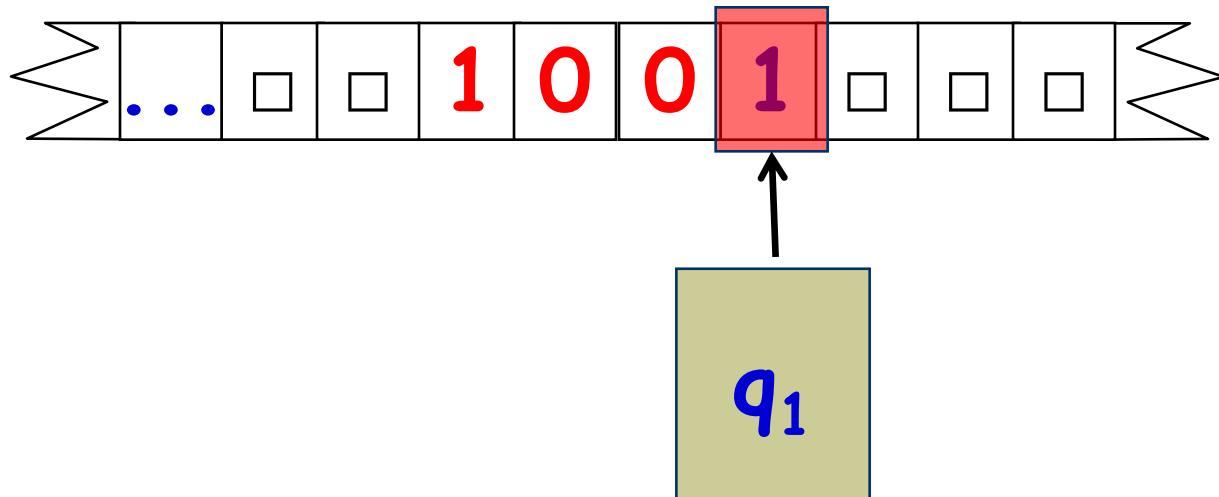
Turing Machine: Example

†	0	1	□
-> q0	(q0, 0, +)	(q1, 1, +)	(qF, 0, =)
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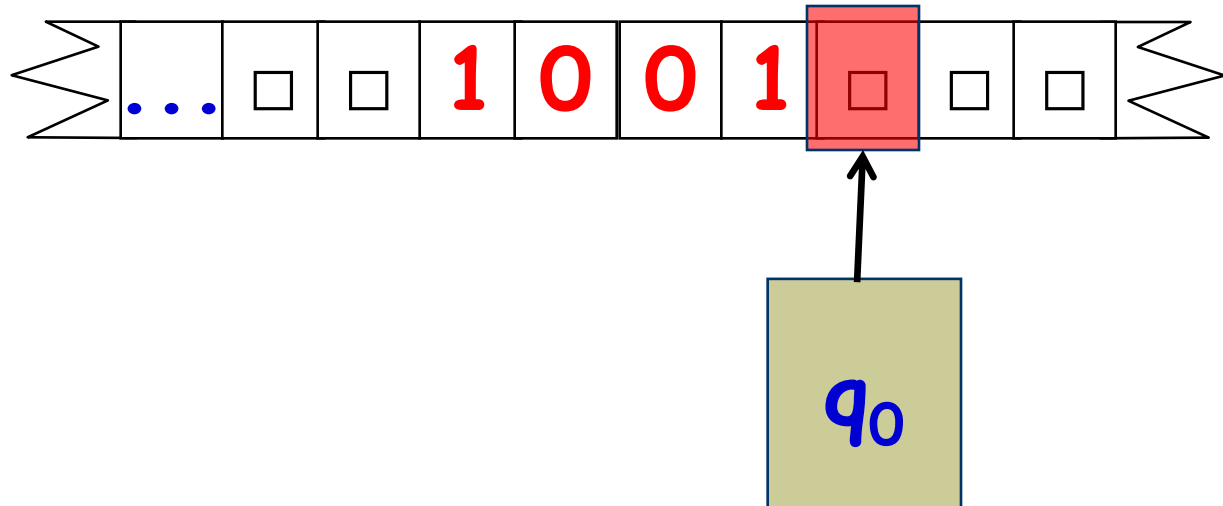
Turing Machine: Example

f	0	1	□
-> q0	(q0, 0, +)	(q1, 1, +)	(qF, 0, =)
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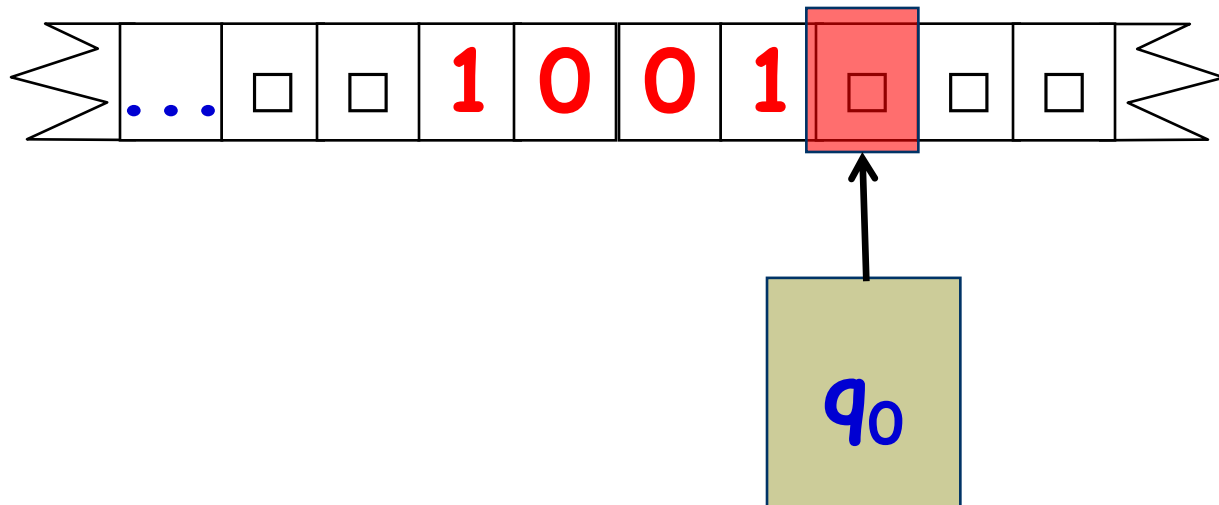
Turing Machine: Example

