

Bachelor in Aerospace Engineering



Universidad
Carlos III de Madrid
www.uc3m.es

**Department of Continuum Mechanics and Structural
Engineering**

Aerospace Structures

Chapter 2. Bending, shear and torsion of thin-walled beams

Bending and shear of open and closed, thin-walled beams (I)



CHAPTER 2. Bending, shear and torsion of thin-walled beams

Bending and shear of open and closed, thin-walled beams (I)

OBJECTIVES

- State the displacement field the cross-section of a in a thin-walled beam
- Calculate normal stresses field in open and closed cross-sections
- Calculate normal stresses field in multi-cell cross-sections

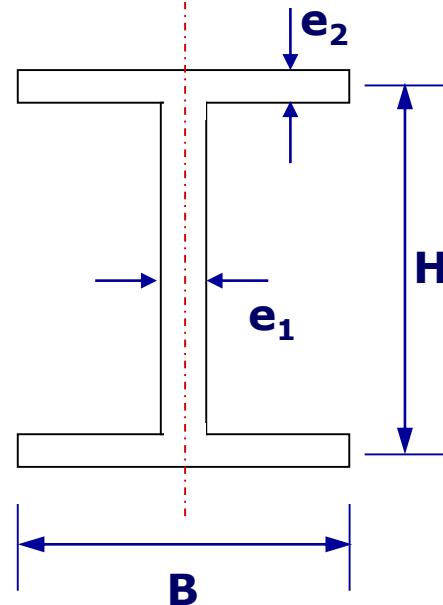


CHAPTER 2. Bending, shear and torsion of thin-walled beams

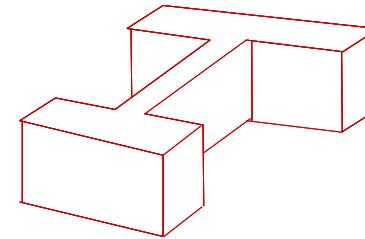
Bending and shear of open and closed, thin-walled beams (I)

- Introduction
- Kinematics
- Normal stresses
- Shear stress field on open sections
- Stress field on single-cell closed sections
- Summary
- References

Cross-section of a thin-walled beam

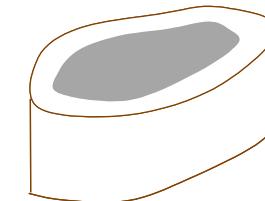


OPEN

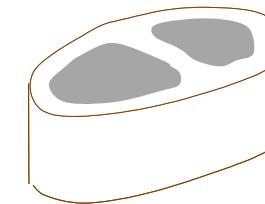


CLOSED

Single cell

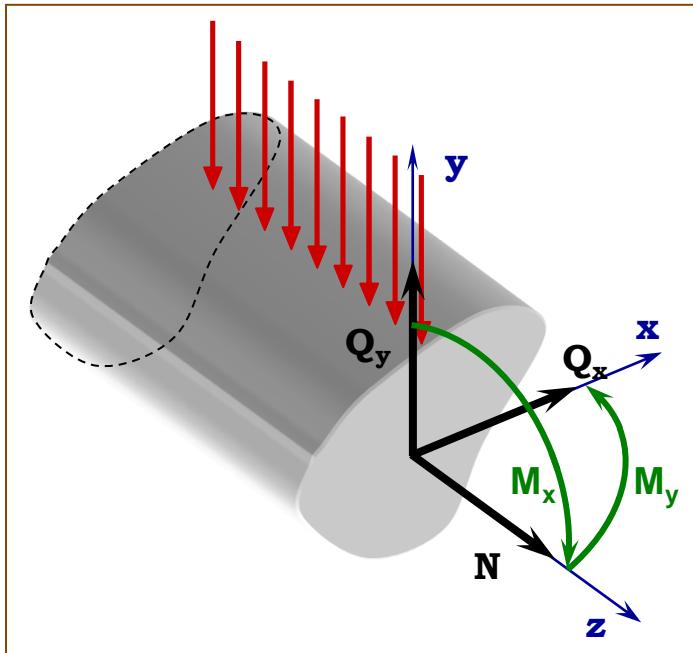


Multi cell



$$e_1, e_2 \leq \frac{1}{10} \cdot \text{Min}\{H, B\}$$

Previous knowledge (Introduction to Structural Analysis)



Displacements field

$$u_z(x, y, z) = u_z^o(z) - \phi_y(z) \cdot x + \phi_x(z) \cdot y$$

Normal stresses

$$\sigma_z = \frac{N}{A} + \frac{y \cdot I_y - x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \cdot M_x + \frac{y \cdot P_{xy} - x \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \cdot M_y$$

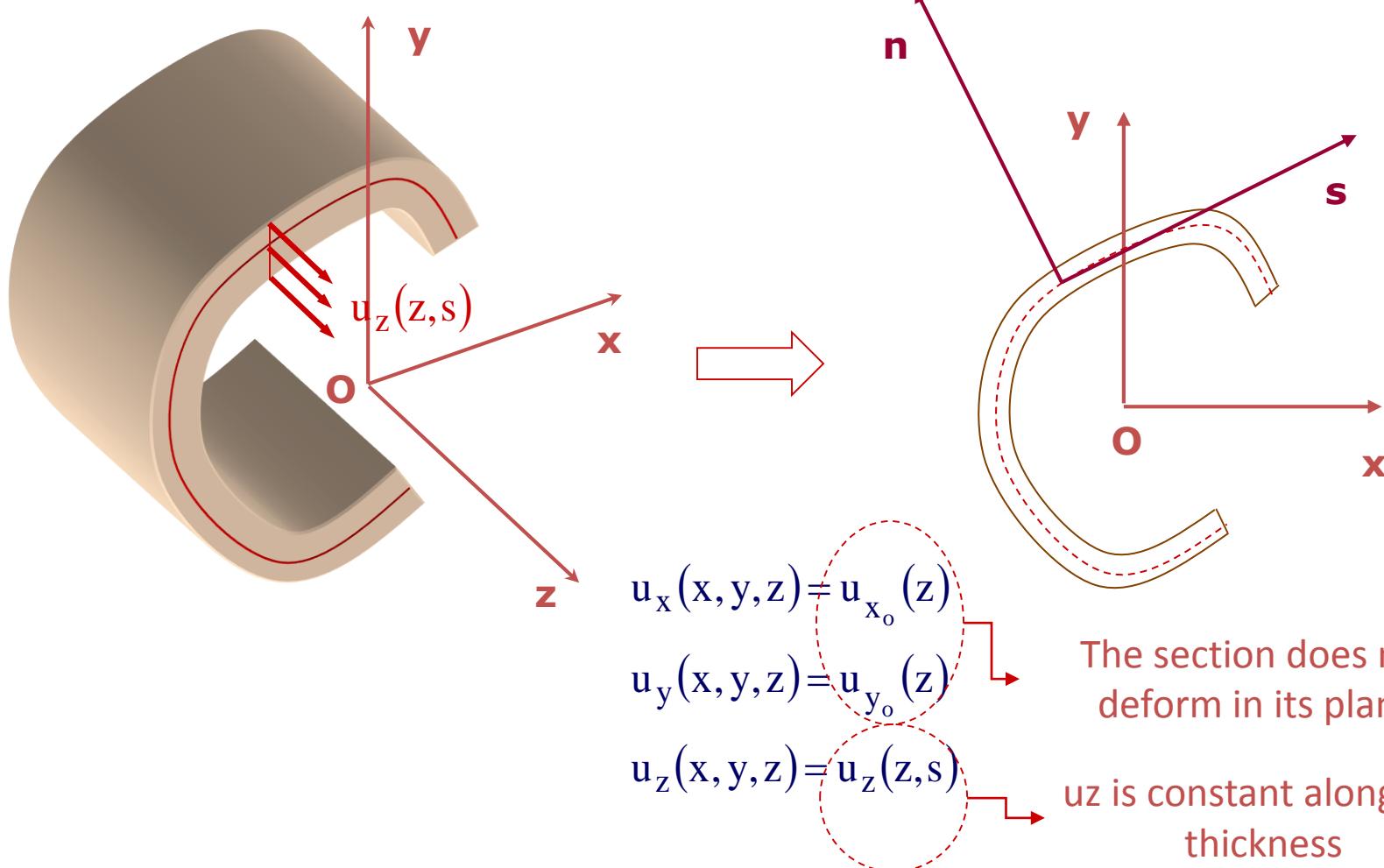
Neutral axis

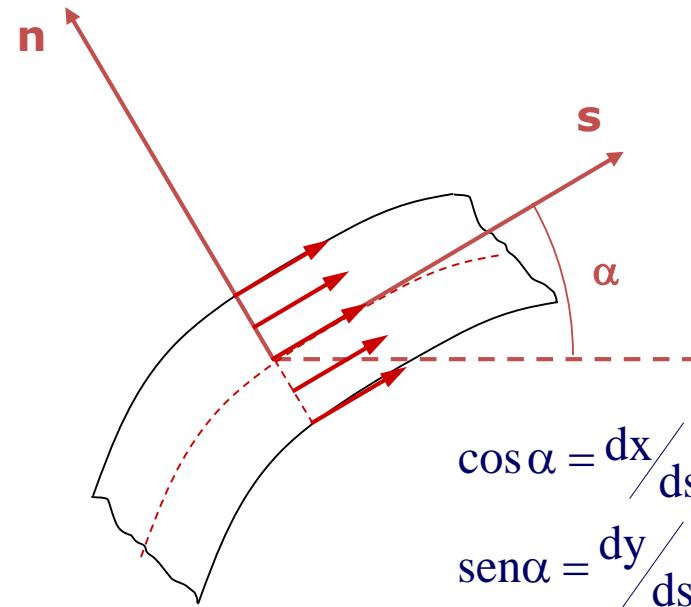
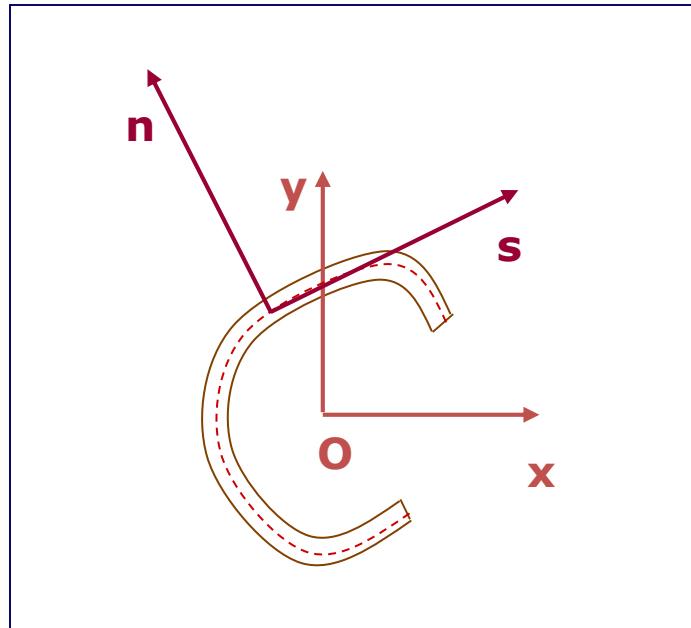
$$\frac{N}{A} + \frac{y \cdot I_y - x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \cdot M_x + \frac{y \cdot P_{xy} - x \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \cdot M_y = 0$$

