Bachelor in Aerospace Engineering

Universidad Carlos III de Madrid
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Department of Continuum Mechanics and Structural Engineering

Aerospace Structures

Chapter 2. Bending, shear and torsion of thin-walled beams

Bending and shear of open and closed, thin-walled beams (I)

CHAPTER 2. Bending, shear and torsion of thin-walled beams

Bending and shear of open and closed, thin-walled beams (I)

OBJECTIVES

- State the displacement field the cross-section of a in a thin-walled beam
- Calculate normal stresses field in open and closed cross-sections
- Calculate normal stresses field in multi-cell cross-sections

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

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CHAPTER 2. Bending, shear and torsion of thin-walled beams

Bending and shear of open and closed, thin-walled beams (I)

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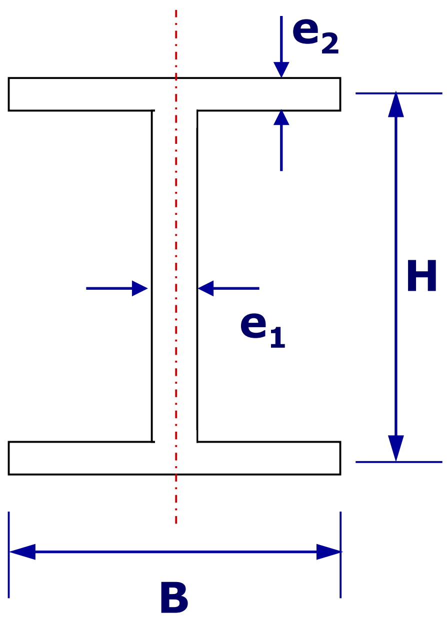
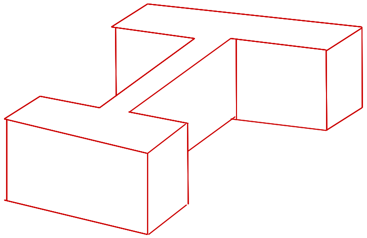
Cross-section of a thin-walled beam



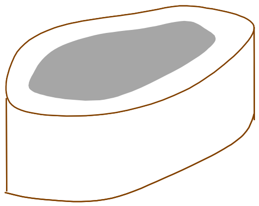
$$e_1 \ , \ e_2 \leq \frac{1}{10} \cdot \mathrm{Min}\{H, B\}$$

OPEN

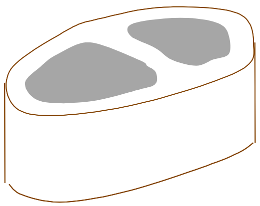


CLOSED

Single cell



Multi cell



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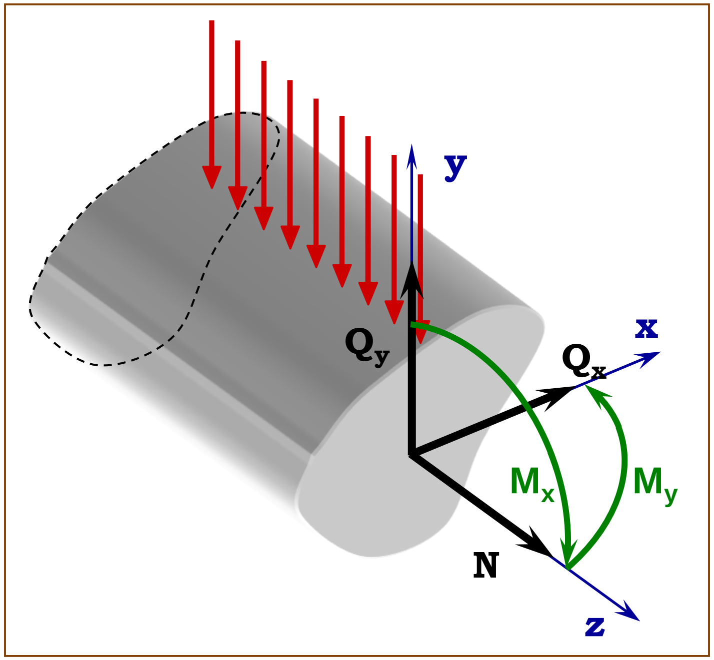
Previous knowledge (Introduction to Structural Analysis)



Normal stresses

$$\sigma_z = \frac{N}{A} + \frac{y \cdot I_y - x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \cdot M_x + \frac{y \cdot P_{xy} - x \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \cdot M_y$$

Neutral axis

$$\frac{N}{A} + \frac{y \cdot I_y - x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \cdot M_x + \frac{y \cdot P_{xy} - x \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \cdot M_y = 0$$

Shear stresses

$$\tau_{yz} = \frac{Q_y}{a_o(y)} \cdot \left(\frac{m^{A_1}{}_{ex} \cdot I_y - m^{A_1}{}_{ey} \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \right) - \frac{Q_x}{a_o(y)} \cdot \left(\frac{m^{A_1}{}_{ex} \cdot P_{xy} - m^{A_1}{}_{ey} \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \right)$$

$$\tau_{xz} = \frac{Q_y}{b_o(x)} \cdot \left(\frac{m^{A_2}{}_{ex} \cdot I_y - m^{A_2}{}_{ey} \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \right) - \frac{Q_x}{b_o(x)} \cdot \left(\frac{m^{A_2}{}_{ex} \cdot P_{xy} - m^{A_2}{}_{ey} \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \right)$$

Displacements field

$$u_z(x, y, z) = u_z^o(z) - \phi_y(z) \cdot x + \phi_x(z) \cdot y$$

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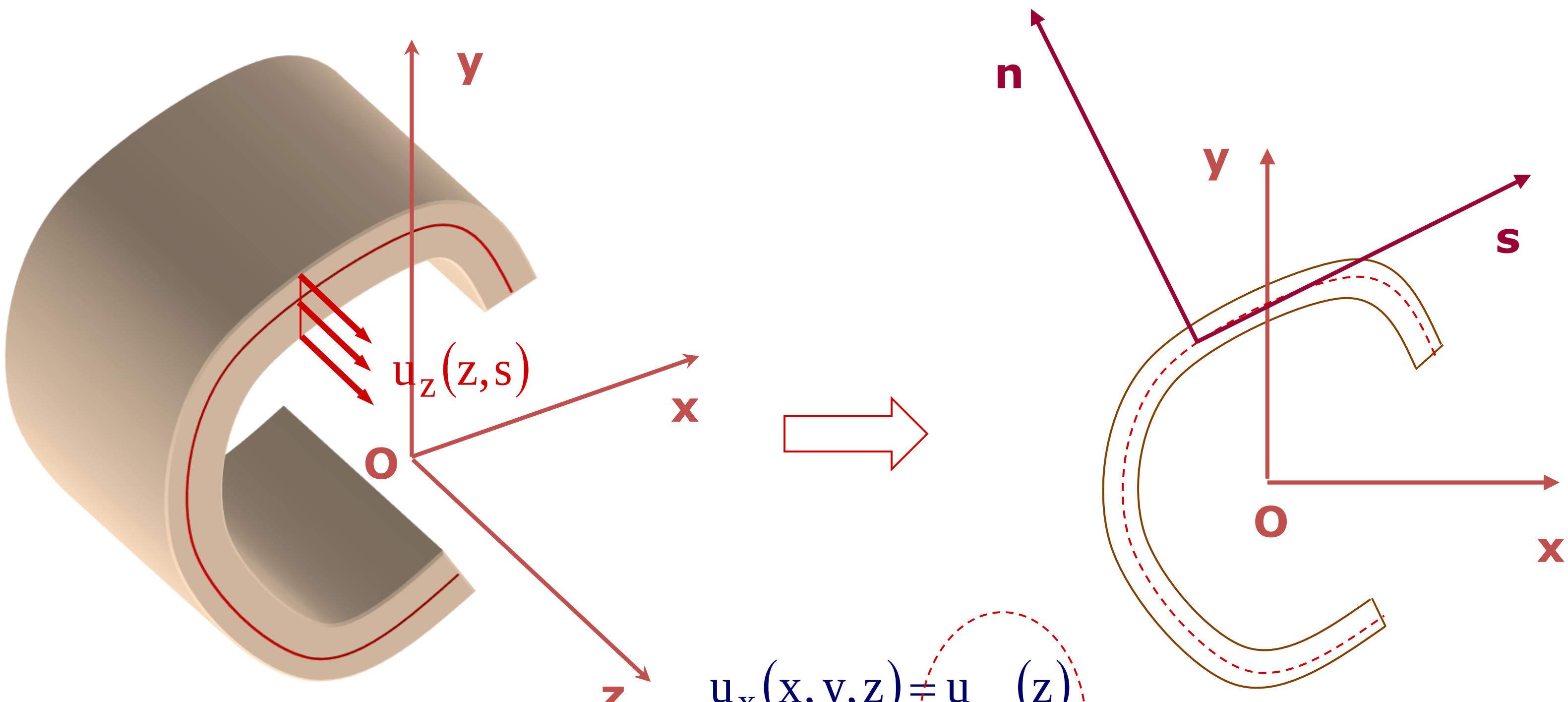
Kinematics



$$u_x(x,y,z) = u_{x_o}(z)$$

$$u_y(x,y,z) = u_{y_o}(z)$$

The section does not deform in its plane

$$u_z(x,y,z) = u_z(z,s)$$

uz is constant along the thickness

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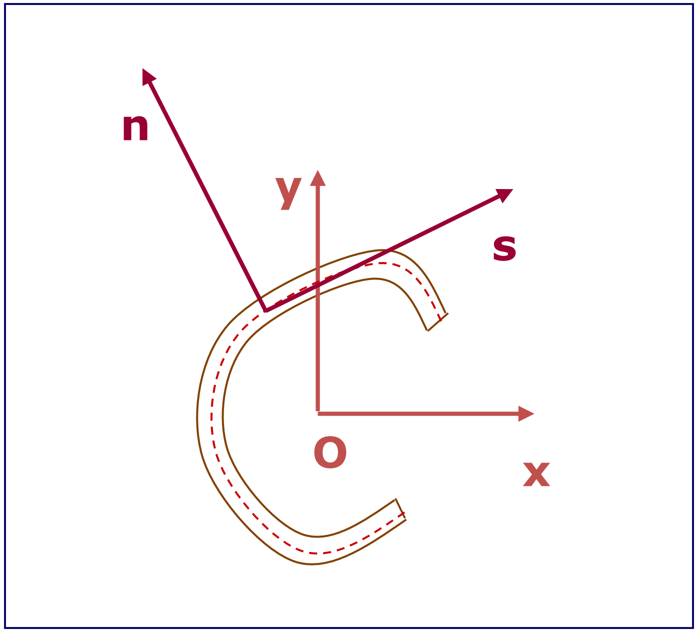
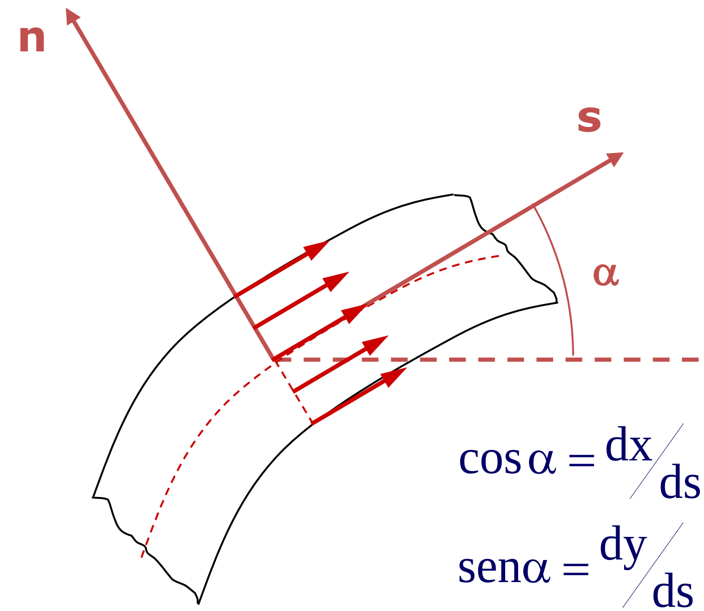
Kinematics





$$\cos\alpha = {dx}/{ds}$$

$$\operatorname{sen}\alpha = {dy}/{ds}$$

$$\gamma_{sz}(z,s) = \frac{\partial u_s(z,s)}{\partial z} + \frac{\partial u_z(z,s)}{\partial s}$$

$$u_s(z,s) = u_{x_o}(z)\cdot\cos\alpha + u_{y_o}(z)\cdot\operatorname{sen}\alpha$$

$$\boxed{u_z(z,s) = u_{z_o}(z) - \frac{du_{x_o}}{dz}\cdot x(s) - \frac{du_{y_o}}{dz}\cdot y(s) + \cancel{\int_0^s \gamma_{sz}(z,s)\cdot ds}}$$