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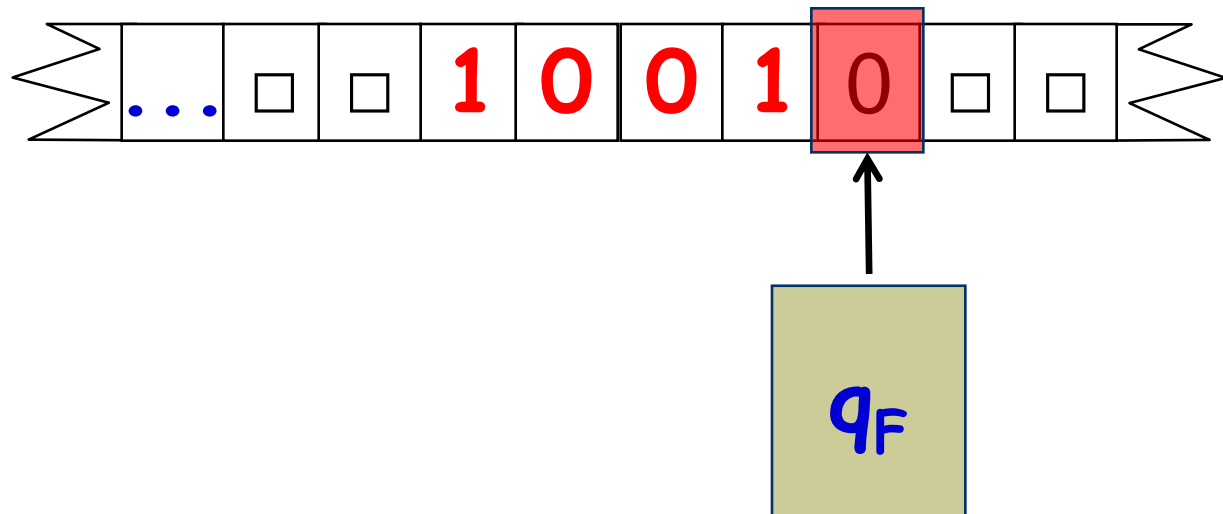
Turing Machine: Example

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Turing Machine: Example

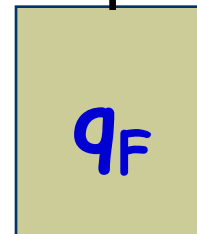
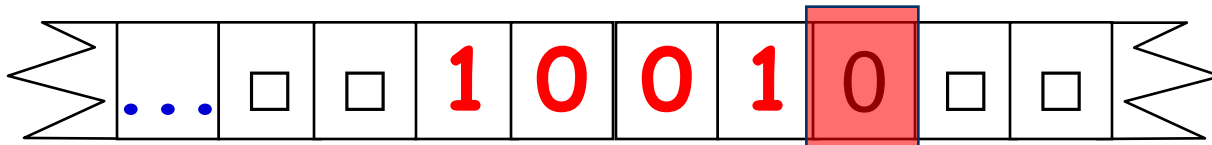
T	U	I	□
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* qF			



Turing Machine: Example

$\rightarrow q_0$	$(q_0, 0, +)$	$(q_1, 1, +)$	$(q_F, 0, =)$
q_1	$(q_1, 0, +)$	$(q_0, 1, +)$	$(q_F, 1, =)$
$* q_F$			

0 \rightarrow even number of 1s
1 \rightarrow odd number of 1s



Situations without transitions \rightarrow Stop



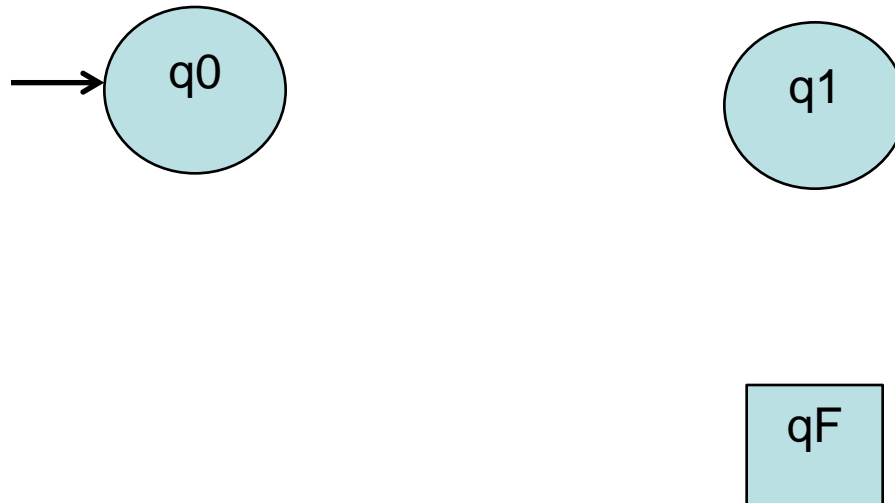
Turing Machine: Graph Representation

- The transition function can be described using also a diagram with states, i.e. a graph in which:
 - Nodes represent states.
 - Arches represent transitions between states.
 - Each arch is labelled including the requisites and effects of each transition (initial symbol, symbol that is rewritten, and direction to move the input header).