Shear stresses

$$\tau_{yz} = \frac{Q_y}{a_o(y)} \cdot \left(\frac{m_{ex}^{A_1} \cdot I_y - m_{ey}^{A_1} \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \right) - \frac{Q_x}{a_o(y)} \cdot \left(\frac{m_{ex}^{A_1} \cdot P_{xy} - m_{ey}^{A_1} \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \right)$$

$$\tau_{xz} = \frac{Q_y}{b_o(x)} \cdot \left(\frac{m_{ex}^{A_2} \cdot I_y - m_{ey}^{A_2} \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \right) - \frac{Q_x}{b_o(x)} \cdot \left(\frac{m_{ex}^{A_2} \cdot P_{xy} - m_{ey}^{A_2} \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \right)$$





$$\gamma_{sz}(z,s) = \frac{\partial u_s(z,s)}{\partial z} + \frac{\partial u_z(z,s)}{\partial s}$$



$$u_z(z,s) = u_{z_o}(z) - \frac{du_{x_o}}{dz} \cdot x(s) - \frac{du_{y_o}}{dz} \cdot y(s) + \int_0^s \gamma_{sz}(z,s) \cdot ds$$

Normal stresses

$$u_z(z, s) = u_{z_o}(z) + \phi_x \cdot x(s) - \phi_y \cdot y(s)$$

$$\varepsilon_z(z, s) = \frac{\partial u_z(z, s)}{\partial z}$$

Constitutive equation

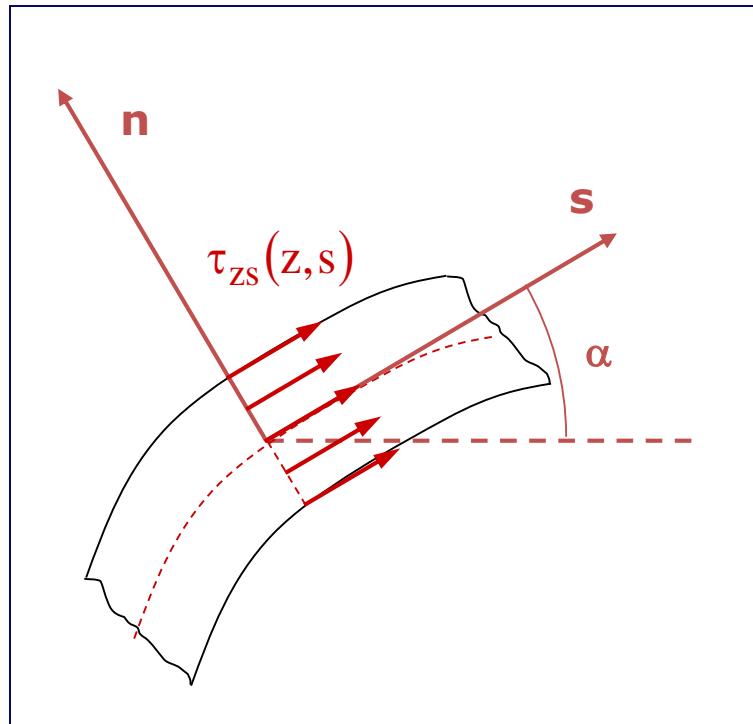
$$\sigma_z(z, s) = E \cdot \left(\frac{du_{z_o}(z)}{dz} + \frac{d\phi_x(z)}{dz} \cdot x(s) - \frac{d\phi_y(z)}{dz} \cdot y(s) \right)$$

Neglecting normal
nonaxial
stresses

Operating as in solid sections

$$\sigma_z(x, y, z) = \frac{N}{A} + \frac{y \cdot I_y - x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \cdot M_x + \frac{y \cdot P_{xy} - x \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \cdot M_y$$

Shear stresses



$$\tau_{zs}(z,s) = \text{cte}$$

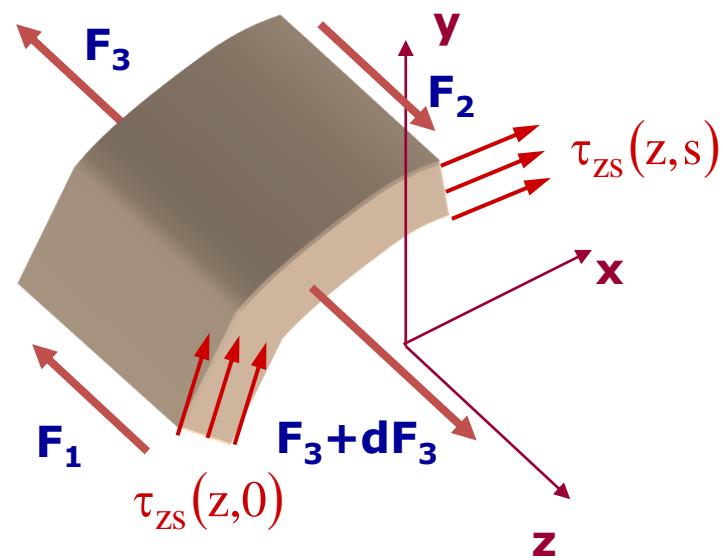
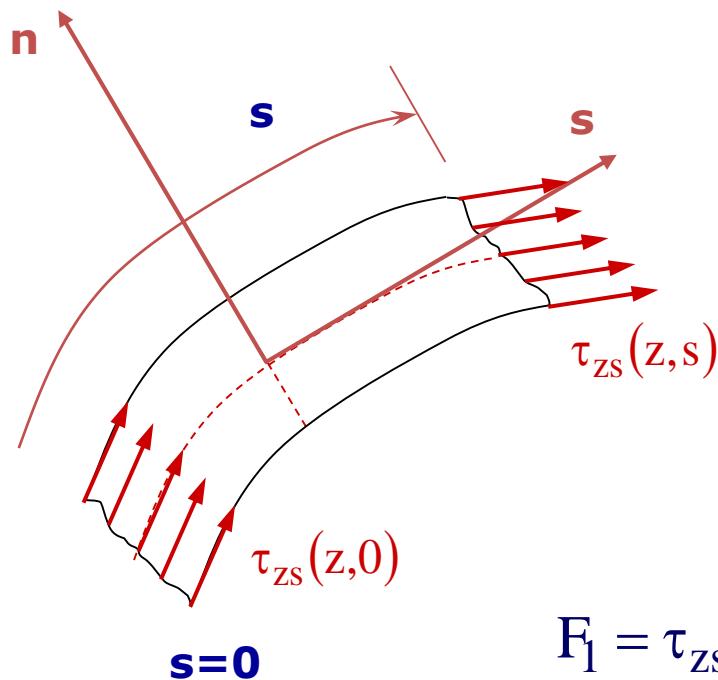
$$\tau_{sn}(z,s) = 0$$

No deformations in cross-section

$$\tau_{zn}(z,s) \approx 0$$

Due to small thickness

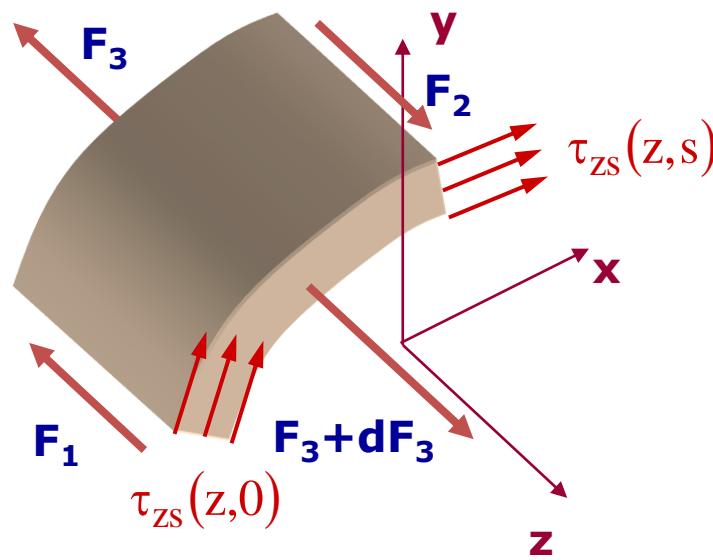
Shear stresses



$$F_1 = \tau_{zs}(z,0) \cdot e(0) \cdot dz$$

$$F_2 = \tau_{zs}(z,s) \cdot e(s) \cdot dz$$

$$F_3 = \int_{A^*} \sigma_z(z,s) \cdot dA$$



Equilibrium equations
in z direction

$$F_3 + dF_3 - F_3 - F_1 + F_2 = 0$$

$$\tau_{zs}(z,s) \cdot e(s) = \tau_{zs}(z,0) \cdot e(0) - \frac{\partial}{\partial z} \int_{A^*} \sigma_z \cdot dA$$

$$\tau_{zs}(z,s) \cdot e(s) = \tau_{zs}(z,0) \cdot e(0) - \frac{\partial}{\partial z} \int_{A^*} \sigma_z \cdot dA$$

Equilibrium
equations

Expression of
normal stresses

$$\tau_{zs}(z,s) \cdot e(s) = \tau_{zs}(z,0) \cdot e(0) - K_y \cdot Q_y + K_x \cdot Q_x$$

The distributed loads in z direction are neglected

$$\tau_{zs}(z,s) \cdot e(s) = \tau_{zs}(z,0) \cdot e(0) - \frac{\partial}{\partial z} \int_{A^*} \sigma_z \cdot dA$$

$$\frac{dM_x}{dz} - Q_y + m_x = 0$$

$$\frac{dQ_x}{dz} + q_x = 0$$

$$\frac{dM_y}{dz} + Q_x + m_y = 0$$

$$\frac{dQ_y}{dz} + q_y = 0$$

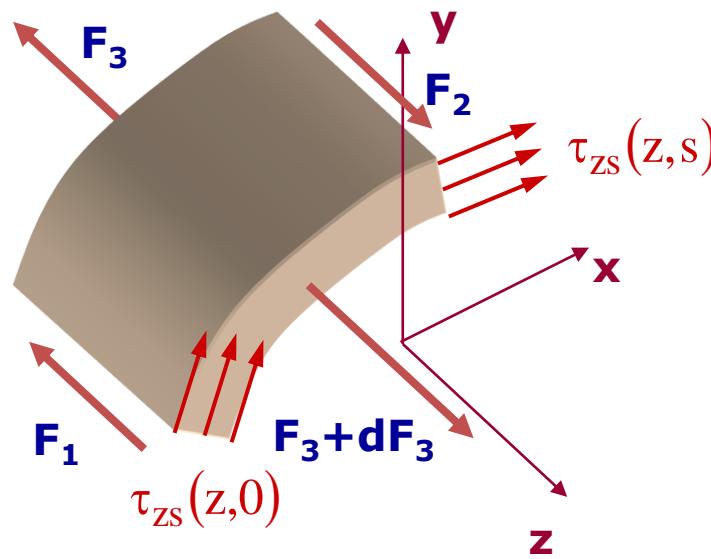
$$\frac{dM_T}{dz} + m_z = 0$$

$$\frac{dN}{dz} + q_z = 0$$

$$\begin{aligned} \sigma_z(x, y, z) = & \frac{N}{A} + \frac{y \cdot I_y - x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \cdot M_x + \\ & + \frac{y \cdot P_{xy} - x \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \cdot M_y \end{aligned}$$

$$\tau_{zs}(z, s) \cdot e(s) = \tau_{zs}(z, 0) \cdot e(0) - K_y \cdot Q_y + K_x \cdot Q_x$$