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Normal stresses

$$u_z(z,s) = u_{z_o}(z) + \phi_x \cdot x(s) - \phi_y \cdot y(s)$$

$$\varepsilon_z(z,s) = \frac{\partial u_z(z,s)}{\partial z}$$

Constitutive equation

Neglecting normal nonaxial stresses

$$\sigma_z(z,s) = E \cdot \left(\frac{du_{z_o}(z)}{dz} + \frac{d\phi_x(z)}{dz} \cdot x(s) - \frac{d\phi_y(z)}{dz} \cdot y(s) \right)$$

Operating as in solid sections

$$\boxed{\sigma_z(x,y,z) = \frac{N}{A} + \frac{y \cdot I_y - x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \cdot M_x + \frac{y \cdot P_{xy} - x \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \cdot M_y}$$

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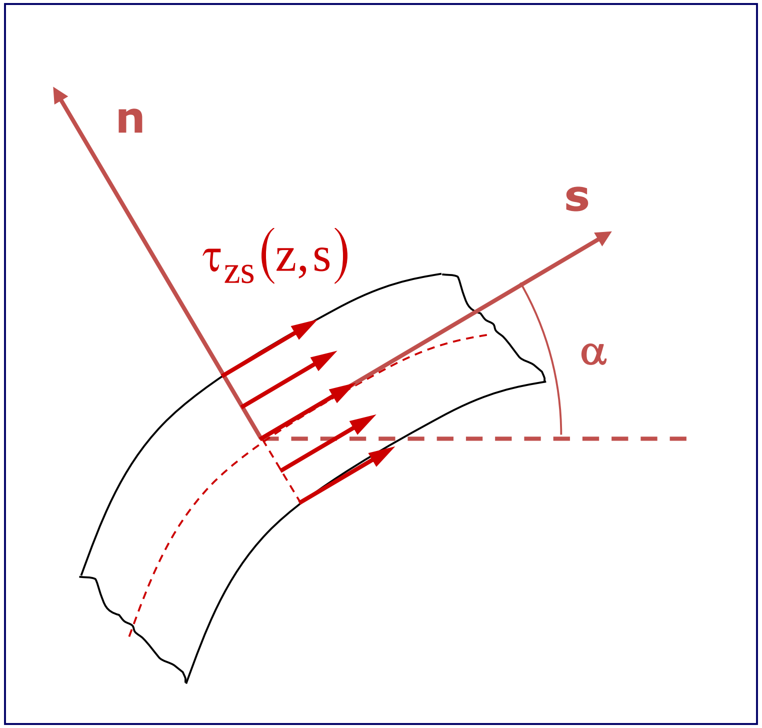
Shear stresses



$$\tau_{zs}(z,s) = cte$$

$$\tau_{sn}(z,s) = 0$$

No deformations in cross-section

$$\tau_{zn}(z,s) \approx 0$$

Due to small thickness

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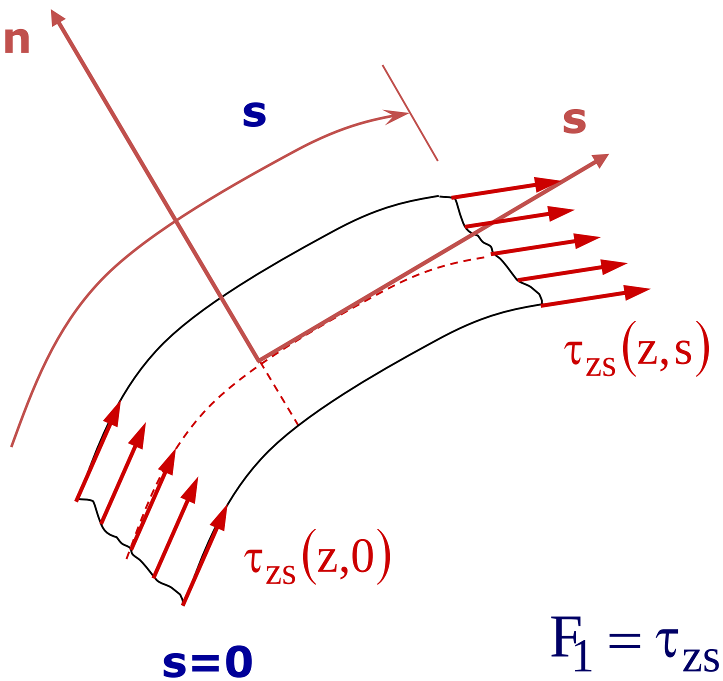
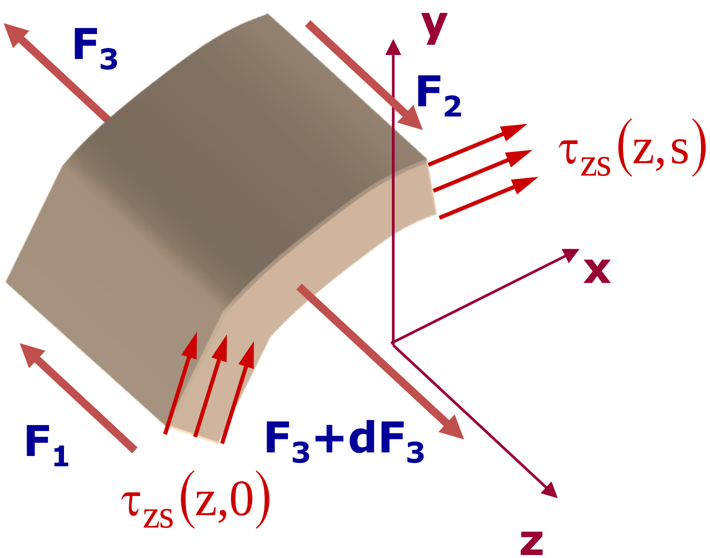
Shear stresses





$$F_1 = \tau_{zs}(z,0)\cdot e(0)\cdot dz$$

$$F_2 = \tau_{zs}(z,s)\cdot e(s)\cdot dz$$

$$F_3 = \int_{A^*} \sigma_z(z,s)\cdot dA$$

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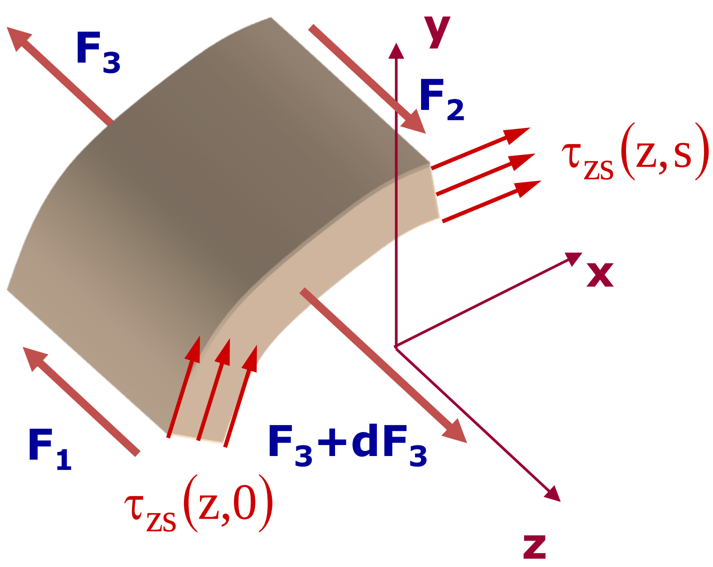
Shear stresses



Equilibrium equations
in z direction

$$F_3 + dF_3 - F_3 - F_1 + F_2 = 0$$

$$\tau_{zs}(z,s)\cdot e(s) = \tau_{zs}(z,0)\cdot e(0) - \frac{\partial}{\partial z}\int_{A^*}\sigma_z \cdot dA$$

Shear stress

$$\tau_{zs}(z,s)\cdot e(s)=\tau_{zs}(z,0)\cdot e(0)-\frac{\partial}{\partial z}\int_{A^*}\sigma_z\cdot dA$$

Equilibrium equations

Espression of normal stresses

$$\tau_{zs}(z,s)\cdot e(s)=\tau_{zs}(z,0)\cdot e(0)-K_y\cdot Q_y+K_x\cdot Q_x$$

The distributed loads in z direction are neglected

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Shear stresses

$$\tau_{zs}(z,s)\cdot e(s)=\tau_{zs}(z,0)\cdot e(0)-\frac{\partial}{\partial z}\int_{A^*}\sigma_z\cdot dA$$

$$\frac{dM_x}{dz}-Q_y+m_x=0 \qquad \frac{dQ_x}{dz}+q_x=0$$

$$\frac{dM_y}{dz}+Q_x+m_y=0 \qquad \frac{dQ_y}{dz}+q_y=0$$

$$\frac{dM_T}{dz}+m_z=0 \qquad \frac{dN}{dz}+q_z=0$$

$$\sigma_z(x,y,z)=\frac{N}{A}+\frac{y\cdot I_y-x\cdot P_{xy}}{I_x\cdot I_y-P_{xy}^2}\cdot M_x+\frac{y\cdot P_{xy}-x\cdot I_x}{I_x\cdot I_y-P_{xy}^2}\cdot M_y$$

$$\tau_{zs}(z,s)\cdot e(s)=\tau_{zs}(z,0)\cdot e(0)-K_y\cdot Q_y+K_x\cdot Q_x$$

The distributed loads in z direction are neglected

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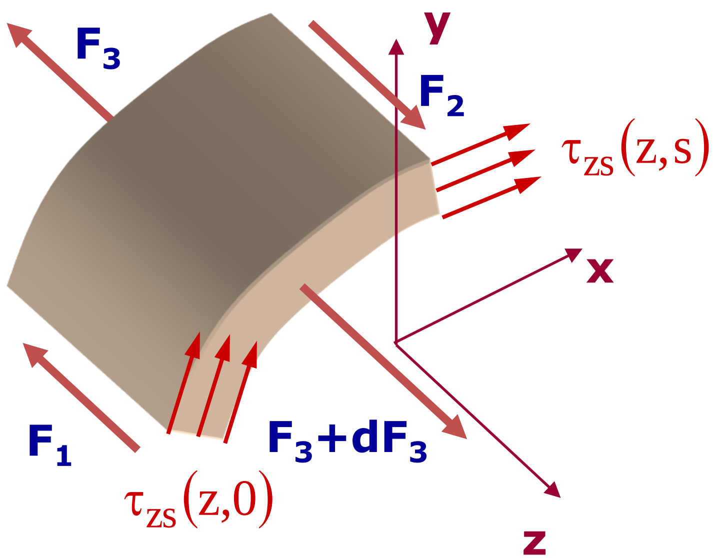
Shear stresses



$$K_y = \frac{m_{ex}^{*} \cdot I_y - m_{ey}^{*} \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^{2}}$$

$$K_x = \frac{m_{ex}^{*} \cdot P_{xy} - m_{ey}^{*} \cdot I_x}{I_x \cdot I_y - P_{xy}^{2}}$$

$$m_{ex}^{*} = \int_{A^*} y(s) \cdot dA = \int_0^s y(s) \cdot e(s) \cdot ds$$

$$m_{ey}^{*} = \int_{A^*} x(s) \cdot dA = \int_0^s x(s) \cdot e(s) \cdot ds$$

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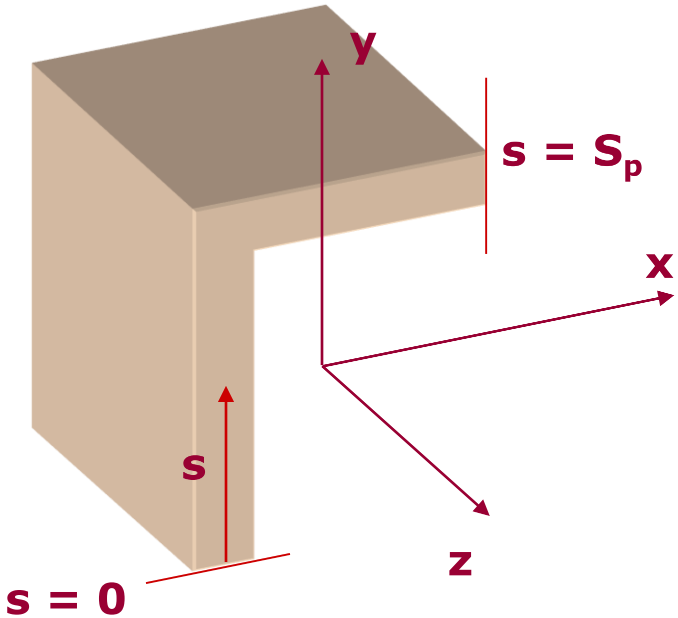
Shear stresses in open sections