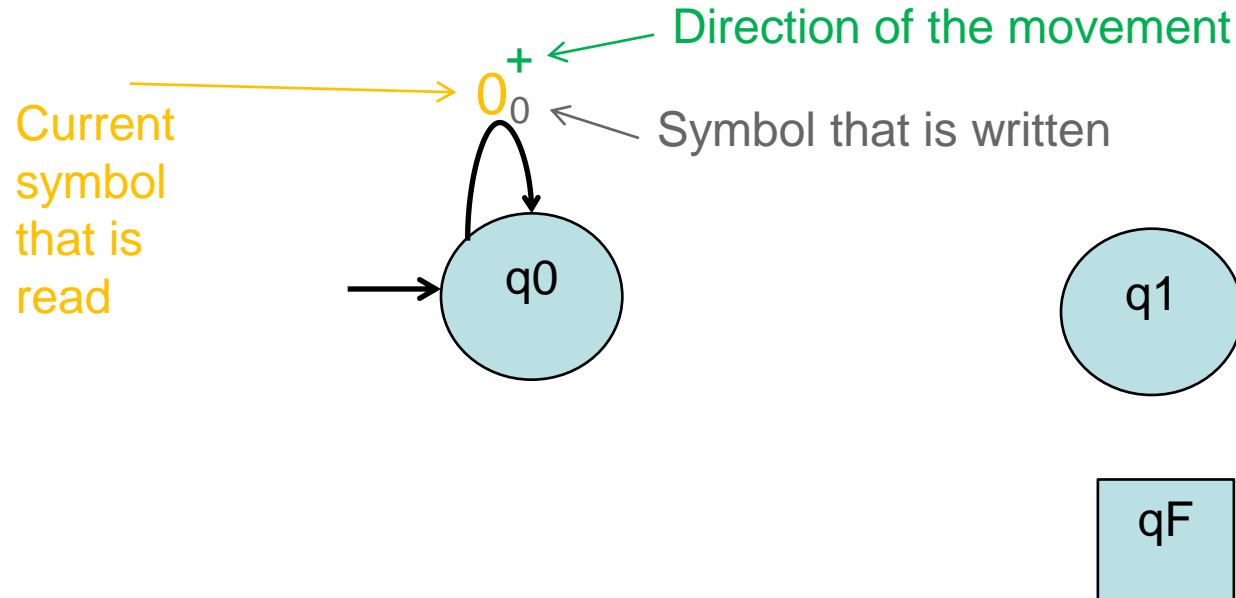
Turing Machine: Graph Representation

	T	U	L	□
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* qF				



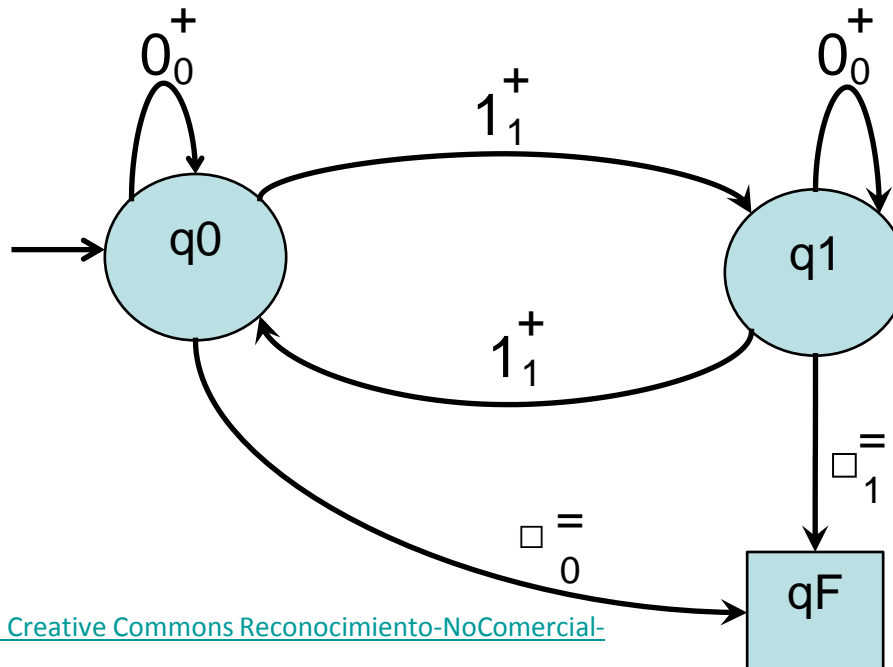
Turing Machine: Graph Representation

$\rightarrow q_0$	$(q_0, 0, +)$	$(q_1, 1, +)$	$(q_F, 0, =)$
q_1	$(q_1, 0, +)$	$(q_0, 1, +)$	$(q_F, 1, =)$
$* q_F$			