The distributed loads in z direction are neglected



$$K_y = \frac{m_{ex}^* \cdot I_y - m_{ey}^* \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2}$$

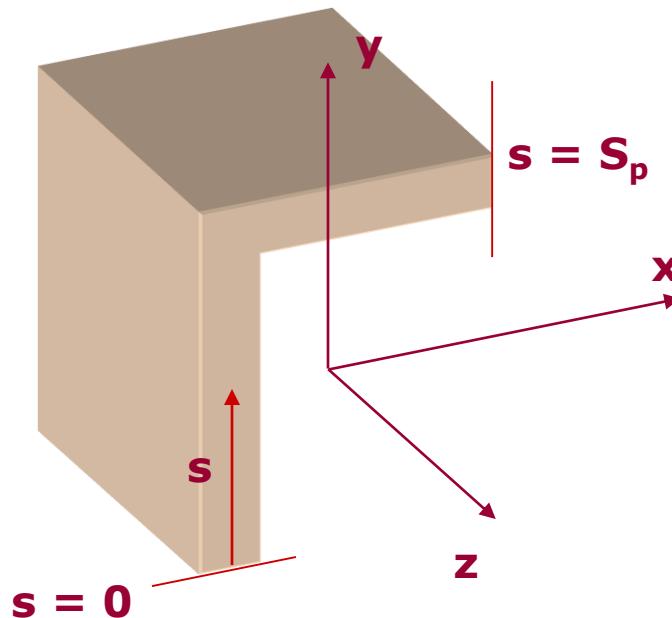
$$K_x = \frac{m_{ex}^* \cdot P_{xy} - m_{ey}^* \cdot I_x}{I_x \cdot I_y - P_{xy}^2}$$

$$m_{ex}^* = \int_{A^*} y(s) \cdot dA = \int_0^s y(s) \cdot e(s) \cdot ds$$

$$m_{ey}^* = \int_{A^*} x(s) \cdot dA = \int_0^s x(s) \cdot e(s) \cdot ds$$

Shear stresses in open sections

Open section



Assumption: no distributed loads in z direction

$$\tau_{zs}(z, 0) \cdot e(0) = 0$$

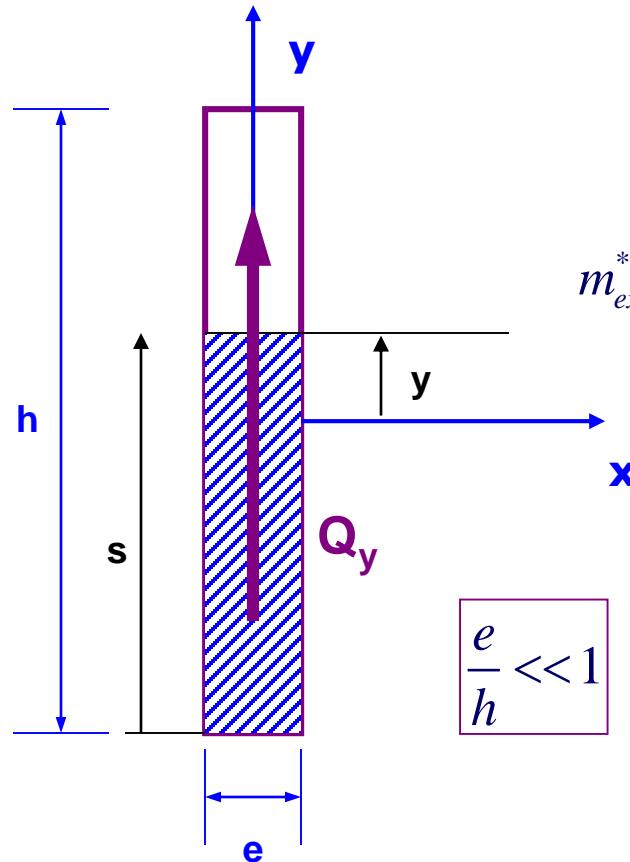
$$\tau_{zs}(z, s) \cdot e(s) = -K_y \cdot Q_y + K_x \cdot Q_x$$

Constant shear flow

$$q(s) = \tau_{zs}(z, s) \cdot e(s)$$

Shear stresses in open sections

Example: open section



$$K_y = \frac{m_{ex}^*}{I_x}$$

$$\tau_{zs}(z, s) \cdot e(s) = -K_y \cdot Q_y$$

$$m_{ex}^* = \int_0^s y(s) \cdot e(s) \cdot ds = \int_0^s \left(s - \frac{h}{2} \right) \cdot e \cdot ds = \frac{s \cdot e}{2} \cdot (s - h)$$

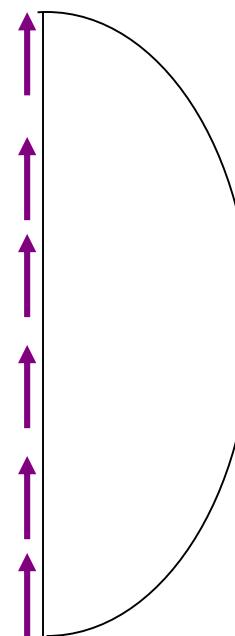
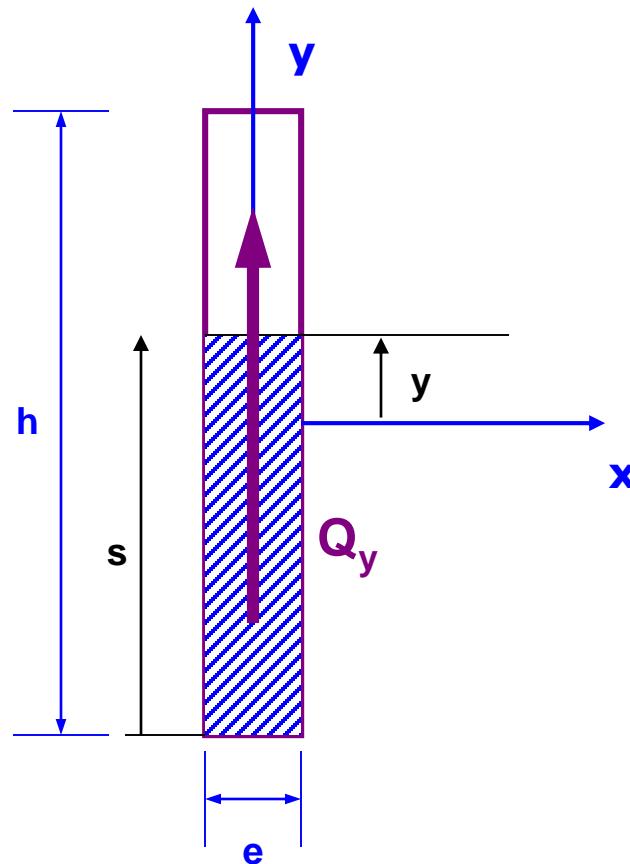
$$q(s) = -\frac{Q_y}{2 \cdot I_x} \cdot s \cdot e \cdot (s - h)$$

$$I_x = \frac{1}{12} \cdot e \cdot h^3$$

$$q(s) = -\frac{6 \cdot Q_y}{h^3} \cdot s \cdot (s - h)$$

Shear stresses in open sections

Example: open section



$$q(s) = \frac{6 \cdot Q_y}{h^3} \cdot s \cdot (h - s)$$

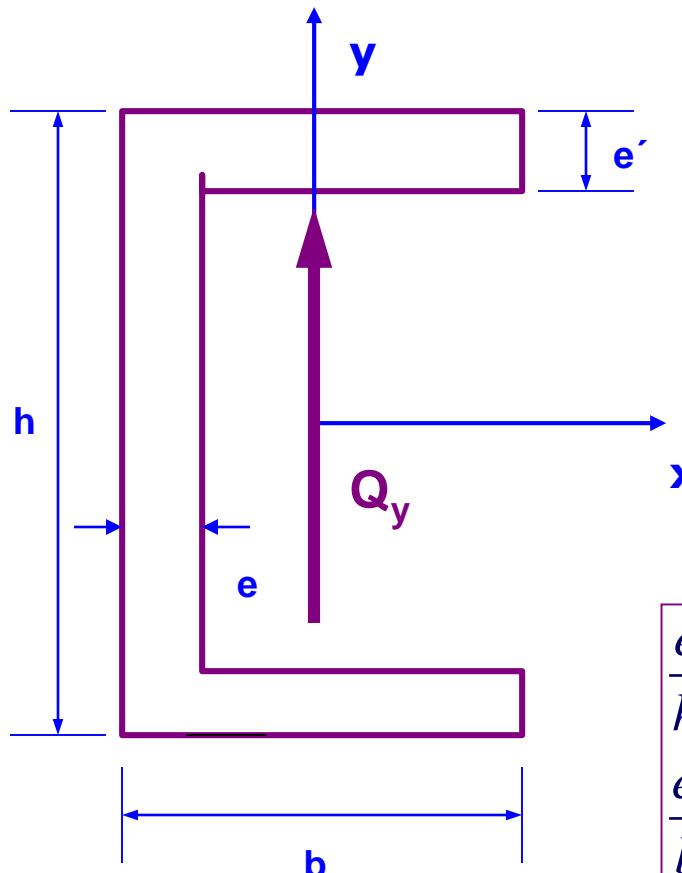
$$s = 0 \quad q(0) = 0$$

$$s = \frac{h}{2} \quad q\left(\frac{h}{2}\right) = \frac{3}{2} \cdot \frac{Q_y}{h}$$