Open section



Assumption: no distributed loads in z direction

$$\tau_{zs}(z,0)\cdot e(0)=0$$

$$\tau_{zs}(z,s)\cdot e(s)=-K_y\cdot Q_y+K_x\cdot Q_x$$

Constant shear flow

$$q(s)=\tau_{zs}(z,s)\cdot e(s)$$

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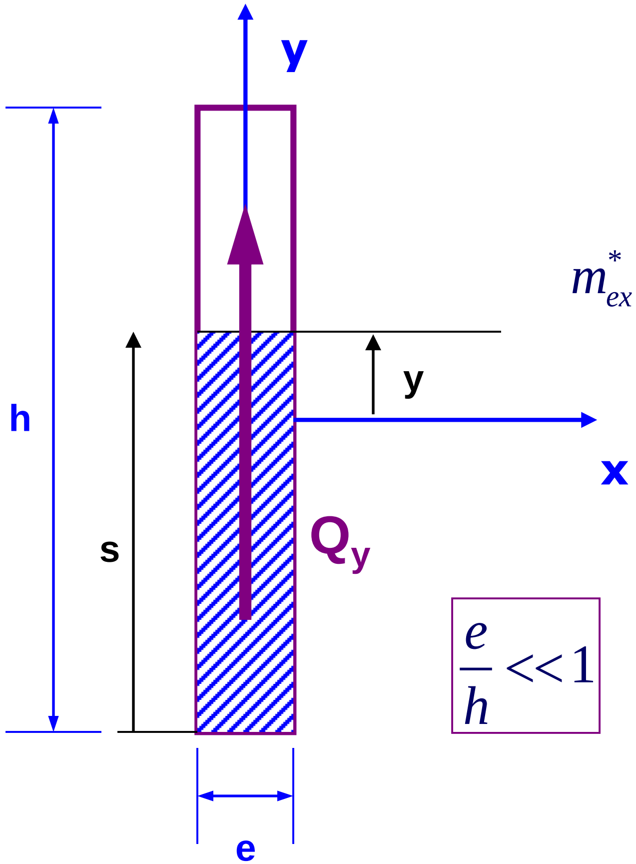
Shear stresses in open sections

Example: open section



$$K_y = \frac{m^*_{ex}}{I_x} \qquad \tau_{zs}(z,s)\cdot e(s) = -K_y \cdot Q_y$$

$$m^*_{ex} = \int_0^s y(s)\cdot e(s)\cdot ds = \int_0^s \left(s - \frac{h}{2}\right)\cdot e\cdot ds = \frac{s\cdot e}{2}\cdot(s-h)$$

$$\boxed{\frac{e}{h} << 1}$$

$$q(s) = -\frac{Q_y}{2\cdot I_x}\cdot s\cdot e\cdot(s-h)$$

$$I_x = \frac{1}{12}\cdot e\cdot h^3$$

$$\boxed{q(s) = -\frac{6\cdot Q_y}{h^3}\cdot s\cdot(s-h)}$$

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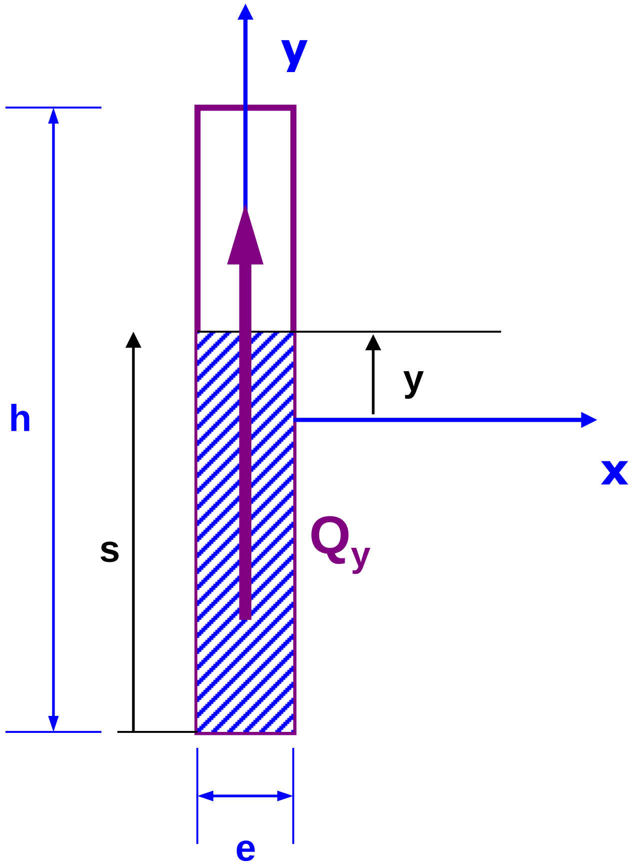
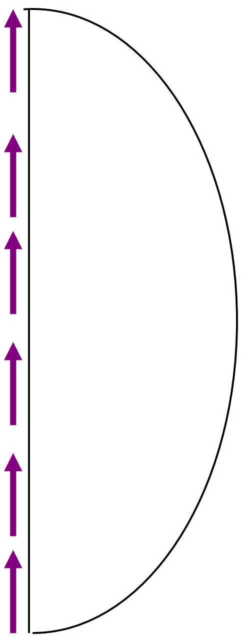
Shear stresses in open sections

Example: open section





$$q(s) = \frac{6 \cdot Q_y}{h^3} \cdot s \cdot (h - s)$$

$s = 0 \qquad q(0) = 0$

$s = \frac{h}{2} \qquad q\left(\frac{h}{2}\right) = \frac{3}{2} \cdot \frac{Q_y}{h}$

$s = h \qquad q(h) = 0$

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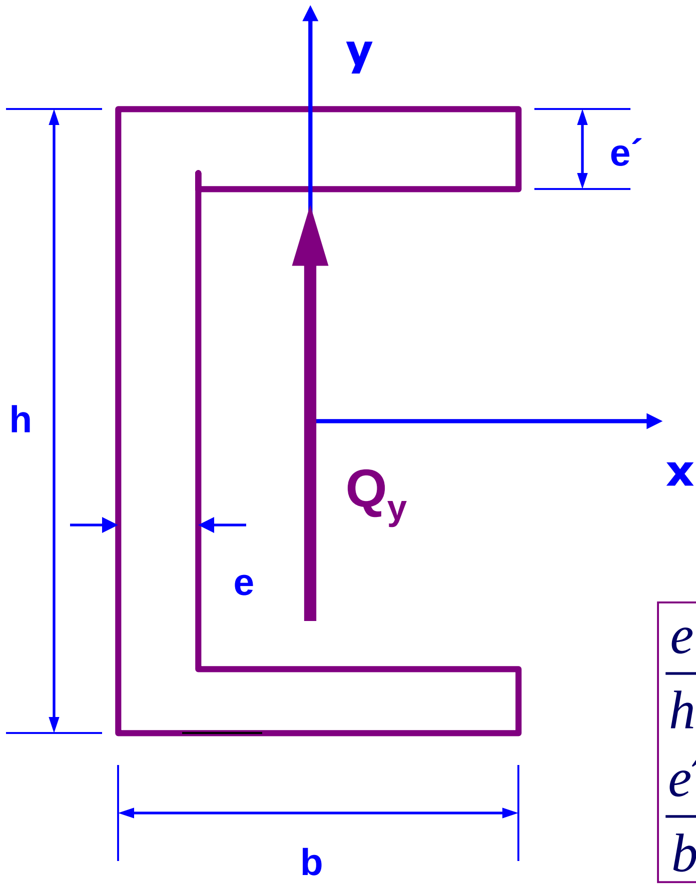
Example: open section



$$\frac{e}{h} << 1$$

$$\frac{e'}{b} << 1$$

$$K_y = \frac{m_{ex}^{*}}{I_x}$$

$$q(s) = -K_y \cdot Q_y$$

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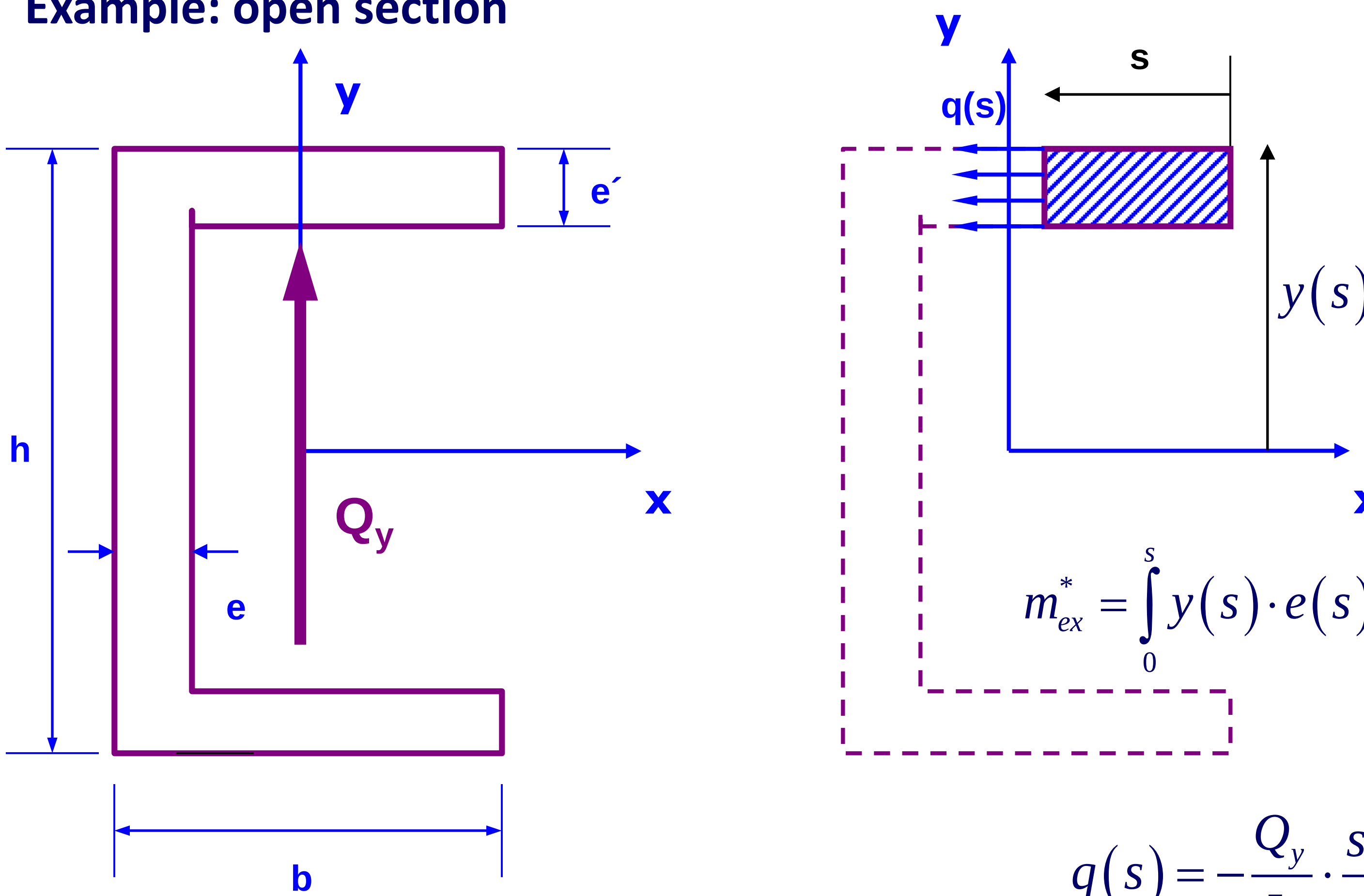
Shear stresses in open sections