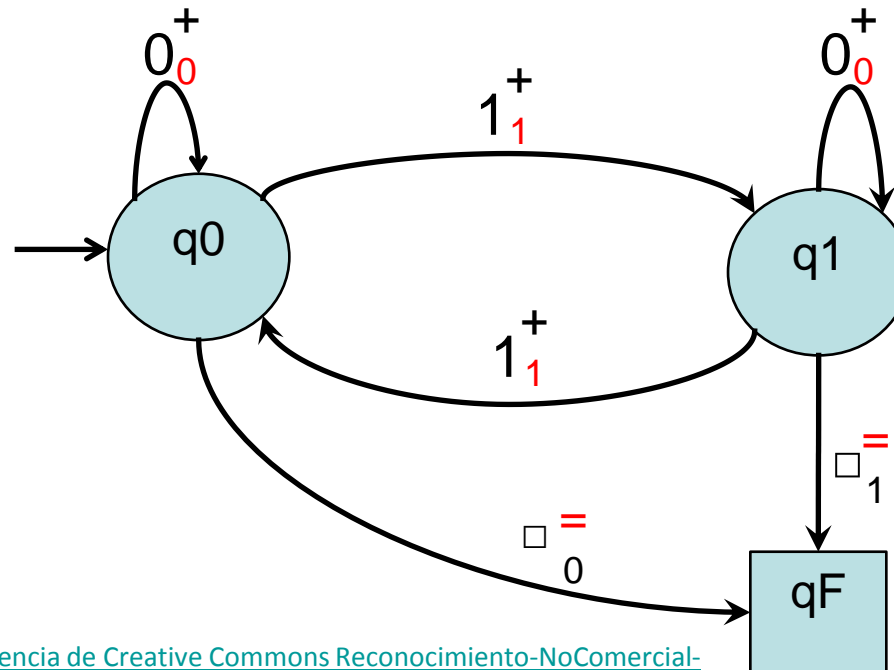
Turing Machine: Graph Representation

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-> q0	(q0, 0, +)	(q1, 1, +)	(qF, 0, =)
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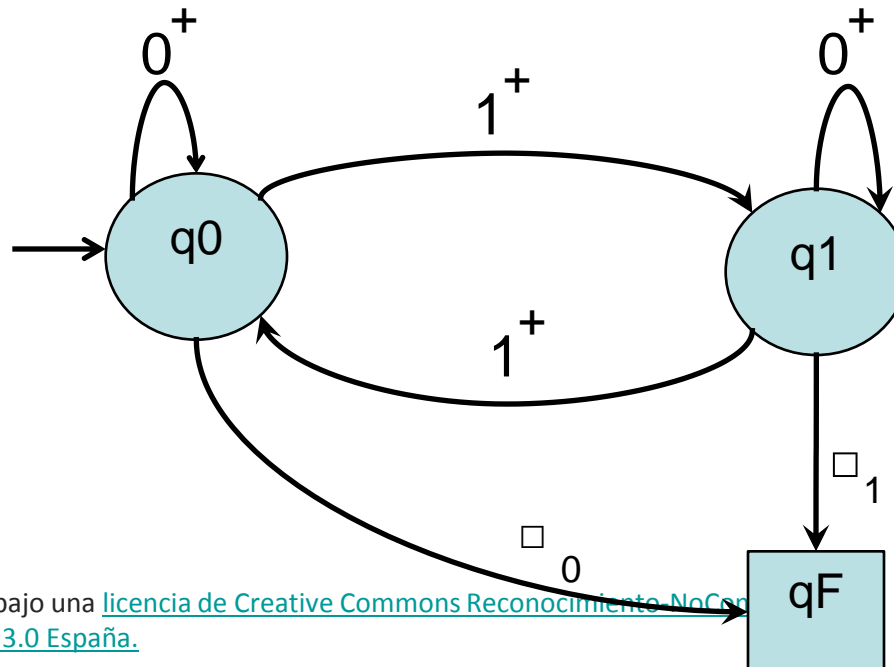
Turing Machine: Graph Representation

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Turing Machine: Graph Representation

f	0	1	□
-> q0	(q0, 0, +)	(q1, 1, +)	(qF, 0, =)
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* qF			





Turing Machine and Languages

- Turing Machine as TRANSDUCER:
 - It modifies the content of the tape,
 - Examples: TM that replaces digits by zeros.

TM that appends a parity bit to the input.

- Turing Machine as RECOGNIZER:
 - TM that recognizes a language.
 - TM that accepts a language.





Turing Machine and Languages

- Turing Machine as TRANSDUCER:
 - **Objective: transform the input (→ Provide the result of an operation).**
 - It verifies:
 - If the input is well-formed, it must finish in a final state.
 - If the input is NOT well-formed, it must finish in a nonfinal state (shows an error in the input word).





Turing Machine and Languages

- Turing Machine as RECOGNIZER
 - **Objective: Decide if the input string is valid or not, following a specific criterion.**
- Two main concepts: RECOGNIZE, ACCEPT
 - A TM **RECOGNIZES** a language L , if for any input in the tape, w , it stops in a final state iff $w \in L$.
 - A TM **ACCEPTS** a language L if, when analyzing a word w , it stops in a final state iff $w \in L$.
 - If the word does not belong to the language, the TM does not need to stop.





- Definition of Turing Machine
- **Variations of Turing Machines**
- Universal Turing Machine
- Additional issues





Variations of Turing Machines

- We have defined a generic Turing Machine.
- The definition of TM admits many variations, but having the same computation capacity, i.e. **all the variations are equivalent.**
- Two Turing Machines, TM_1 y TM_2 are equivalent **if both carry out the same action for all their inputs.**
 - What does equivalent mean for a Turing Machine?
 - ✓ **TM as transducers:** for each possible input, at the beginning of the process, **at the end of this the contents of the tape must be the same.**
 - ✓ **TM as recognizers:** both TM **accepts the same words.**
 - ✓ If for some input w , one TM does not stop, the second one will not stop for such input w .