$$s = h \quad q(h) = 0$$

Shear stresses in open sections

Example: open section

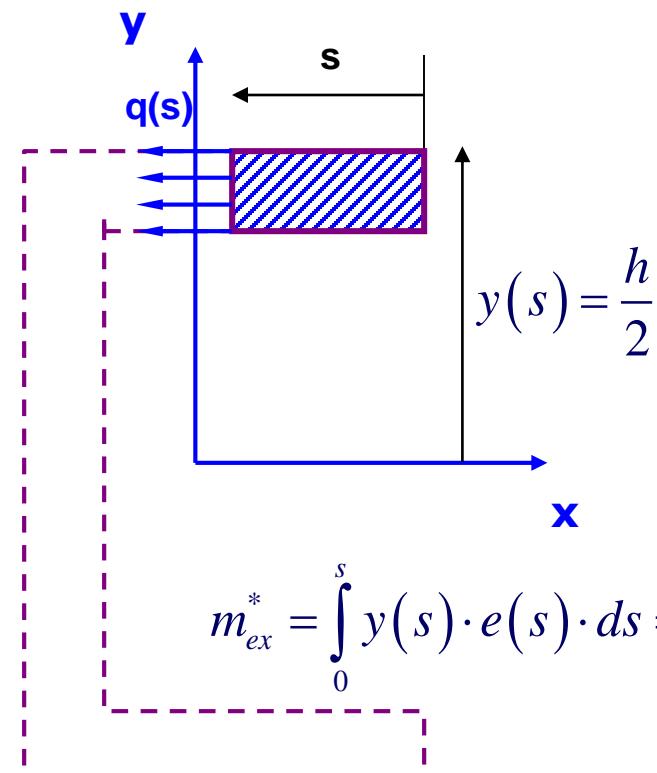
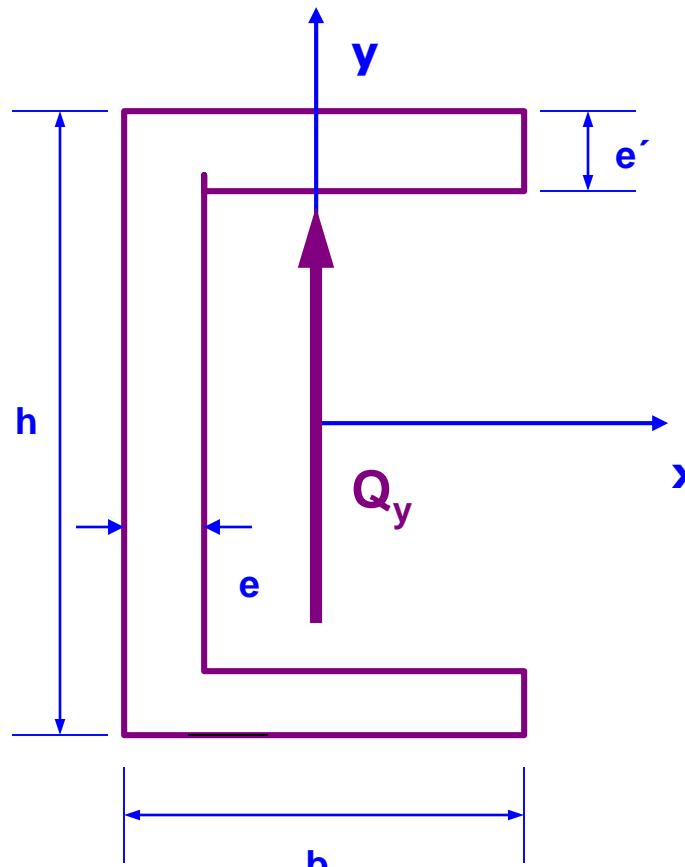


$$\frac{e}{h} \ll 1$$
$$\frac{e'}{b} \ll 1$$

$$K_y = \frac{m_{ex}^*}{I_x}$$
$$q(s) = -K_y \cdot Q_y$$

Shear stresses in open sections

Example: open section

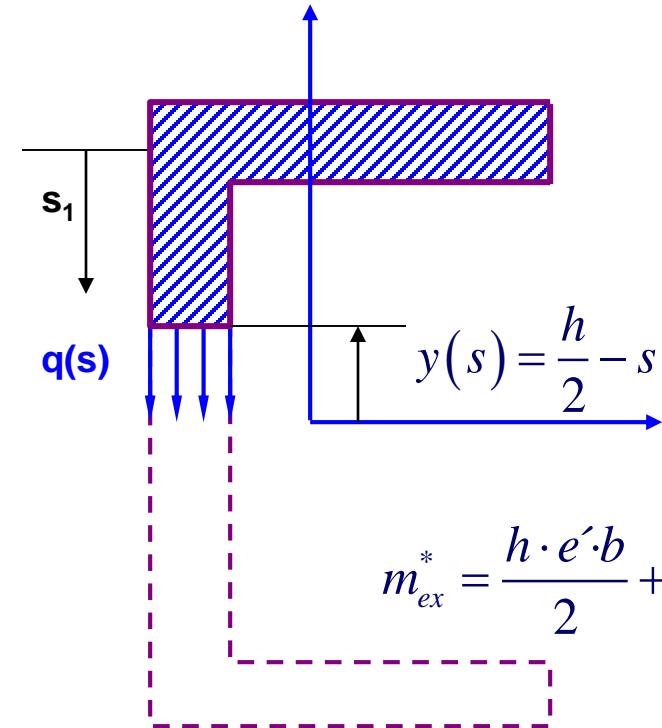
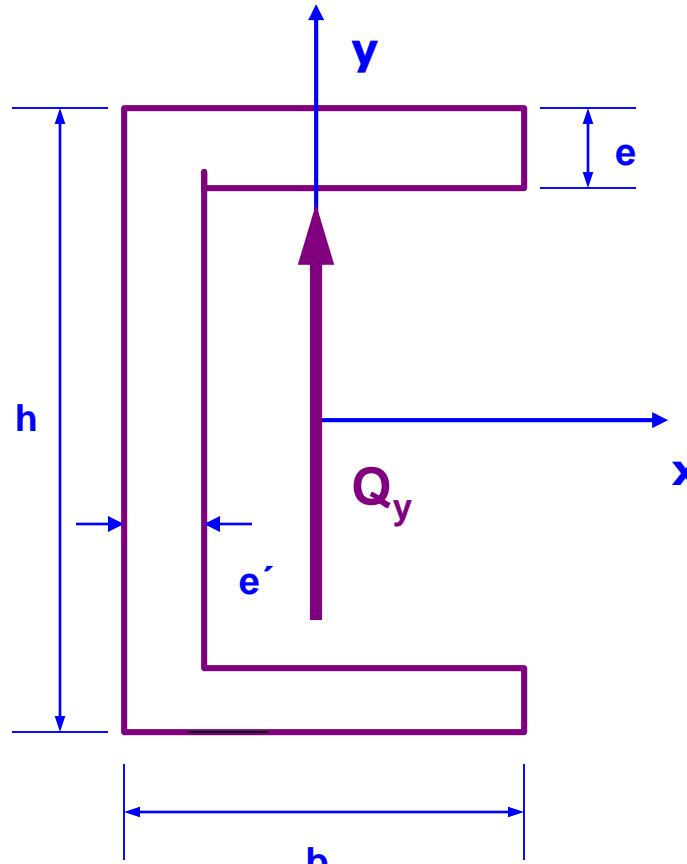


$$m_{ex}^* = \int_0^s y(s) \cdot e(s) \cdot ds = \frac{s \cdot e' \cdot h}{2}$$

$$q(s) = -\frac{Q_y}{I_x} \cdot \frac{s \cdot e' \cdot h}{2}$$

Shear stresses in open sections

Example: open section

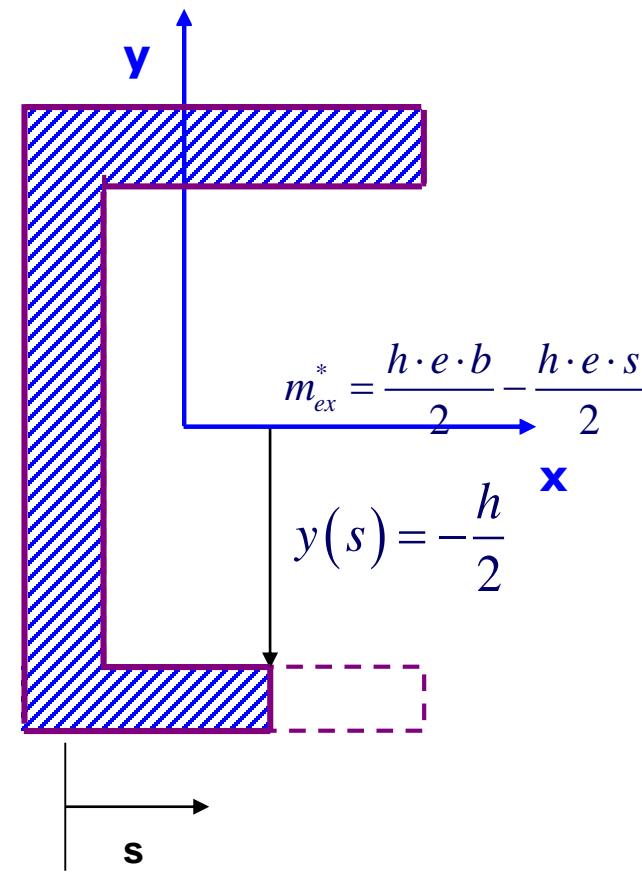
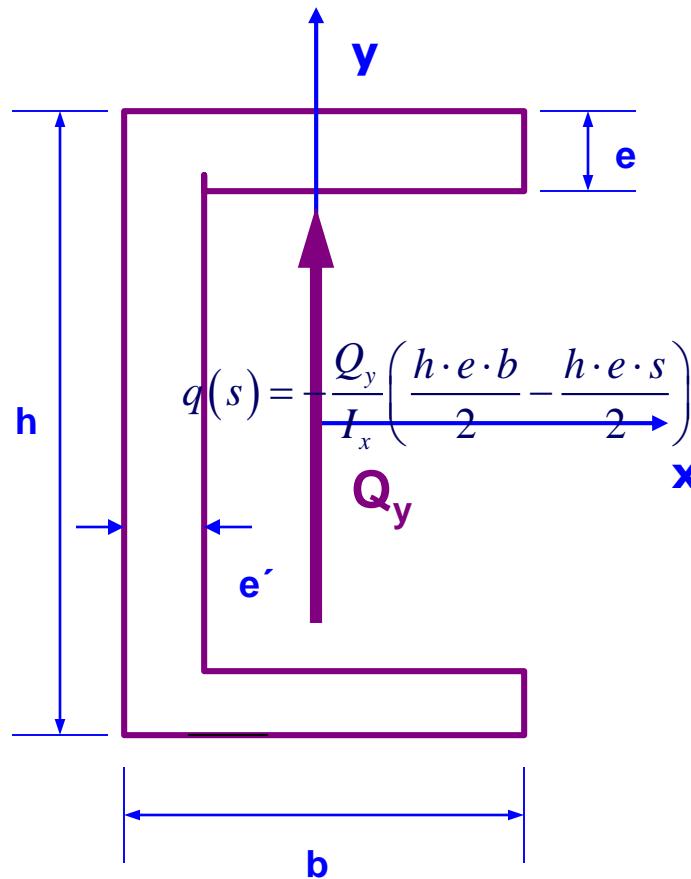


$$m_{ex}^* = \frac{h \cdot e' \cdot b}{2} + \frac{s_1 \cdot e}{2} \cdot (h - s_1)$$

$$q(s) = -\frac{Q_y}{I_x} \cdot \left[\frac{h \cdot e' \cdot b}{2} + \frac{s_1 \cdot e}{2} \cdot (h - s_1) \right]$$