Example: open section



$$y(s) = \frac{h}{2}$$

$$m_{ex}^{*} = \int_{0}^{s} y(s) \cdot e(s) \cdot ds = \frac{s \cdot e' \cdot h}{2}$$

$$q(s) = -\frac{Q_y}{I_x} \cdot \frac{s \cdot e' \cdot h}{2}$$

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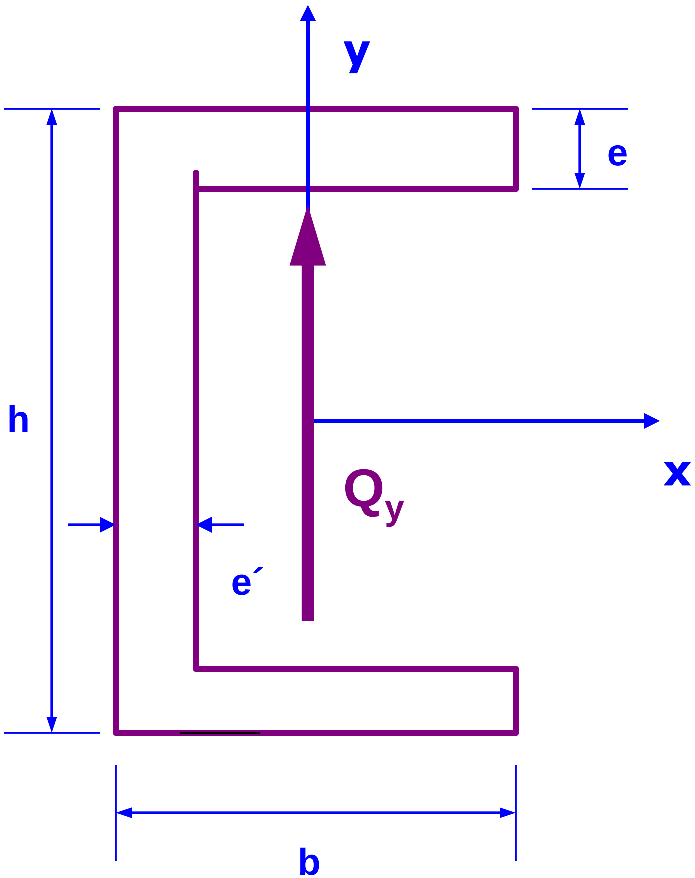
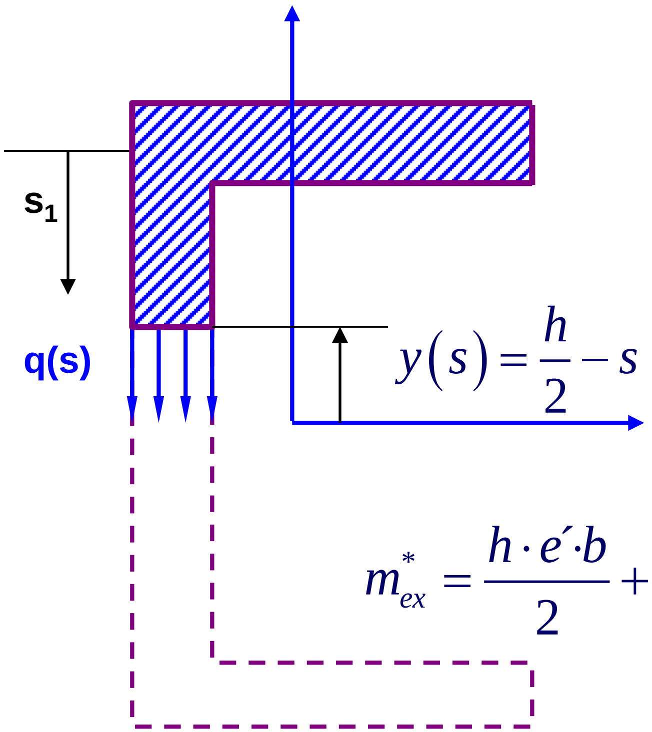
Shear stresses in open sections

Example: open section





$$m_{ex}^{*}=\frac{h\cdot e\acute{}\cdot b}{2}+\frac{s_1\cdot e}{2}\cdot(h-s_1)$$

$$q(s)=-\frac{Q_y}{I_x}\cdot\left[\frac{h\cdot e\acute{}\cdot b}{2}+\frac{s_1\cdot e}{2}\cdot(h-s_1)\right]$$

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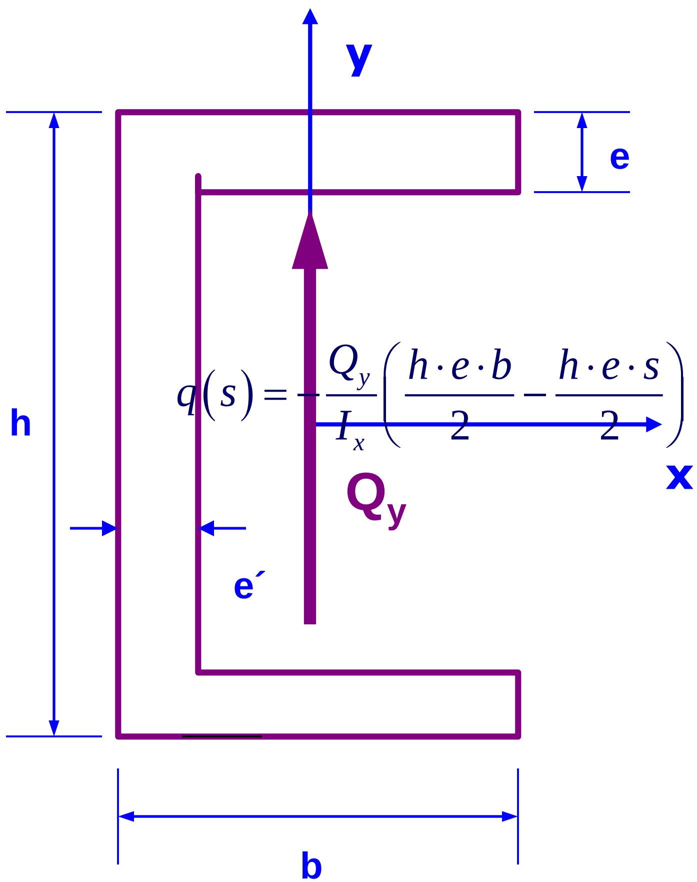
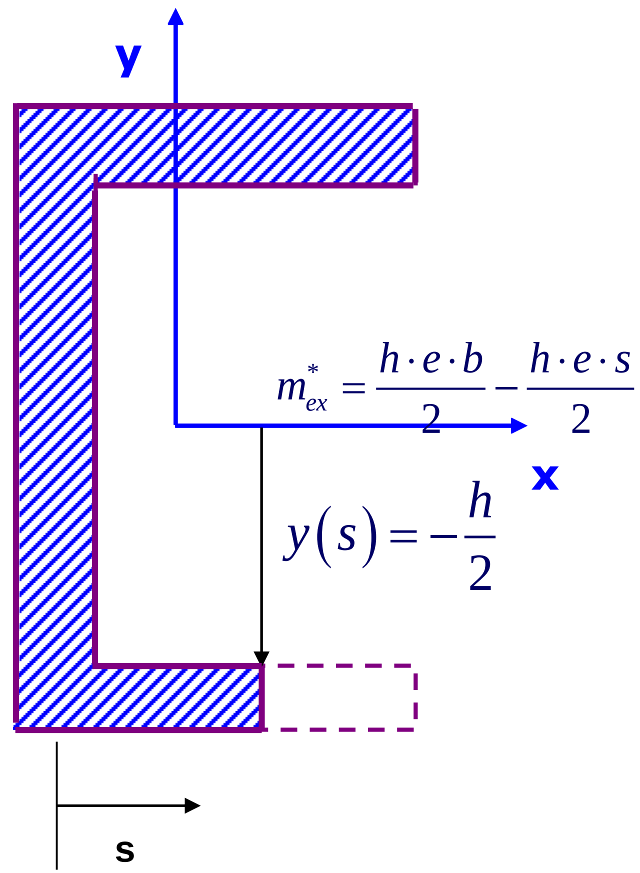
Shear stresses in open sections

Example: open section



$$q(s) = -\frac{Q_y}{I_x}\left(\frac{h\cdot e\cdot b}{2} - \frac{h\cdot e\cdot s}{2}\right)$$