Variations of Turing Machines

- Using the generic TM it is possible to impose restrictions, without they suppose limitations in the computation capacity.
- These restrictions can be imposed on:
 - **The alphabet of the tape: binary TM.** $\Gamma = \{0, 1\}$. (~~No $\Sigma = \{0, 1\}$ y $\Gamma = \{0, 1, b\}$~~).
 - **The structure of the tape: Limited TM.**
 - E.g.: infinite tape only by the right (limited by the left using the input word).
 - **Movements (to write, move and change the state)**
 - E.g.: not to allow that the head remains quiet; not to allow that it writes and changes simultaneously of state; etc.





Turing Machine with Binary Alphabet

Theorem:

Given a generic TM , there is an equivalent TM with binary alphabet in the tape, $\Gamma = \{0,1\}$

Turing Machine with Binary Alphabet

$$M \rightarrow M_{(2)}$$

IMPORTANT: it is $\Gamma = \{0,1\}$ and $\Sigma \subseteq \Gamma$, not ~~$\Sigma = \{0,1\}$ and $\Gamma = \{0,1,b\}$~~

Different descriptions in the bibliography

(According to section 2.2.3, Alfonseca 2007)





Turing Submachines (Subroutines)

- Same concept that functions, methods, procedures (subprograms).
- A TM can, during its execution, invoke to another one.
- **DEFINITION:**
 - A submachine of Turing is a TM that can receive arguments (i.e. symbols) to carry out a specific task specify, and can be invoked with another TM.





Turing Submachines (Subroutines)

- When a TM M_1 invokes another TM M_2 with an argument σ , the parameter in the definition of the submachine is replaced by the symbol by means it has been invoked: $M_2(\sigma)$.
 - $M_2(\sigma)$ is executed:
 - Using the same tape that M_1 .
 - From the initial state of M_2 .
 - With the R/W head of M_2 on the same position where it was in M_1 .





Turing Submachines (Subroutines)

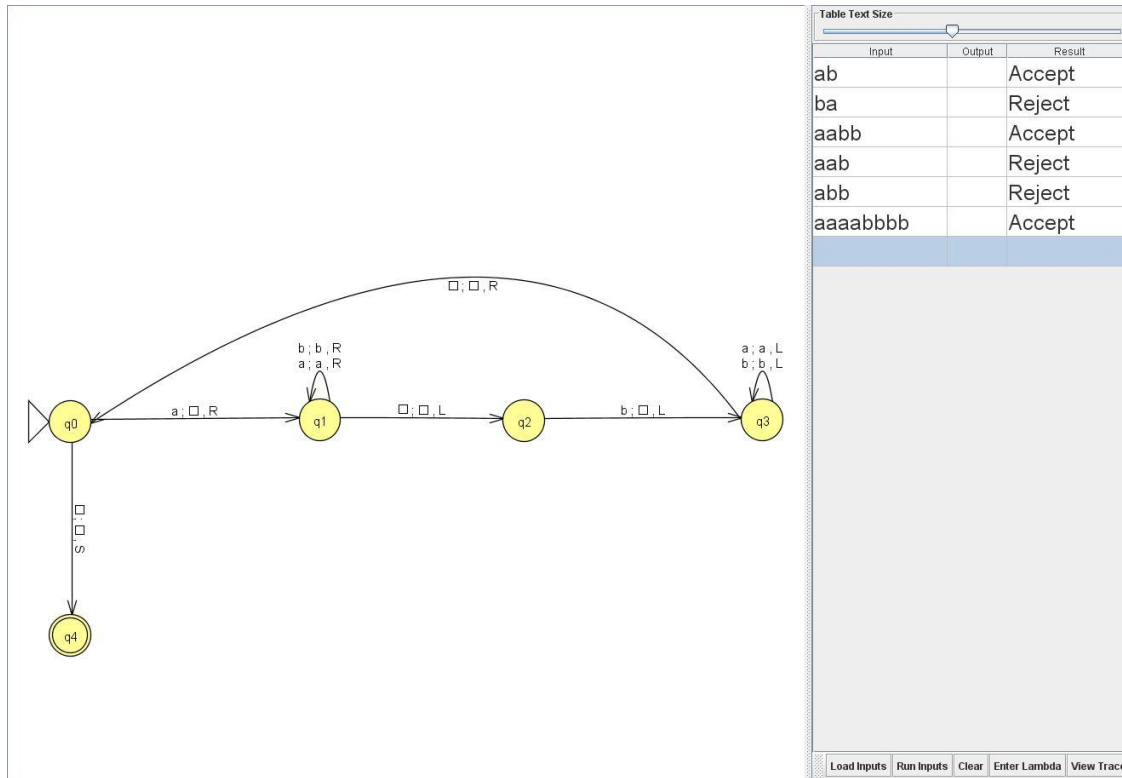
- To use Turing Submachines does not mean that the computability capacity (calculation) of TM is increased.
- Example:

- Design a TM (using submachines) that recognizes the words of the language $L = \{a^n b^n, n \geq 0\}$