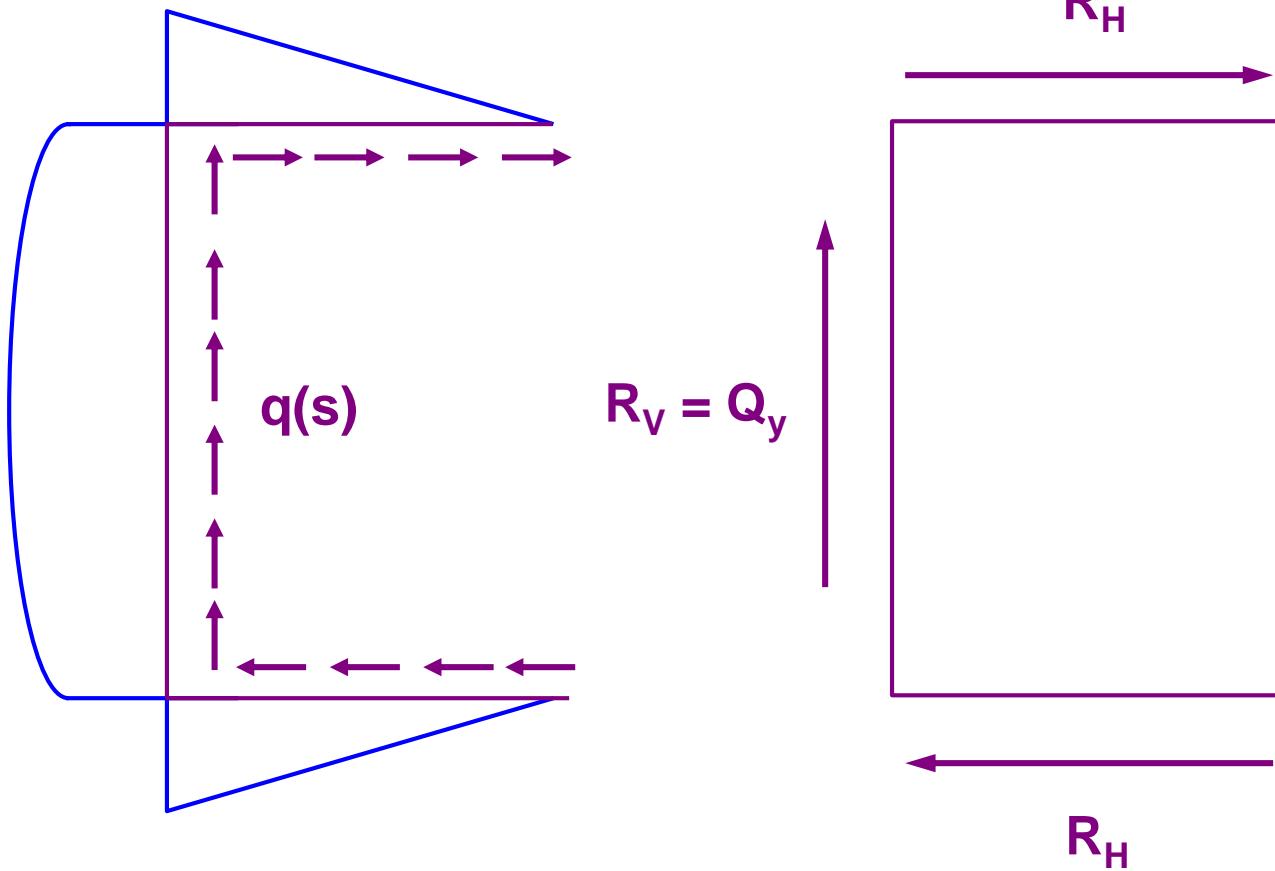
Shear stresses in open sections

Example: open section



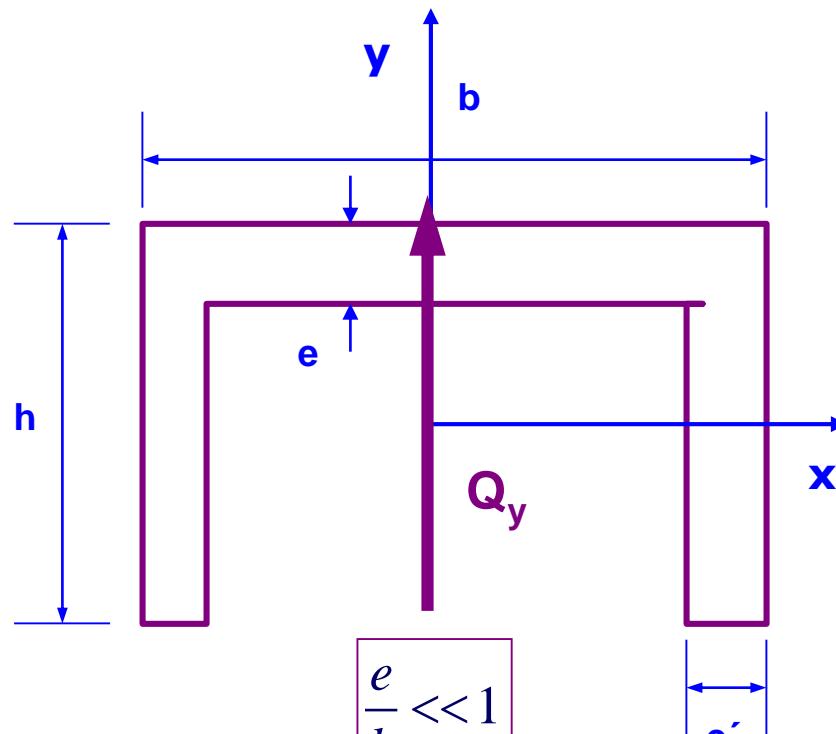
Shear stresses in open sections

Example: open section

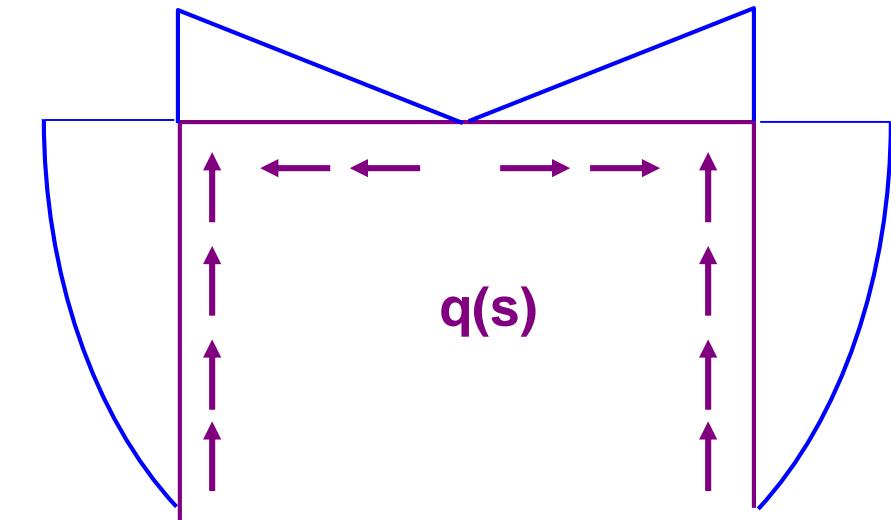


Shear stresses in open sections

Example: open section

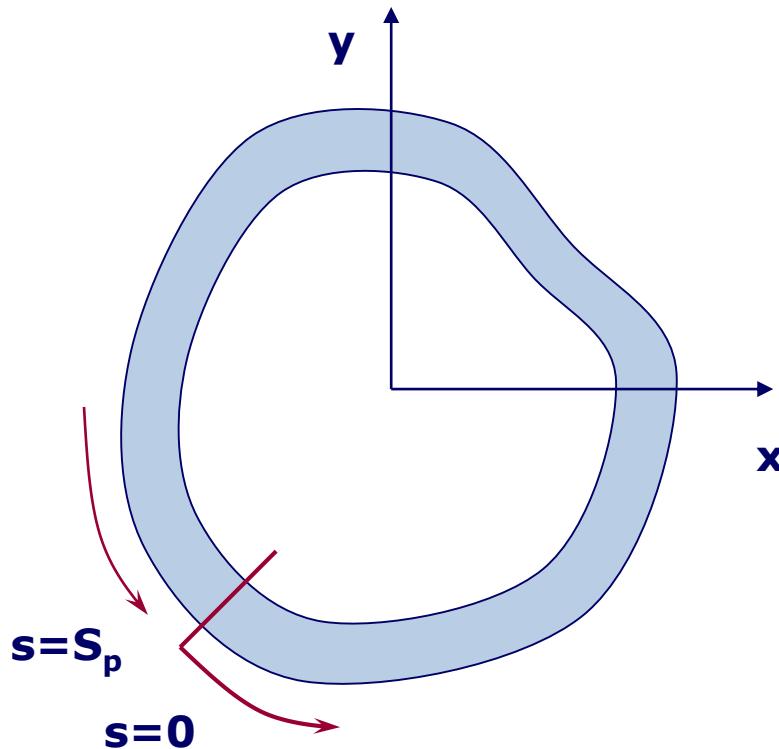


$$\frac{e}{h} \ll 1$$
$$\frac{e'}{b} \ll 1$$



Shear stresses in closed sections

Single-cell closed section

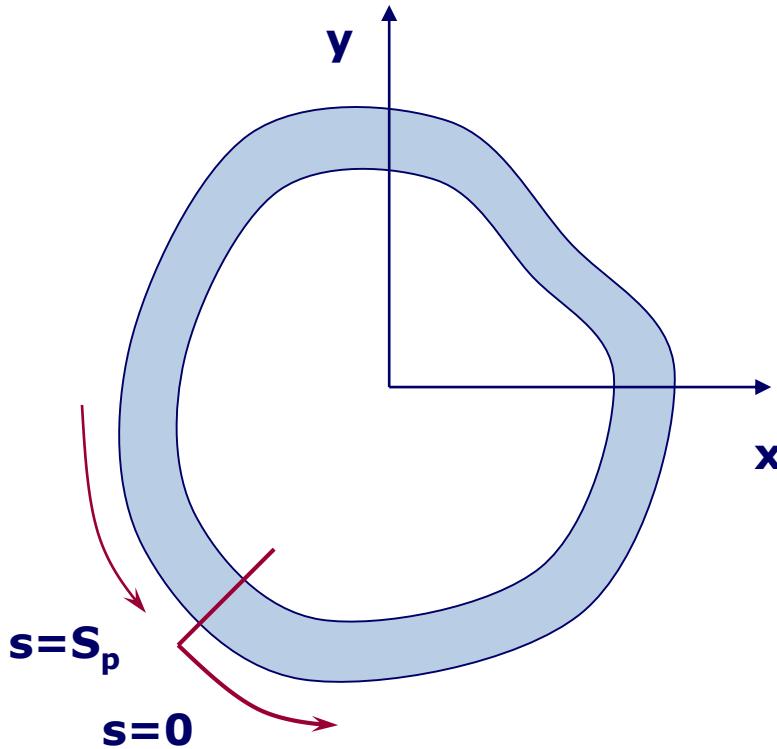


An arbitrary point is the origin of the coordinate s

$$q(s) = q(s)_{open} + q(0)$$

Shear stresses in closed sections

Single-cell closed section



$$q(s) = q(s)_{open} + q(0)$$

$$\int_0^{S_p} \gamma_{zs} \cdot ds = 0$$

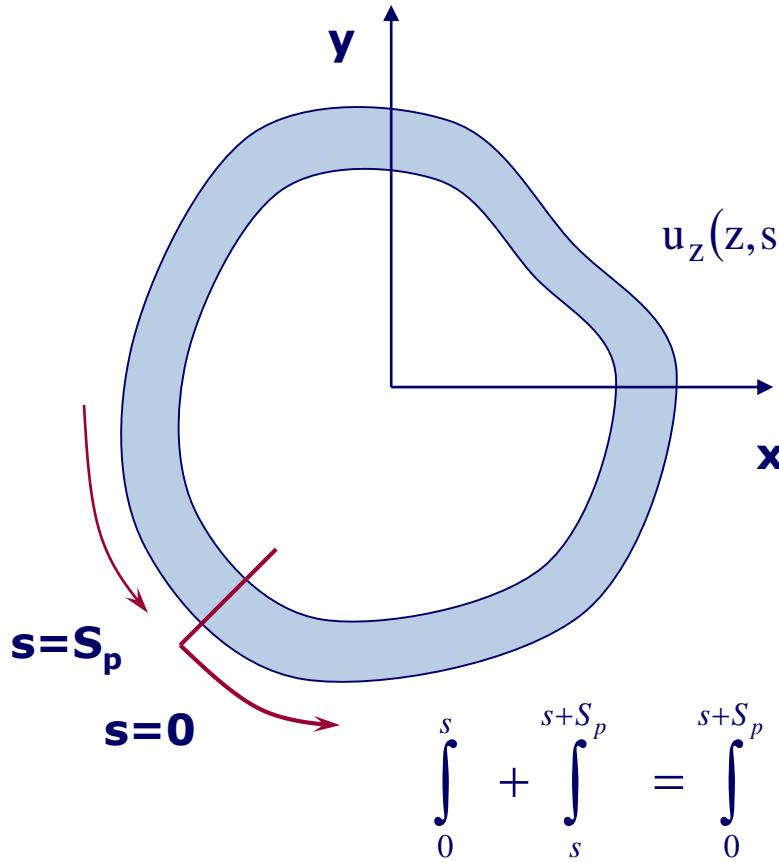
$$\int_0^{S_p} \frac{\tau_{zs}}{G} \cdot ds = 0$$