$$m_{ex}^{*} = \frac{h\cdot e\cdot b}{2} - \frac{h\cdot e\cdot s}{2}$$

$$y(s) = -\frac{h}{2}$$

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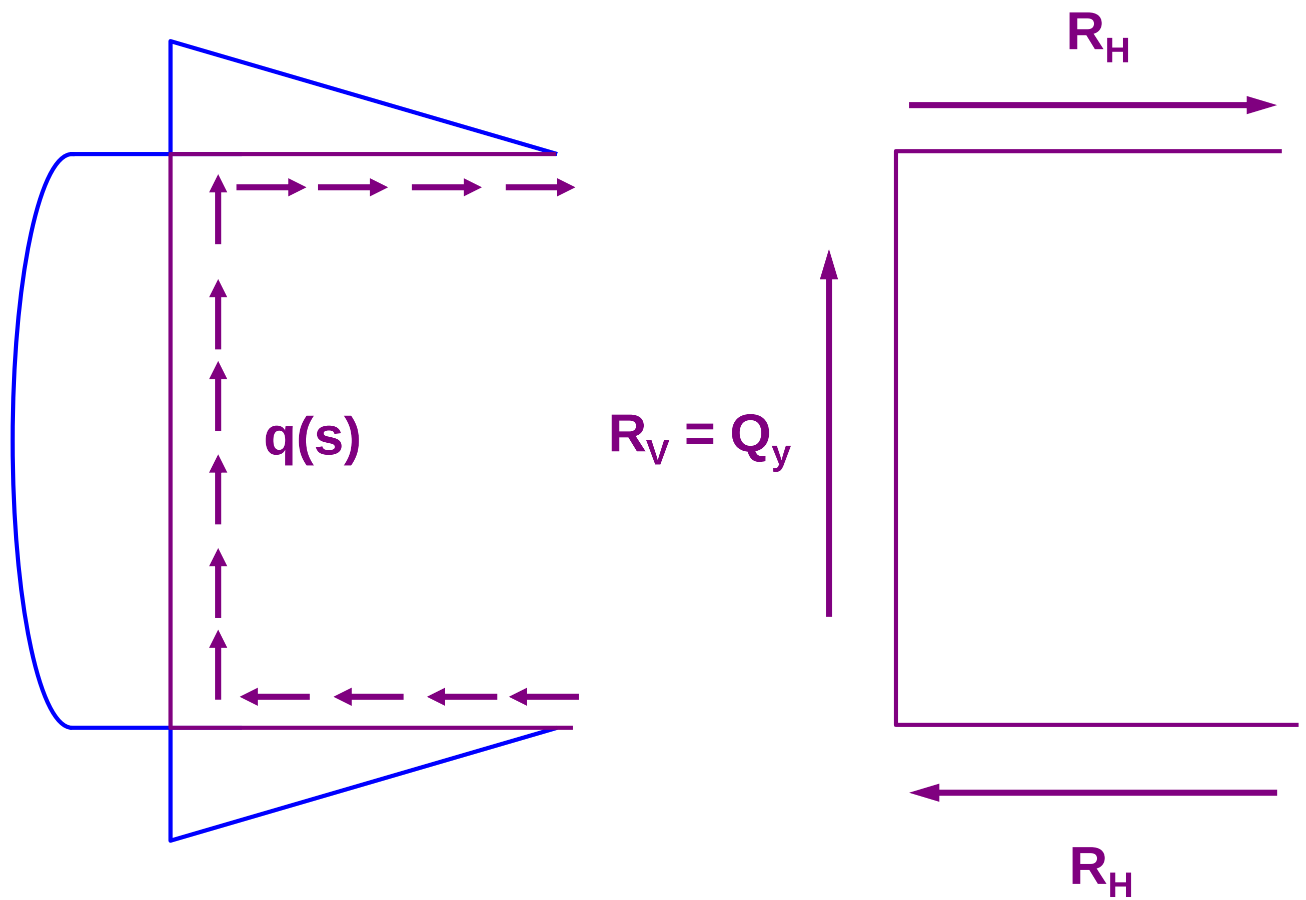
Shear stresses in open sections

Example: open section



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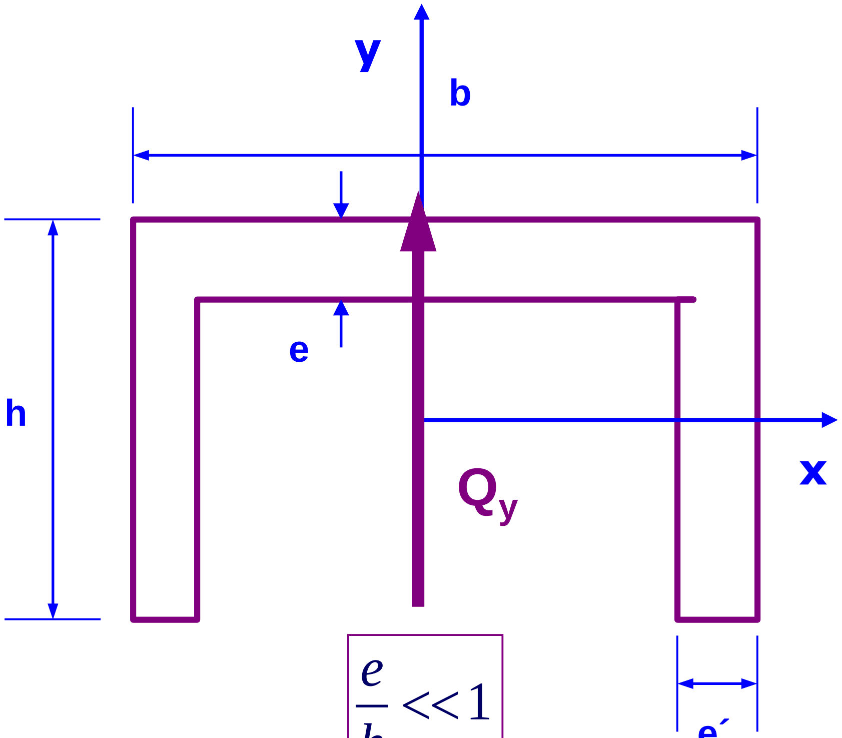
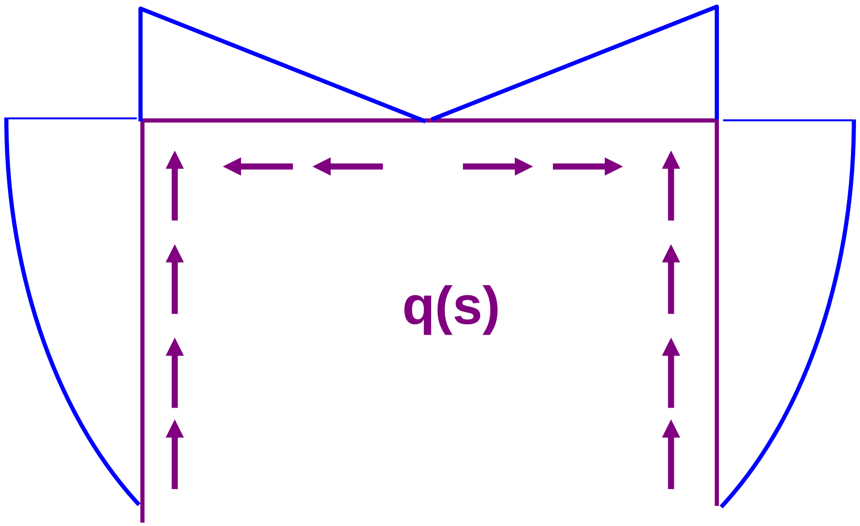
Shear stresses in open sections

Example: open section



$$\frac{e}{h} << 1$$

$$\frac{e'}{b} << 1$$



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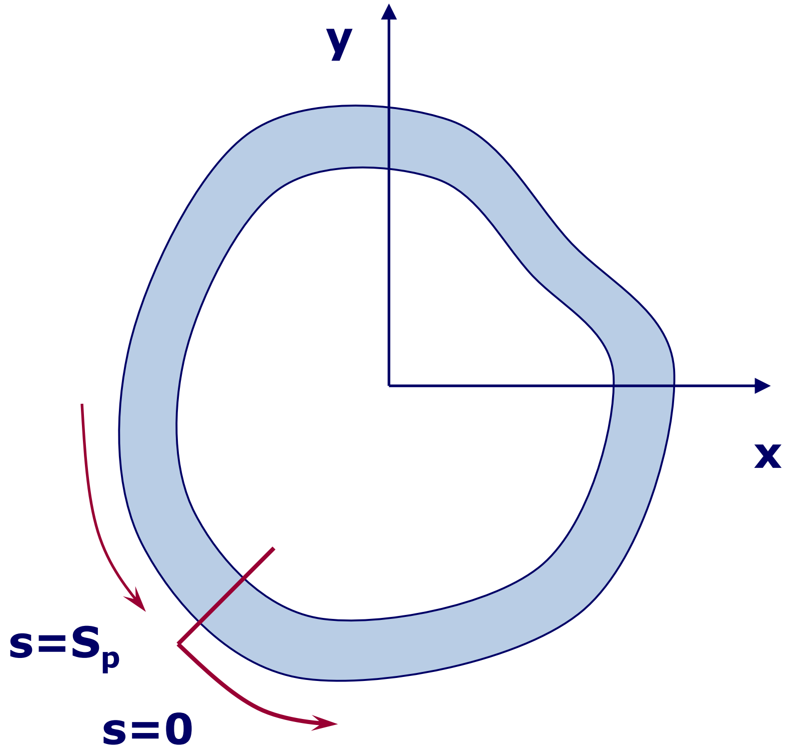
Shear stresses in closed sections

Single-cell closed section



An arbitrary point is the origin of the coordinate *s*

$$\boxed{q(s) = q(s)_{open} + q(0)}$$

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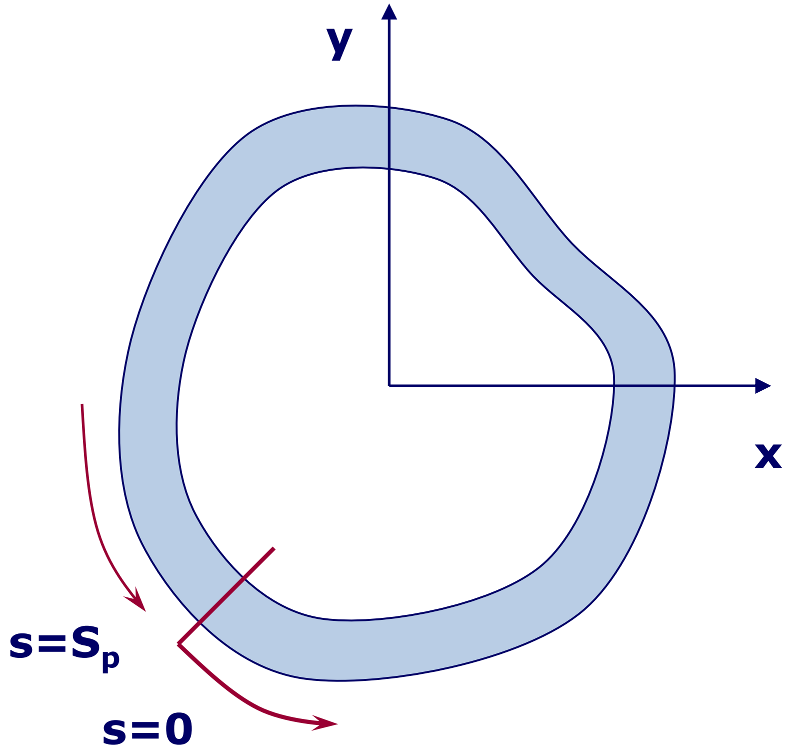
Shear stresses in closed sections

Single-cell closed section



$$q(s) = q(s)_{open} + q(0)$$

$$\int_0^{S_p} \gamma_{zs} \cdot ds = 0$$

$$\int_0^{S_p} \frac{\tau_{zs}}{G} \cdot ds = 0$$

$$\int_0^{S_p} \frac{q(s)_{open}}{G \cdot e(s)} \cdot ds + \int_0^{S_p} \frac{q(0)}{G \cdot e(s)} \cdot ds = 0$$

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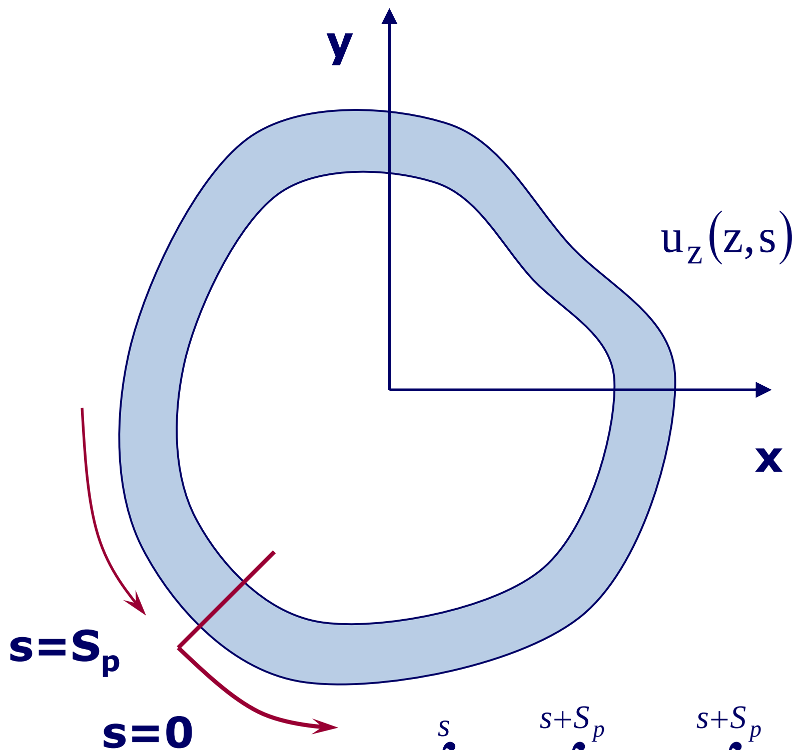
Cálculo de tensiones tangenciales