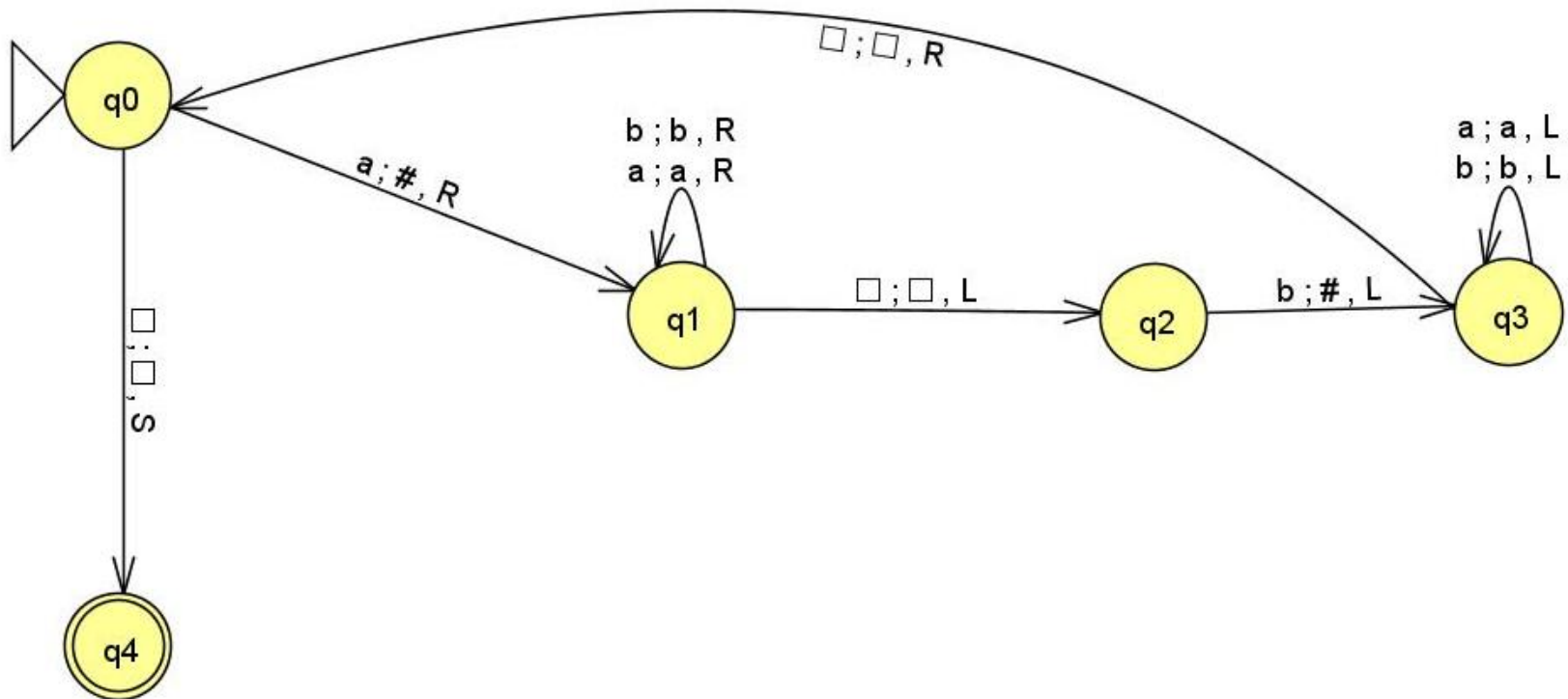
Turing Submachines (Subroutines)

Example: $L = \{a^n b^n, n \geq 0\}$



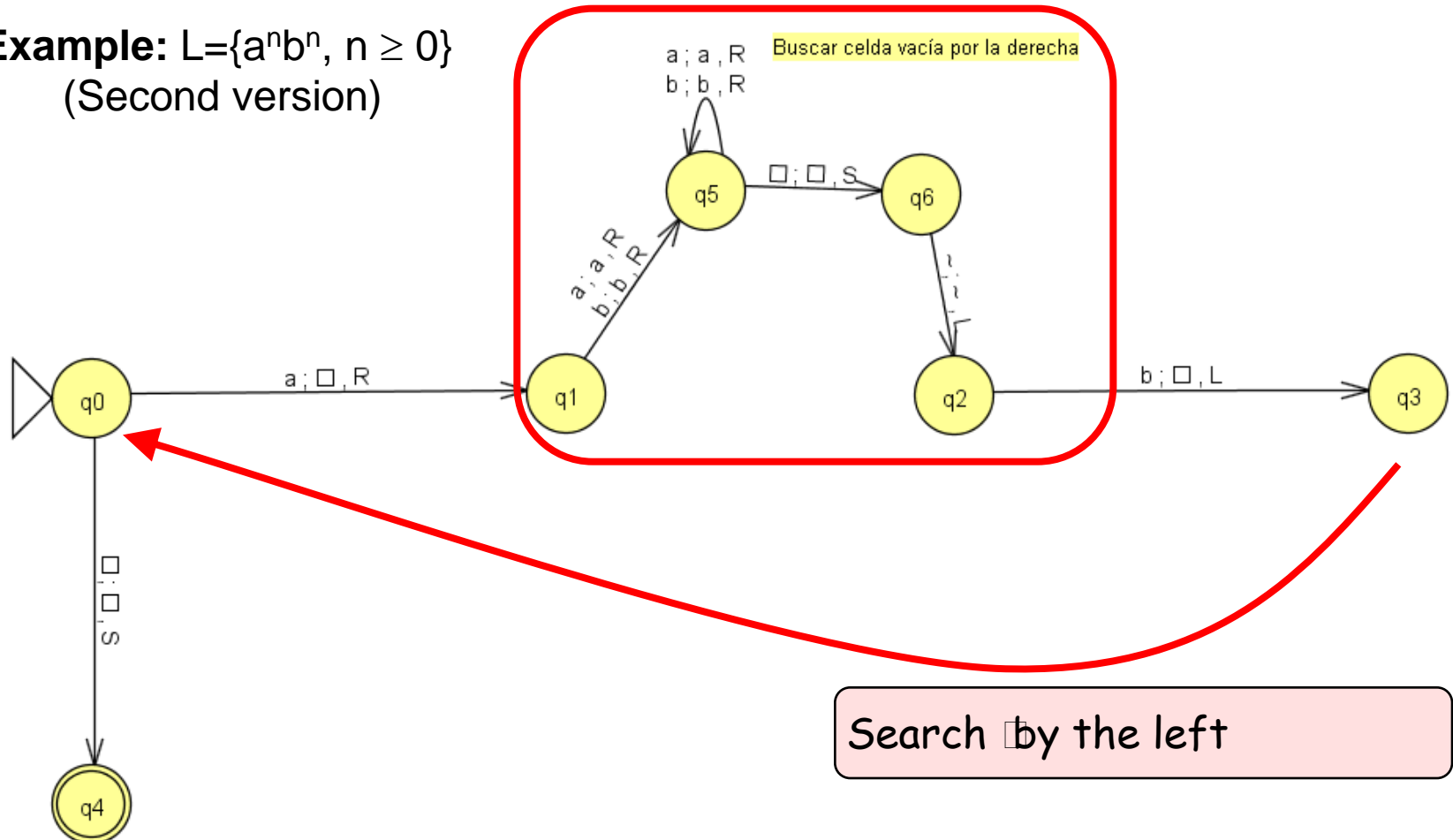
Turing Submachines (Subroutines)