$$\int_0^{S_p} \frac{q(s)_{open}}{G \cdot e(s)} \cdot ds + \int_0^{S_p} \frac{q(0)}{G \cdot e(s)} \cdot ds = 0$$



Cálculo de tensiones tangenciales

Single-cell closed section



$$u_z(z, s) = u_{z_o}(z)$$

$$y(s) = y(s + S_p)$$

$$z(s) = z(s + S_p)$$

$$u_z(z, s) = u_{z_o}(z) - \frac{du_{x_o}}{dz} \cdot x(s) - \frac{du_{y_o}}{dz} \cdot y(s) + \int_0^s \gamma_{sz}(z, s) \cdot ds$$

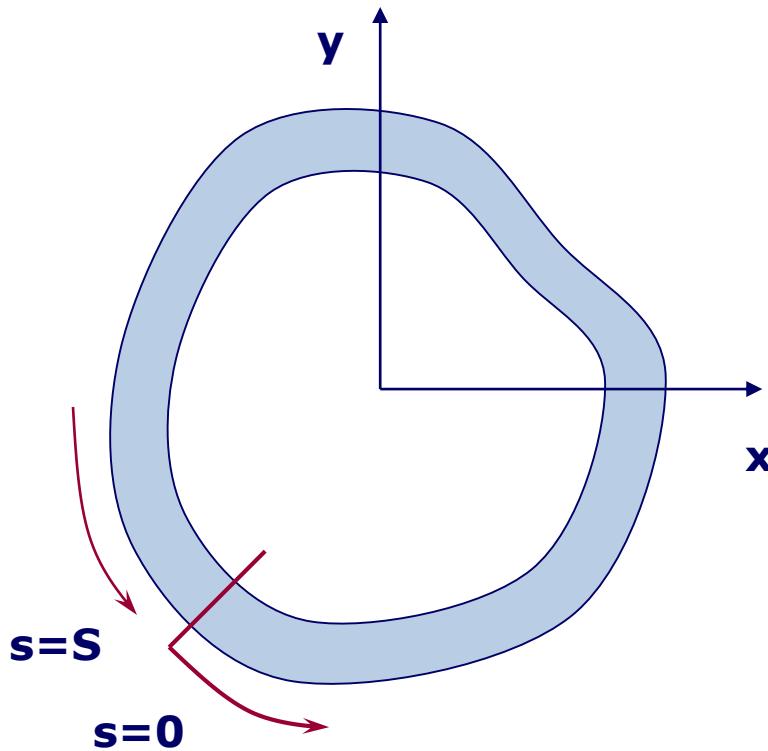
$$\int_0^s \gamma_{sz}(z, s) \cdot ds = \int_0^{s+S_p} \gamma_{sz}(z, s) \cdot ds$$

$$\int_0^s + \int_s^{s+S_p} = \int_0^{s+S_p} = \int_s^{s+S_p} = 0$$

$$\int_0^{S_p} \gamma_{sz}(z, s) \cdot ds = 0$$

Shear stresses in closed sections

Single-cell closed section



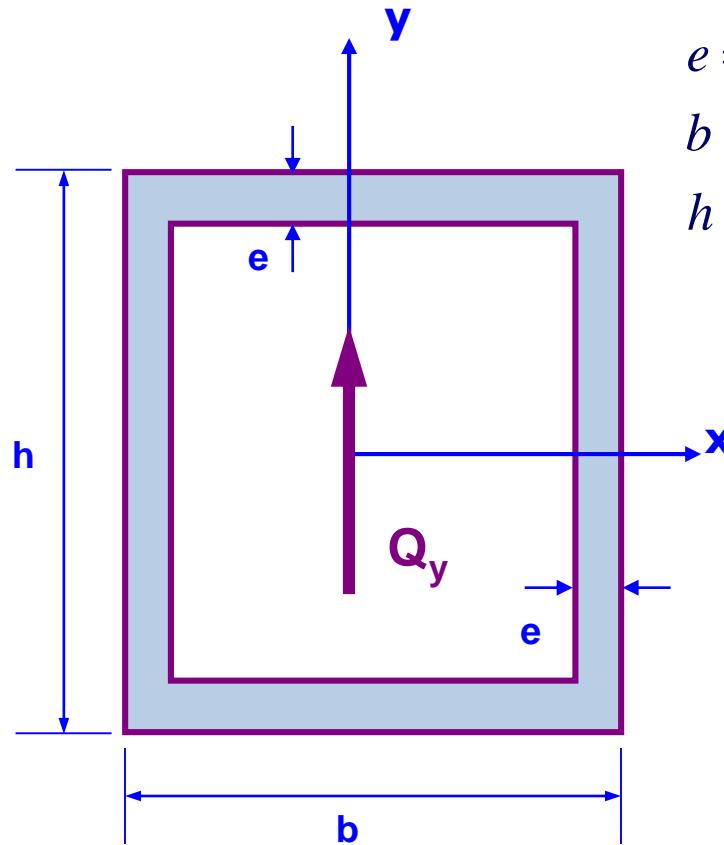
$$q(0) = - \frac{\int_0^S \frac{q(s)_{open}}{G \cdot e(s)} \cdot ds}{\int_0^S \frac{1}{G \cdot e(s)} \cdot ds}$$

$$q(s) = q(s)_{open} + q(0)$$

Shear stresses in closed sections

Single-cell closed section

Example

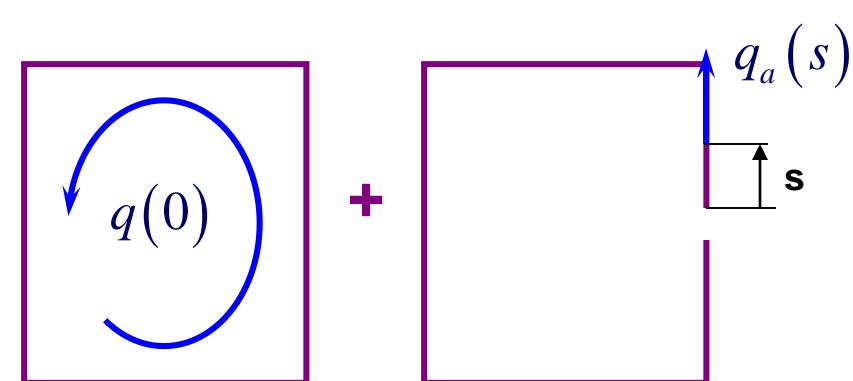


$$e = 0,8 \text{ cm}$$

$$b = 15 \text{ cm}$$

$$h = 20 \text{ cm}$$

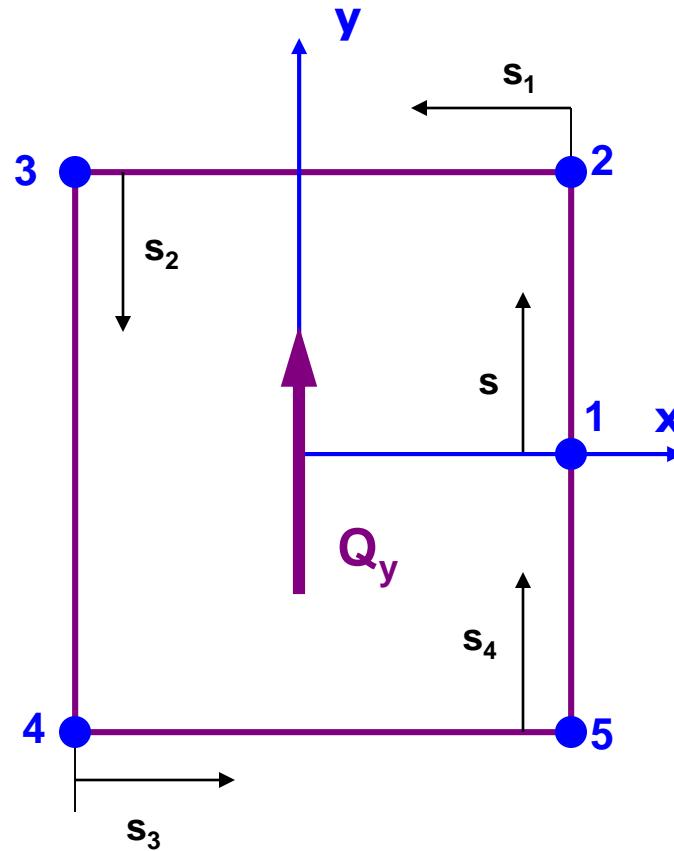
$$q(s) = q_a(s) + q(0)$$



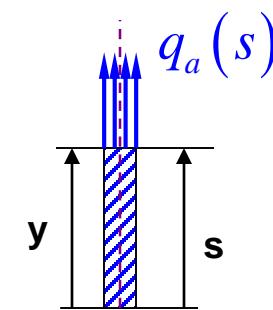
Shear stresses in closed sections

Single-cell closed section

Example



zone 1-2



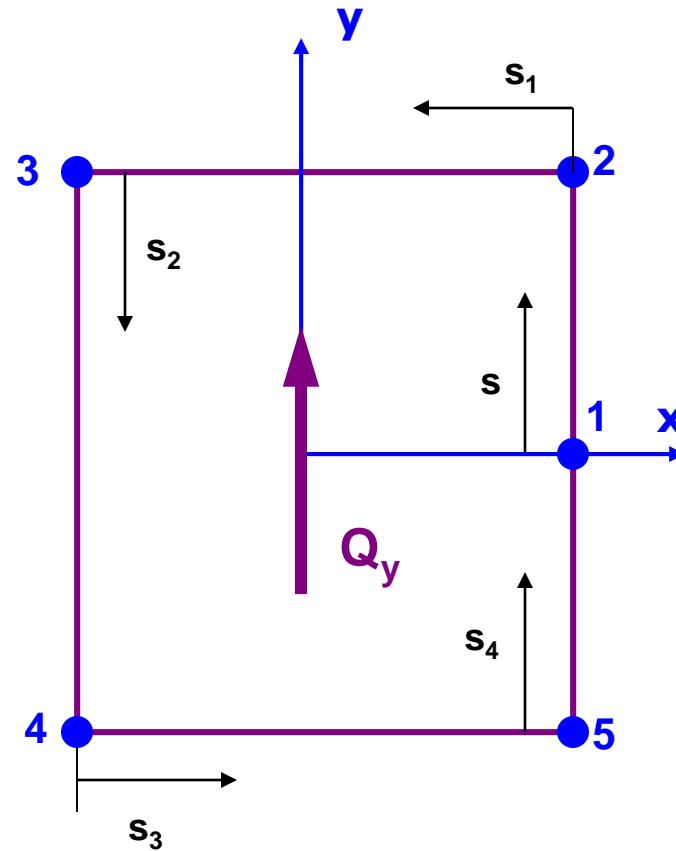
$$m_{ex}^* = \int_0^s y(s) \cdot e(s) \cdot ds = \frac{s^2 \cdot e}{2}$$