Single-cell closed section

$$u_z(z,s) = u_z(z,s+S_p)$$
$$y(s) = y(s+S_p)$$
$$z(s) = z(s+S_p)$$



$$u_z(z,s) = u_{z_o}(z) - \frac{du_{x_o}}{dz}\cdot x(s) - \frac{du_{y_o}}{dz}\cdot y(s) + \int_0^s \gamma_{sz}(z,s)\cdot ds$$

$$\int_0^s \gamma_{sz}(z,s)\cdot ds = \int_0^{s+S_p} \gamma_{sz}(z,s)\cdot ds$$

$$\int_0^s + \int_s^{s+S_p} = \int_0^{s+S_p} = \int_0^s \qquad \int_s^{s+S_p} = 0 \qquad \int_0^{S_p} \gamma_{sz}(z,s)\cdot ds = 0$$

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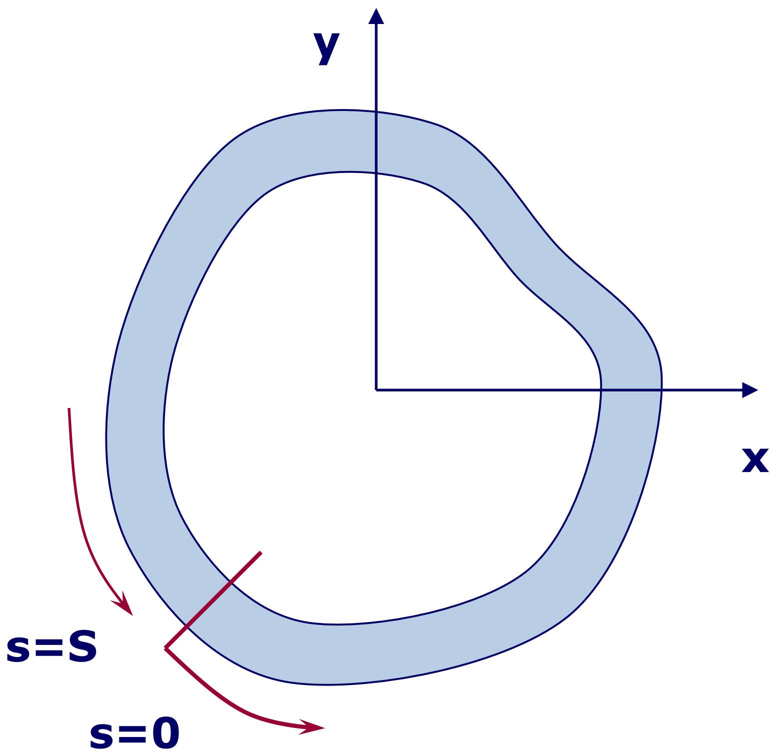
Shear stresses in closed sections

Single-cell closed section



$$q(0) = -\frac{\displaystyle\int_0^S \frac{q(s)_{open}}{G \cdot e(s)} \cdot ds}{\displaystyle\int_0^S \frac{1}{G \cdot e(s)} \cdot ds}$$

$$q(s) = q(s)_{open} + q(0)$$

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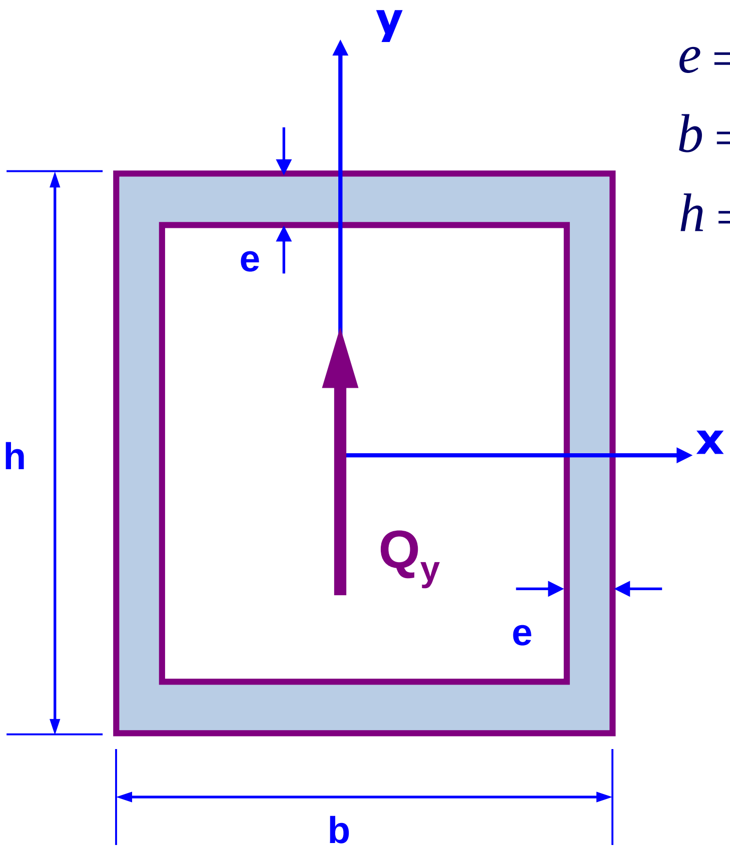
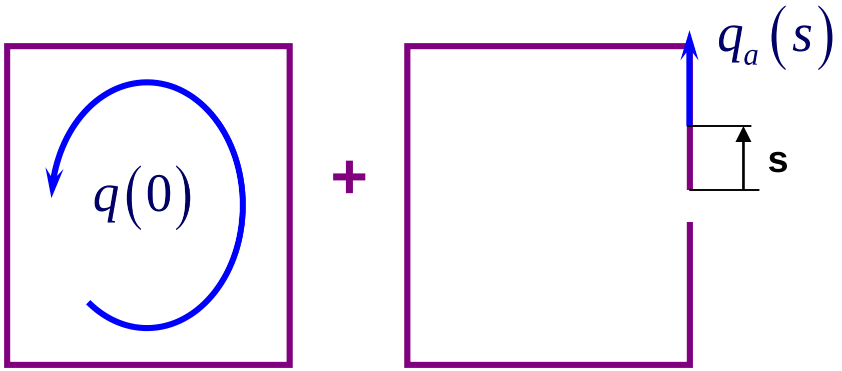
Shear stresses in closed sections

Single-cell closed section
Example



$$e = 0{,}8 \;\; cm$$
$$b = 15 \;\; cm$$
$$h = 20 \;\; cm$$

$$q(s) = q_a(s) + q(0)$$



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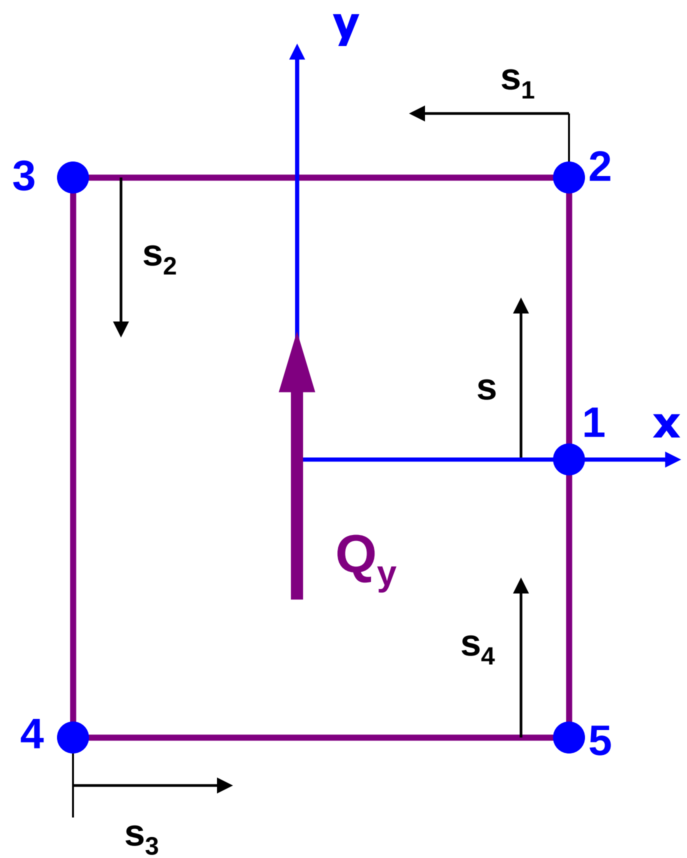
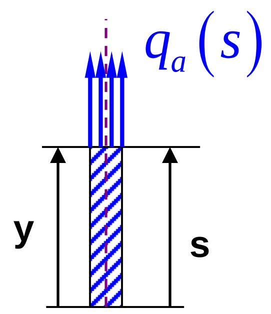
Shear stresses in closed sections

Single-cell closed section

Example



zone 1-2



$$m_{ex}^{*} = \int_{0}^{s} y(s)\cdot e(s)\cdot ds = \frac{s^2\cdot e}{2}$$

$$q_a(s) = -\frac{Q_y}{I_x}\cdot\frac{e\cdot s^2}{2}$$

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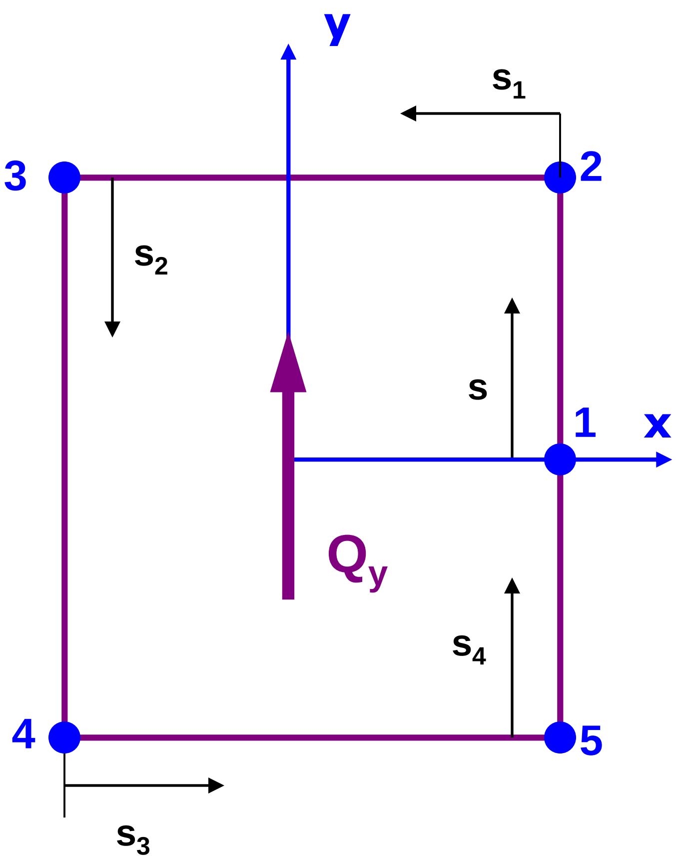
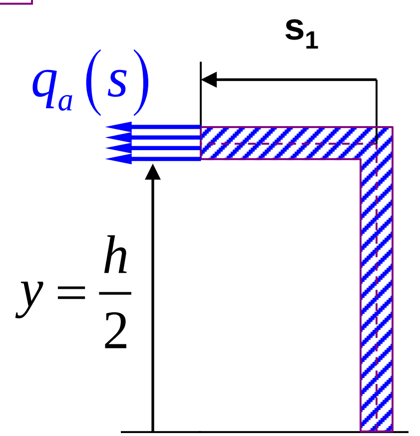
Universidad Carlos III de Madrid