Example: $L = \{a^n b^n, n \geq 0\}$



Turing Submachines (Subroutines)

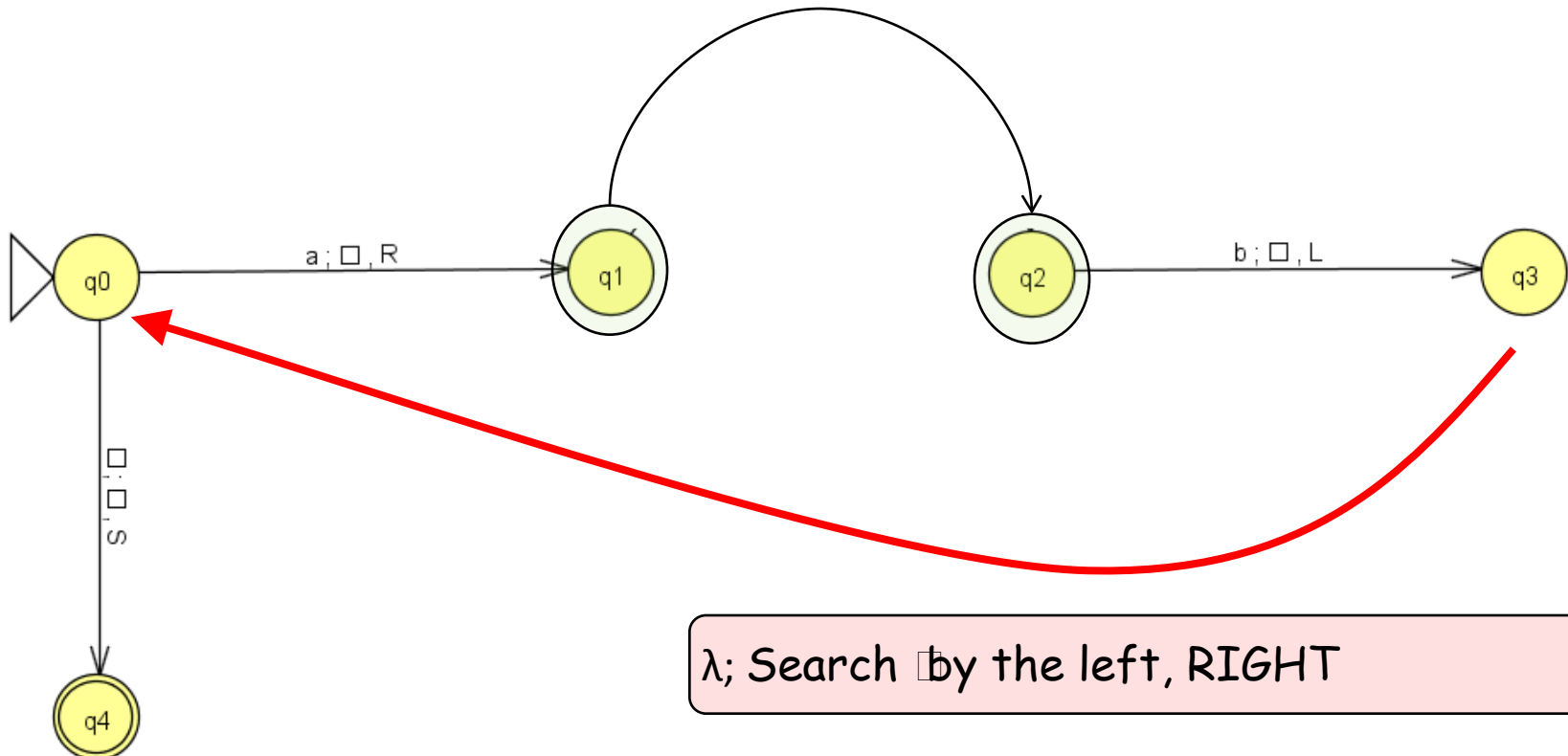
Example: $L = \{a^n b^n, n \geq 0\}$
(Second version)



Turing Submachines (Subroutines)