$$q_a(s) = -\frac{Q_y}{I_x} \cdot \frac{e \cdot s^2}{2}$$

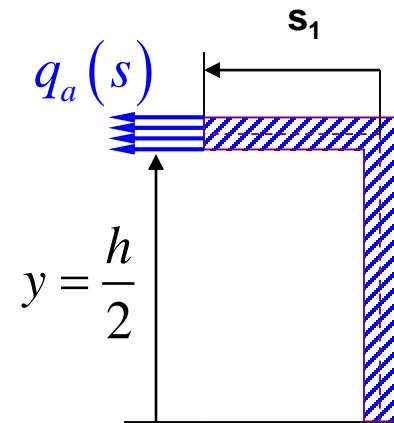
Shear stresses in closed sections

Single-cell closed section

Example



zone 2-3



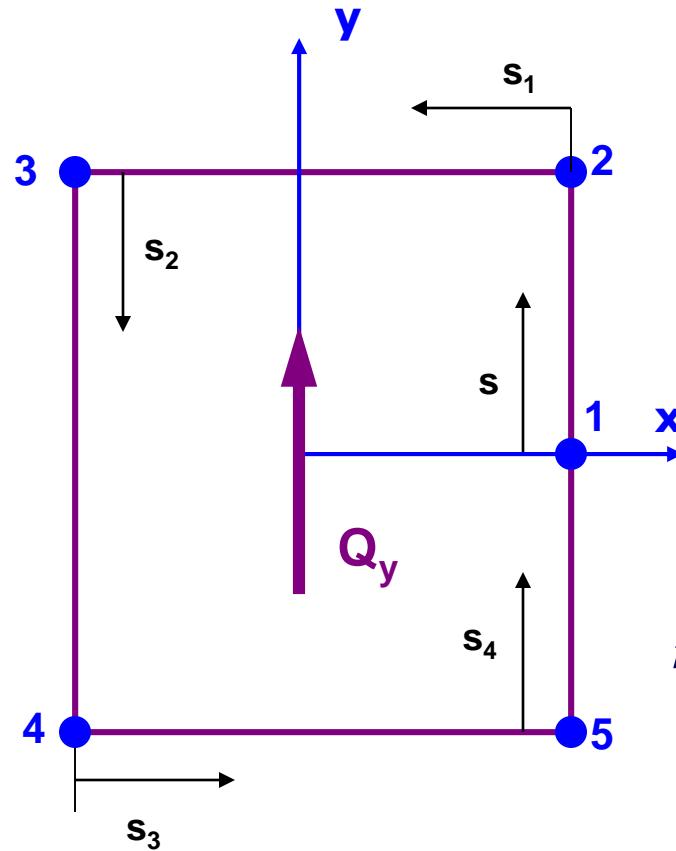
$$m_{ex}^* = \int_0^s y(s) \cdot e(s) \cdot ds = \frac{e \cdot h^2}{8} + \frac{h \cdot e \cdot s_1}{2}$$

$$q(s) = -\frac{Q_y}{I_x} \cdot \left(\frac{e \cdot h^2}{8} + \frac{h \cdot e \cdot s_1}{2} \right)$$

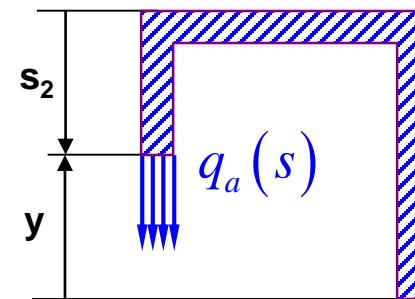
Shear stresses in closed sections

Single-cell closed section

Example



zone 3-4



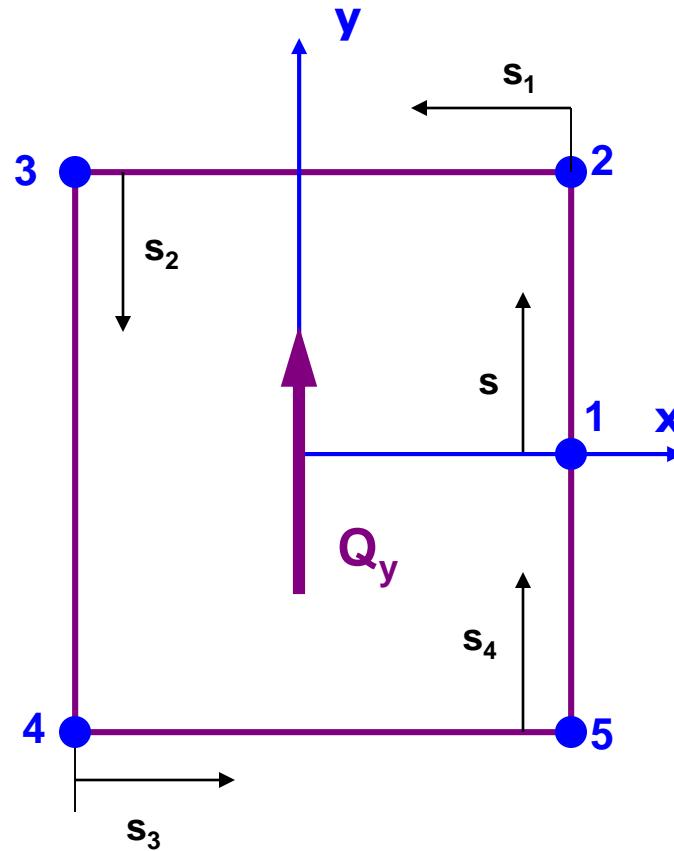
$$m_{ex}^* = \int_0^s y(s) \cdot e(s) \cdot ds = \frac{e \cdot h^2}{8} + \frac{h \cdot e \cdot b}{2} + \frac{e \cdot s_2}{2} \cdot (h - s_2)$$

$$q(s) = -\frac{Q_y}{I_x} \cdot \left[\frac{e \cdot h^2}{8} + \frac{h \cdot e \cdot b}{2} + \frac{e \cdot s_2}{2} \cdot (h - s_2) \right]$$

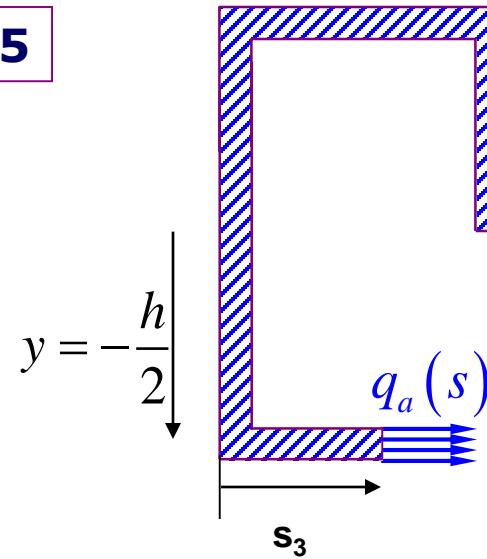
Shear stresses in closed sections

Single-cell closed section

Example



zone 4-5



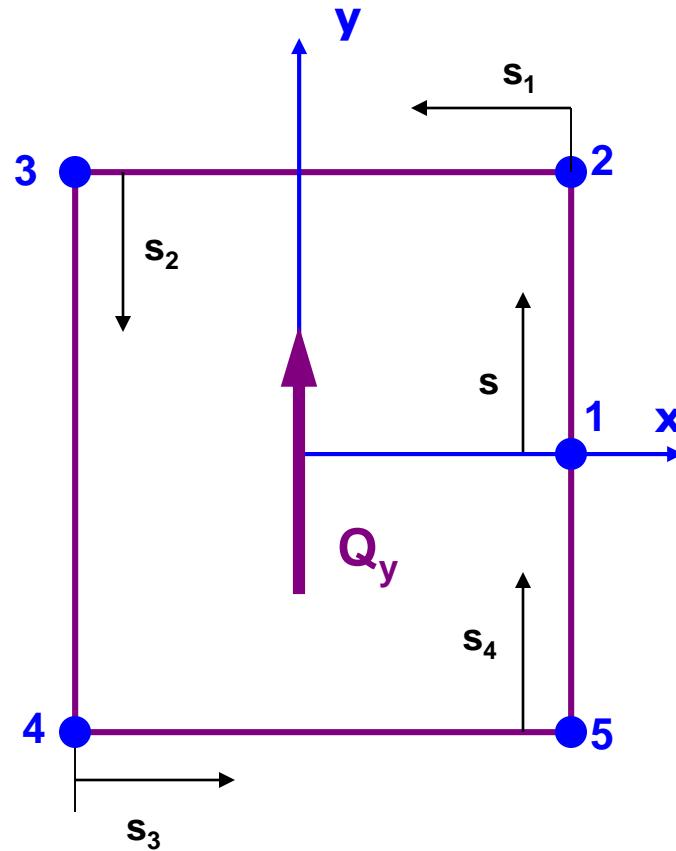
$$m_{ex}^* = \int_0^s y(s) \cdot e(s) \cdot ds = \frac{e \cdot h^2}{8} + \frac{h \cdot e \cdot b}{2} - \frac{h \cdot e}{2} \cdot s_3$$

$$q(s) = -\frac{Q_y}{I_x} \cdot \left[\frac{e \cdot h^2}{8} + \frac{h \cdot e \cdot b}{2} - \frac{h \cdot e}{2} \cdot s_3 \right]$$