Shear stresses in closed sections

Single-cell closed section
Example



zone 2-3



$$m_{ex}^{*}=\int_{0}^{s} y(s)\cdot e(s)\cdot ds=\frac{e\cdot h^{2}}{8}+\frac{h\cdot e\cdot s_{1}}{2}$$

$$q(s)=-\frac{Q_{y}}{I_{x}}\cdot\left(\frac{e\cdot h^{2}}{8}+\frac{h\cdot e\cdot s_{1}}{2}\right)$$

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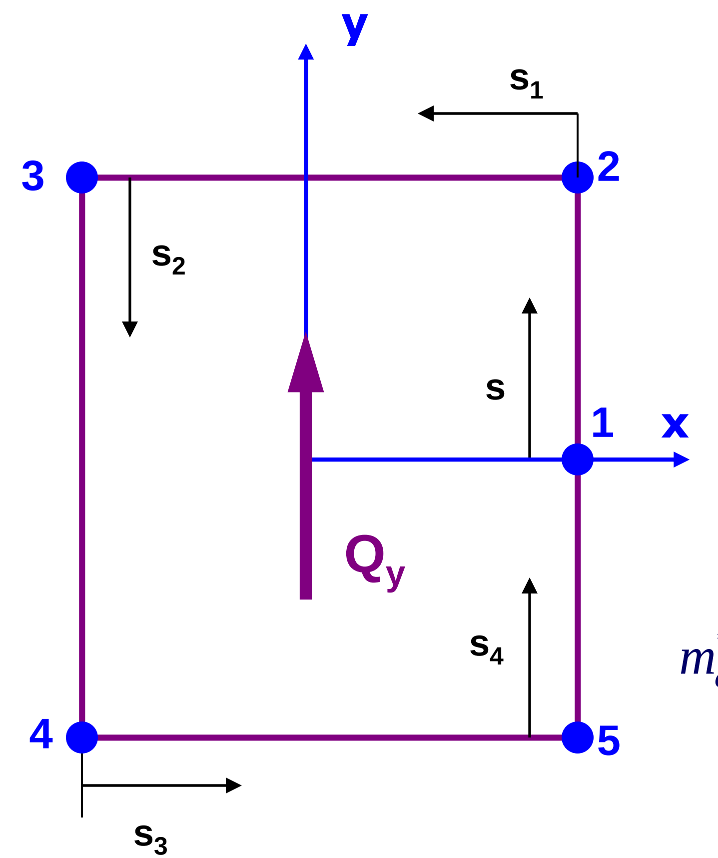
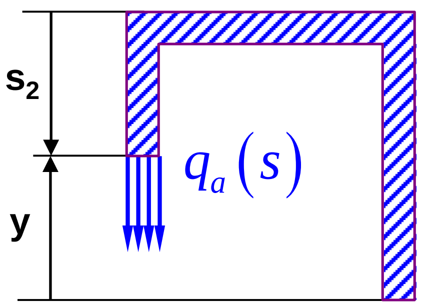
Single-cell closed section
Example



zone 3-4



$$m_{ex}^{*}=\int_{0}^{s} y(s)\cdot e(s)\cdot ds=\frac{e\cdot h^{2}}{8}+\frac{h\cdot e\cdot b}{2}+\frac{e\cdot s_{2}}{2}\cdot(h-s_{2})$$

$$q(s)=-\frac{Q_{y}}{I_{x}}\cdot\left[\frac{e\cdot h^{2}}{8}+\frac{h\cdot e\cdot b}{2}+\frac{e\cdot s_{2}}{2}\cdot(h-s_{2})\right]$$

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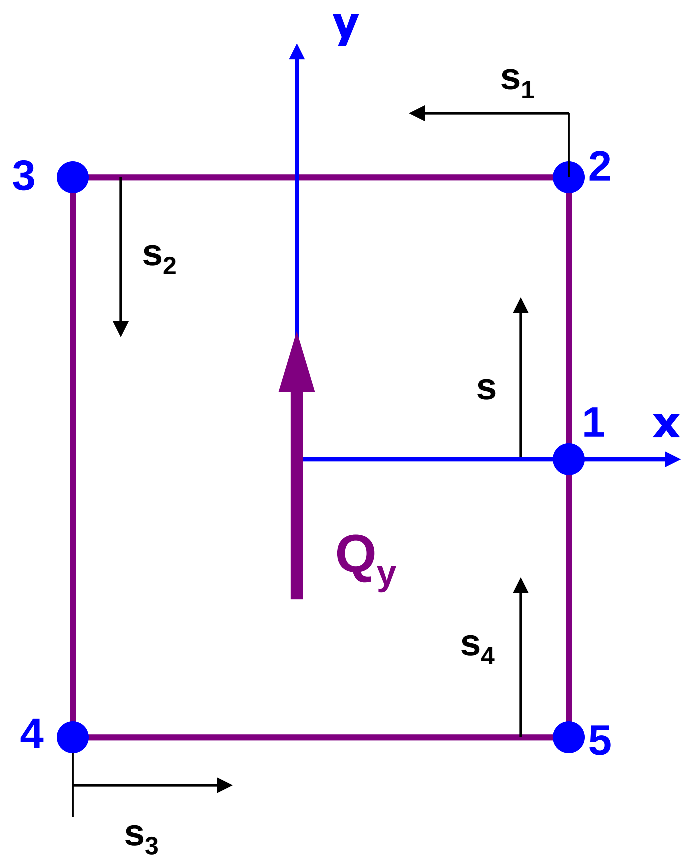
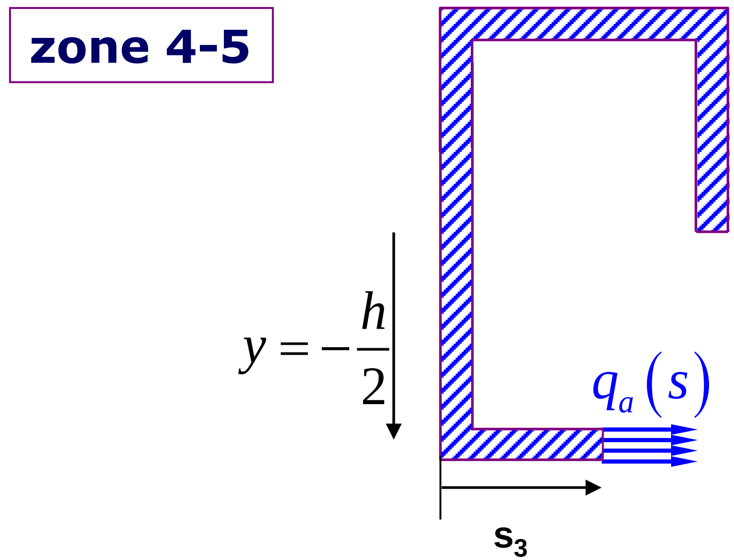
Shear stresses in closed sections

Single-cell closed section
Example





$$m_{ex}^{*}=\int_{0}^{s} y(s)\cdot e(s)\cdot ds=\frac{e\cdot h^2}{8}+\frac{h\cdot e\cdot b}{2}-\frac{h\cdot e}{2}\cdot s_3$$

$$q(s)=-\frac{Q_y}{I_x}\cdot\left[\frac{e\cdot h^2}{8}+\frac{h\cdot e\cdot b}{2}-\frac{h\cdot e}{2}\cdot s_3\right]$$

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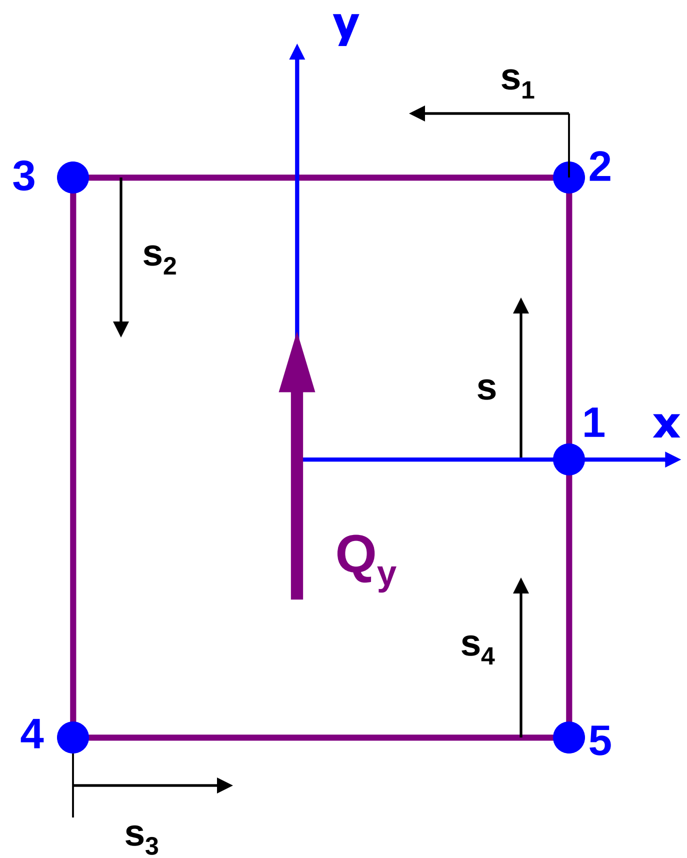
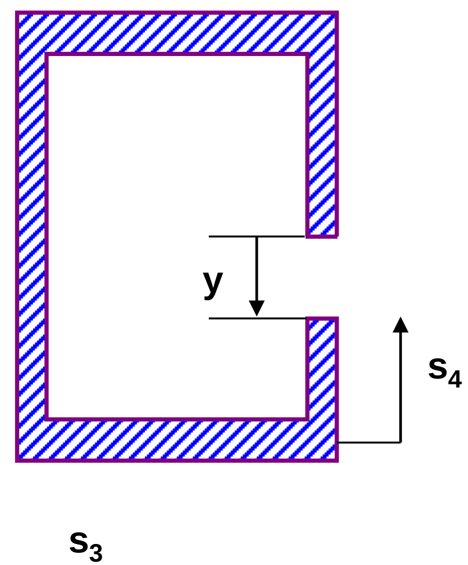
Shear stresses in closed sections

Single-cell closed section
Example



zone 5-1



$$m_{ex}^{*}=\int_0^s y(s)\cdot e(s)\cdot ds=\frac{e\cdot h^2}{8}++\frac{s_4\cdot e}{2}\cdot(s_4-h)$$

$$q(s)=-\frac{Q_y}{I_x}\cdot\left[\frac{e\cdot h^2}{8}+\frac{s_4\cdot e}{2}\cdot(s_4-h)\right]$$

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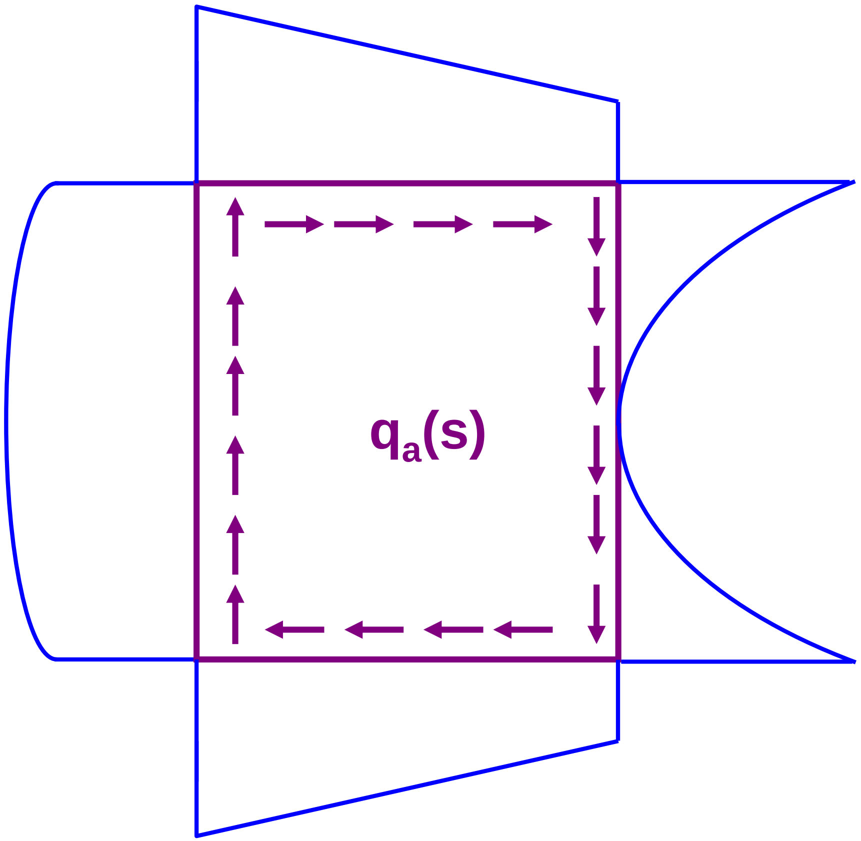
Shear stresses in closed sections