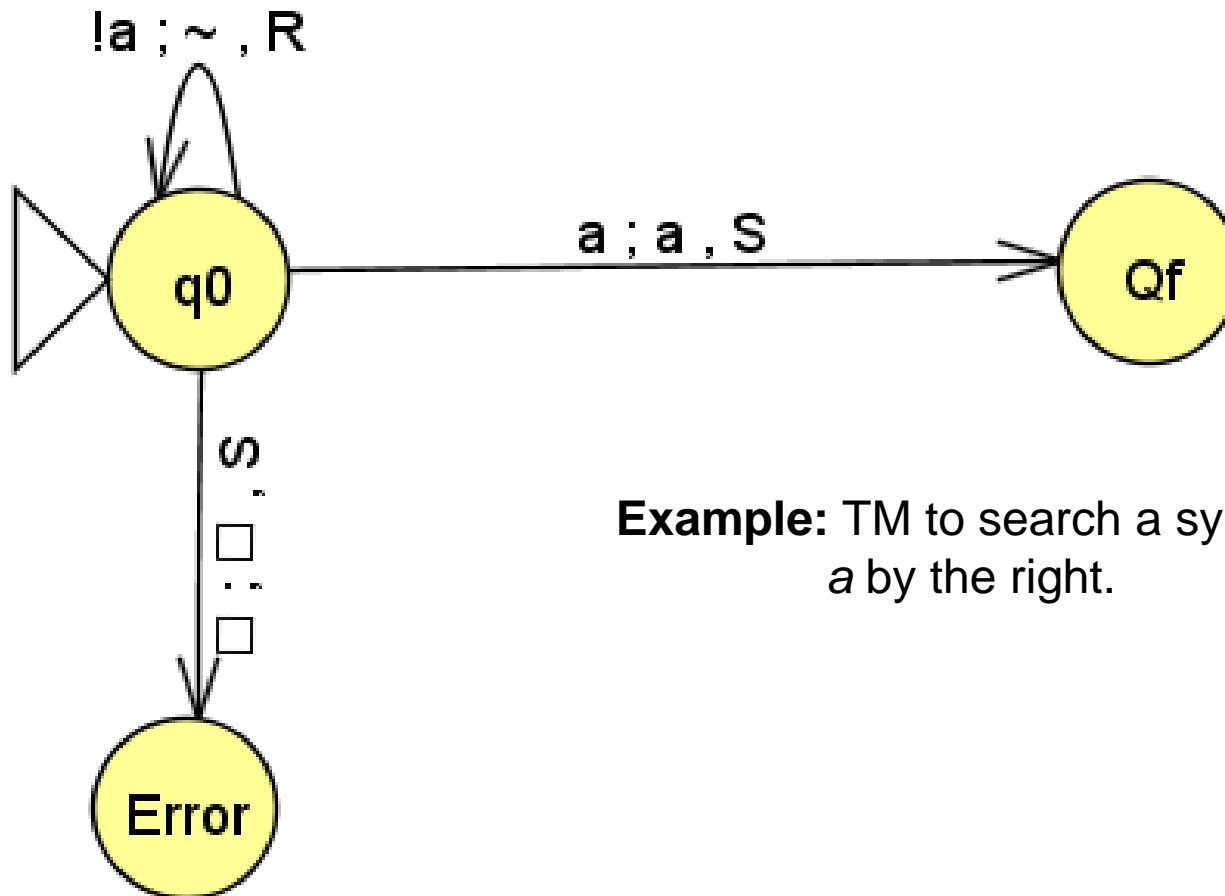
Example: $L = \{a^n b^n, n \geq 0\}$
(using submachines)

λ ; Search \square by the right, LEFT



λ ; Search \square by the left, RIGHT

Turing Submachines (Subroutines)



Example: TM to search a symbol a by the right.



- Definition of Turing Machine
- Variations of Turing Machines
- **Universal Turing Machine**
- Additional issues





Universal Turing Machine

- It is possible to define a TM that simulates the execution of EVERY TURING MACHINE.

This machine is called

UNIVERSAL TURING MACHINE (U)





Universal Turing Machine

- **Universal Turing machine: U**
 - Input:
 - Description of the TM.
 - Input string (word): w .
 - What does $U(TM, w)$ simulate?
 - It simulates the operation of TM, transition by transition.
 - If TM accepts w , then U stops in a final state for that word.
 - If TM does not accept w , U does not stop or it stops in a non-final state for this word.





Universal Turing Machine

- **Universal Turing Machine: U**
 - It simulates a TM.
 - What does it happen if TM never stops?





Universal Turing Machine: Undecidability

- HALTING PROBLEM
 - Remember: *“The output of the machine—i.e., the solution to a mathematical query—can be read from the system once the TM has stopped.”*
 - However, in the case of Gödel’s undecidable propositions, the machine would never stop, and this became known as the “halting problem.”
 - It is not possible to decide if, given an input word w , the machine will stop or will be always running.
 - There is not a TM that answers this question.





- Definition of Turing Machine
- Variations of Turing Machines
- Universal Turing Machine
- **Additional issues**





- The solution of the problem is the transition table, but this is not enough to complete an exercise.
- It is essential to think and describe the algorithm before to be implemented, as well as the behavior of each one of the states.
- It is precise to evaluate the machine with a significant set of inputs. There are maybe “difficult” cases that have to be considered.
- Many TM operates on input tapes with a specific configuration, that normally we do not verify.





- Any solution is not valid, you must think the algorithm very well and verify that it works for every case.
- If TM must operate as if it had “memory”, additional states can be incorporated (in more complicated machines, the input data can be written in specific positions of the tape).
- If a TM must remember positions within the tape, they usually use additional markers.
- If the task can be divided in several sequential subtasks (or subprogram), then do it.