Single-cell closed section
Example



$$q(0) = -\frac{\int_0^{2\cdot(h+b)} q_a(s)\cdot ds}{2\cdot(h+b)}$$

$$q(0) = 100\cdot\frac{Q_y}{I_x}$$

$$q(s) = q_a(s) + 100\cdot\frac{Q_y}{I_x}$$

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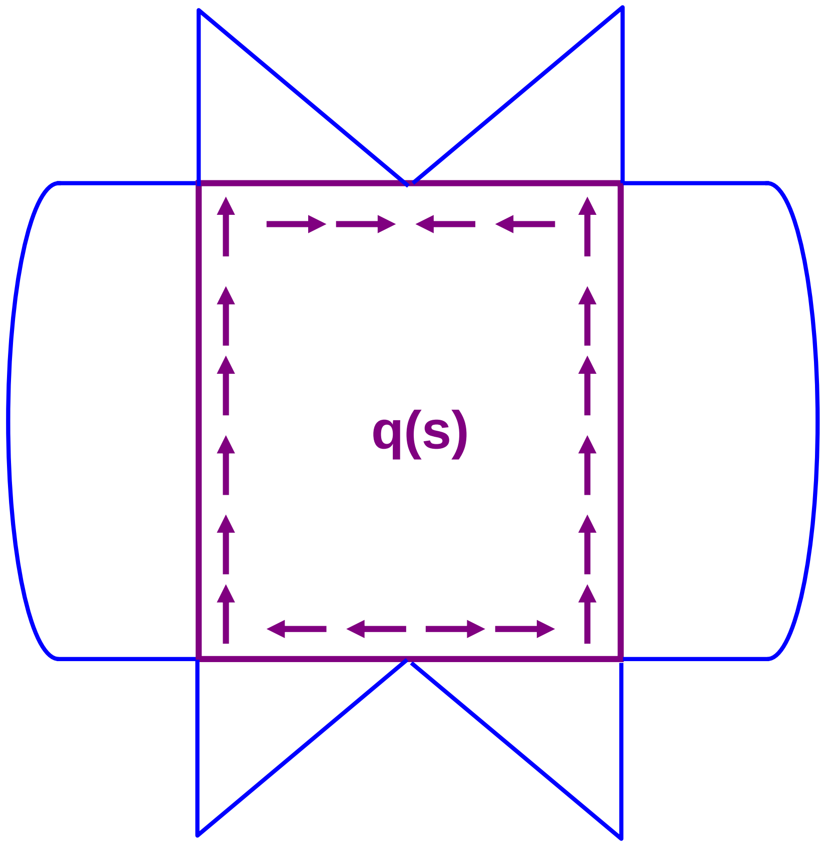
Shear stresses in closed sections

Single-cell closed section
Example



$$q(0) = -\frac{\int_0^{2\cdot(h+b)} q_a(s)\cdot ds}{2\cdot(h+b)}$$

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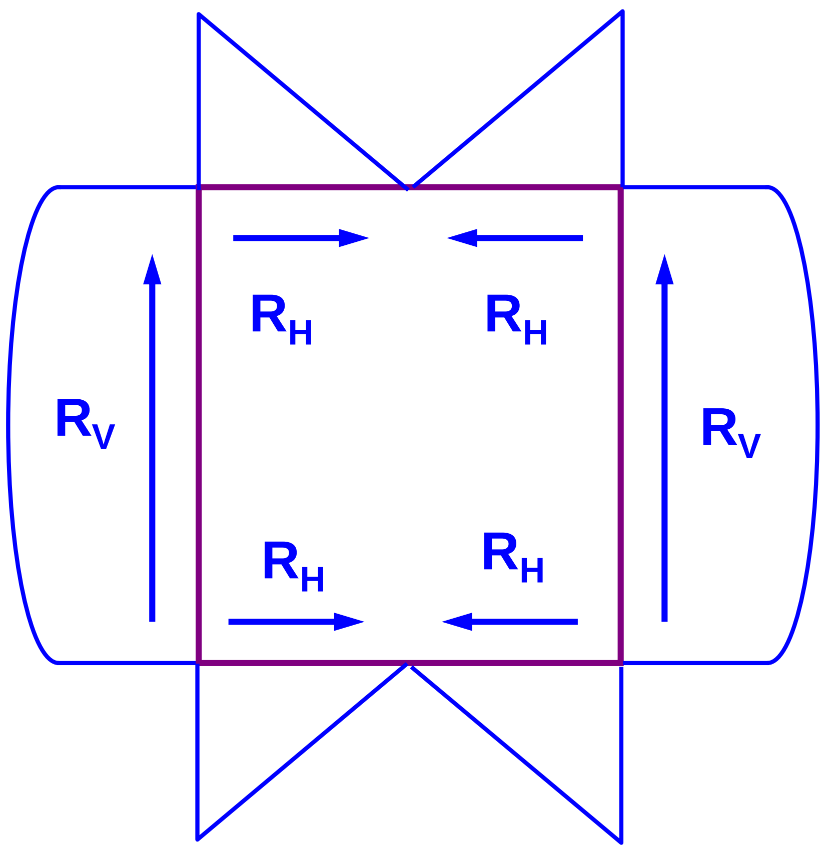
Shear stresses in closed sections

Single-cell closed section
Example



$$q(0) = -\frac{\int_0^{2\cdot(h+b)} q_a(s)\cdot ds}{2\cdot(h+b)}$$

$$q(0) = 100\cdot\frac{Q_y}{I_x}$$

$$q(s) = q_a(s) + 100\cdot\frac{Q_y}{I_x}$$

$$R_V = \frac{Q_y}{2}$$

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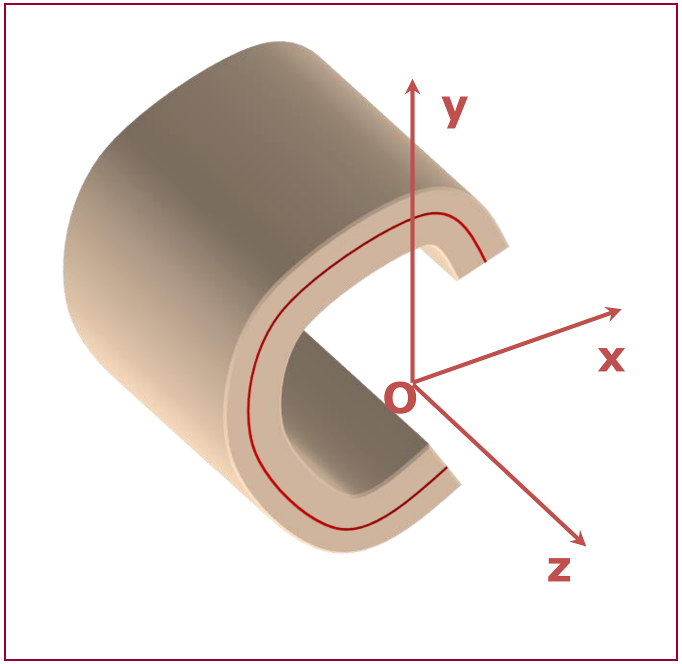
Summary



Displacements

$$u_x(x,y,z) = u_{x_o}(z)$$

$$u_y(x,y,z) = u_{y_o}(z)$$

$$u_z(x,y,z) = u_z(z,s)$$

Normal stresses

$$\sigma_z(x,y,z) = \frac{N}{A} + \frac{y \cdot I_y - x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \cdot M_x + \frac{y \cdot P_{xy} - x \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \cdot M_y$$

Shear stresses

$$q(s) = q(0) - K_y \cdot Q_y + -K_x \cdot Q_x$$

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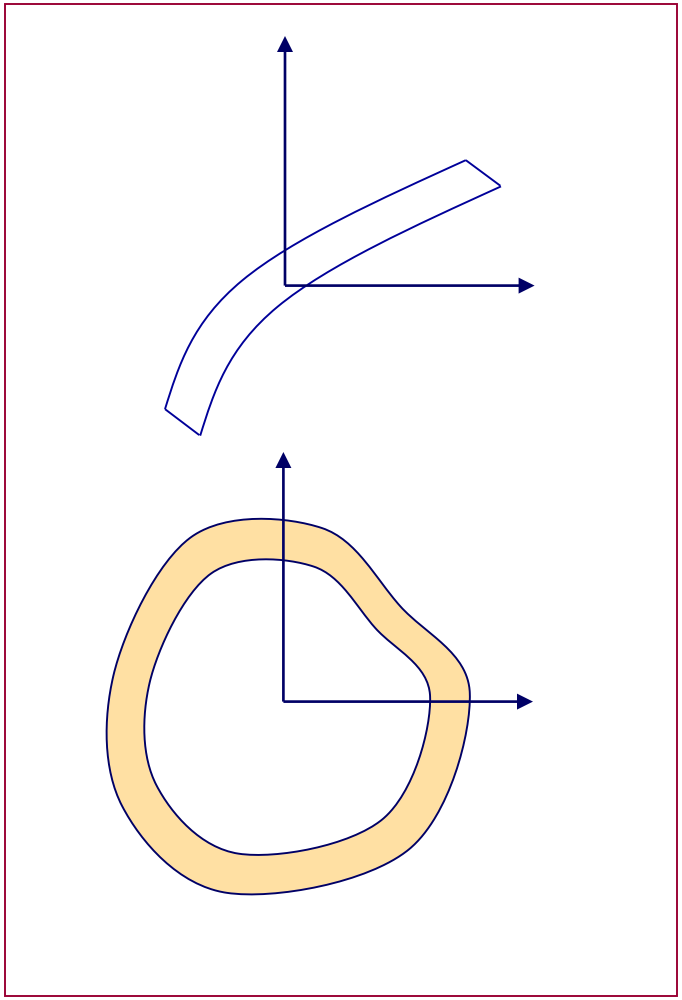
Summary



Open section

$$q(s) = q(0) - K_y \cdot Q_y + -K_x \cdot Q_x$$

Closed sections

$$q(s) = q(s)_{open} + q(0)$$

$$q(0) = -\frac{\displaystyle\int_0^S \frac{q(s)_{open}}{G \cdot e(s)} \cdot ds}{\displaystyle\int_0^S \frac{1}{G \cdot e(s)} \cdot ds}$$

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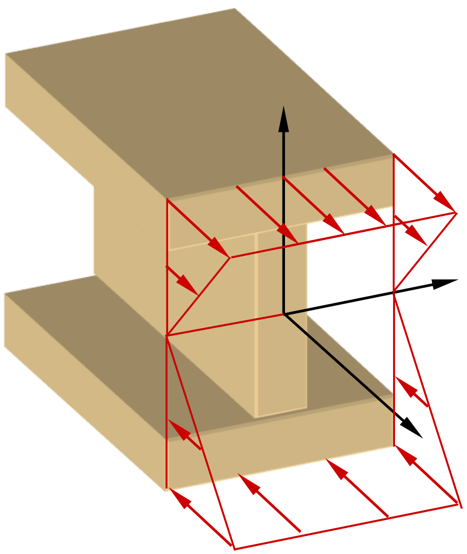

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