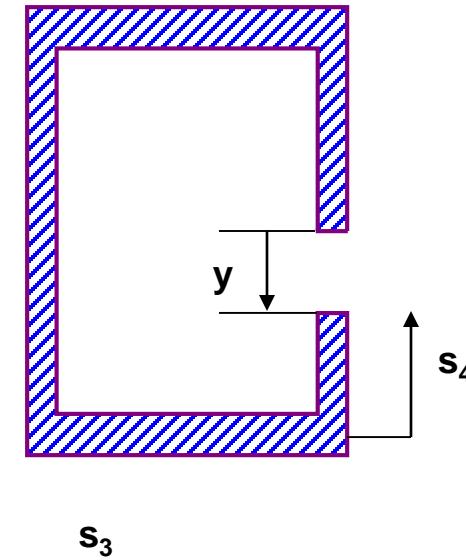
Shear stresses in closed sections

Single-cell closed section

Example



zone 5-1



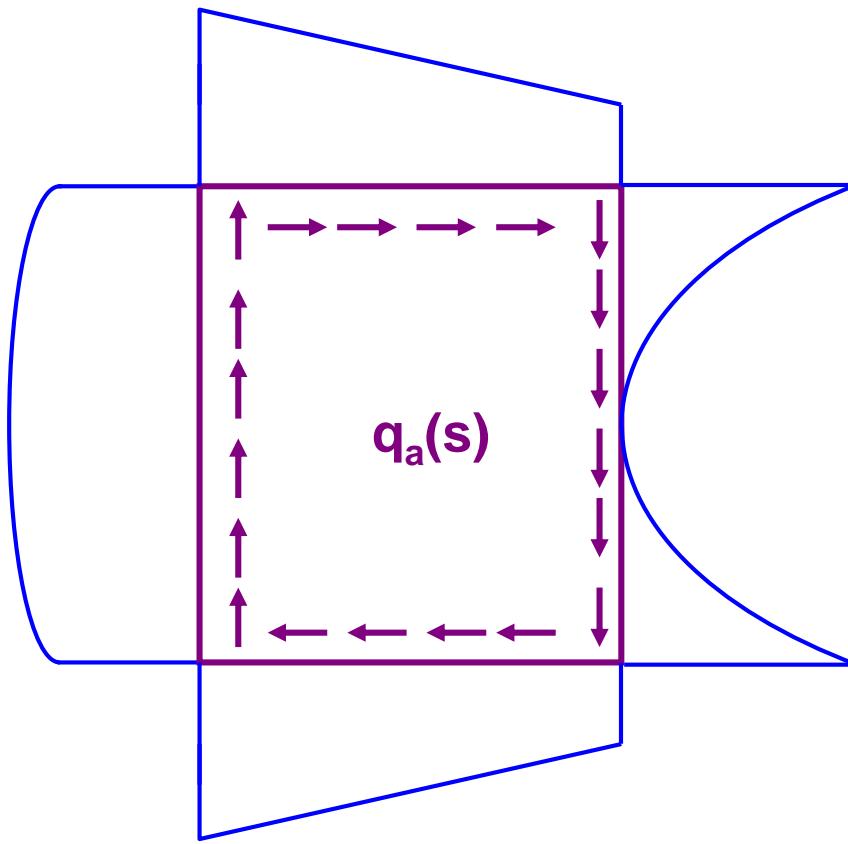
$$m_{ex}^* = \int_0^s y(s) \cdot e(s) \cdot ds = \frac{e \cdot h^2}{8} + \frac{s_4 \cdot e}{2} \cdot (s_4 - h)$$

$$q(s) = -\frac{Q_y}{I_x} \cdot \left[\frac{e \cdot h^2}{8} + \frac{s_4 \cdot e}{2} \cdot (s_4 - h) \right]$$

Shear stresses in closed sections

Single-cell closed section

Example



$$q(0) = -\frac{\int_0^{2(h+b)} q_a(s) \cdot ds}{2 \cdot (h + b)}$$

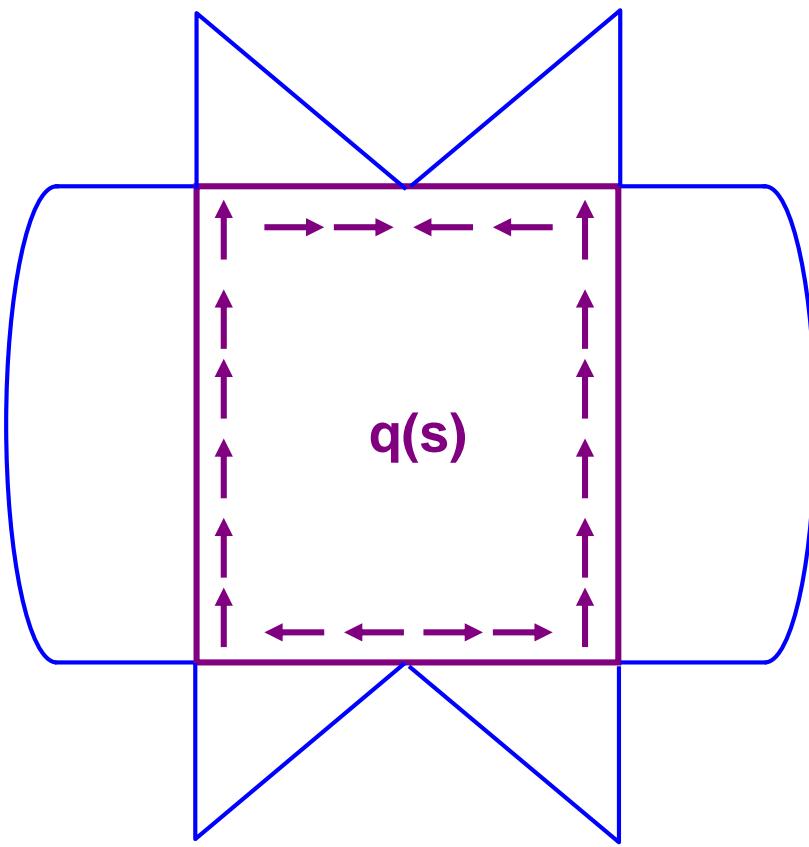
$$q(0) = 100 \cdot \frac{Q_y}{I_x}$$

$$q(s) = q_a(s) + 100 \cdot \frac{Q_y}{I_x}$$

Shear stresses in closed sections

Single-cell closed section

Example



$$q(0) = -\frac{\int_0^{2(h+b)} q_a(s) \cdot ds}{2 \cdot (h + b)}$$

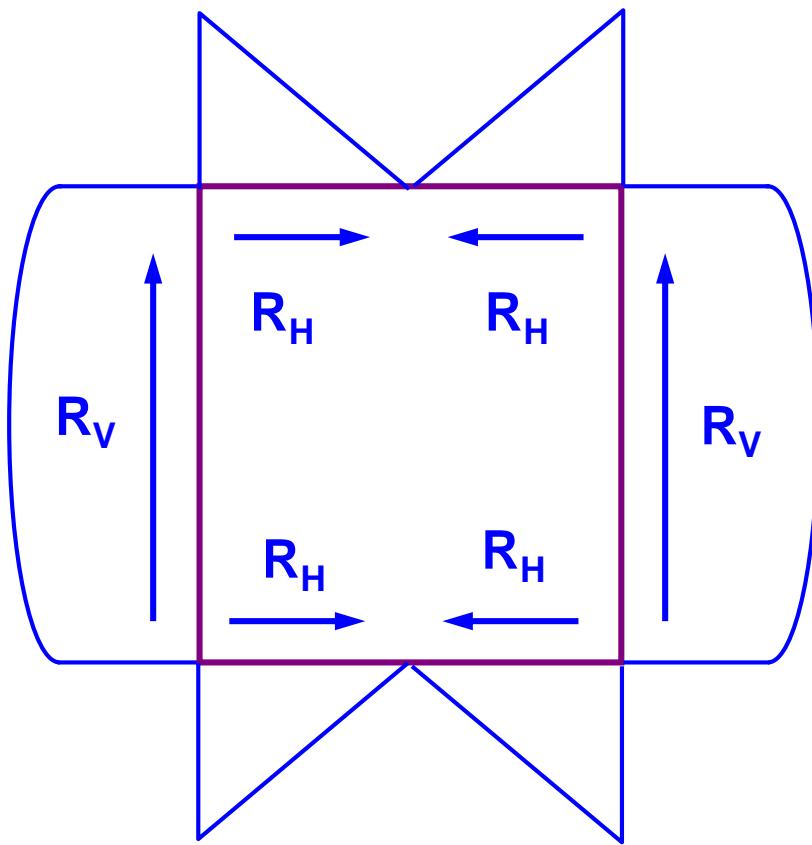
$$q(0) = 100 \cdot \frac{Q_y}{I_x}$$

$$q(s) = q_a(s) + 100 \cdot \frac{Q_y}{I_x}$$

Shear stresses in closed sections

Single-cell closed section

Example

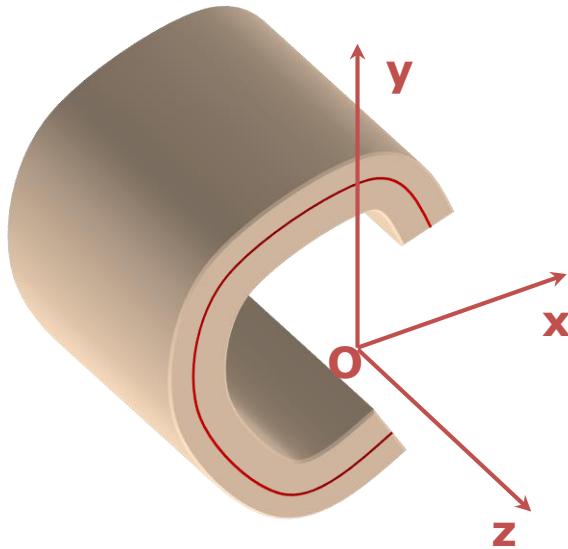


$$q(0) = -\frac{\int_0^{2(h+b)} q_a(s) \cdot ds}{2 \cdot (h + b)}$$

$$q(0) = 100 \cdot \frac{Q_y}{I_x}$$

$$q(s) = q_a(s) + 100 \cdot \frac{Q_y}{I_x}$$

$$R_V = \frac{Q_y}{2}$$



Displacements

$$u_x(x, y, z) = u_{x_0}(z)$$

$$u_y(x, y, z) = u_{y_0}(z)$$

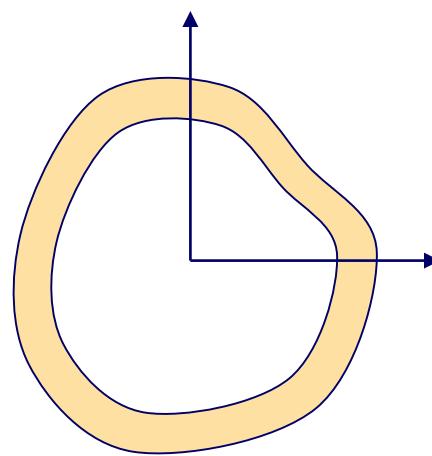
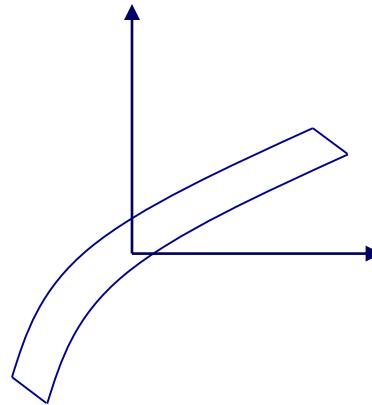
$$u_z(x, y, z) = u_z(z, s)$$

Normal stresses

$$\sigma_z(x, y, z) = \frac{N}{A} + \frac{y \cdot I_y - x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2} \cdot M_x + \frac{y \cdot P_{xy} - x \cdot I_x}{I_x \cdot I_y - P_{xy}^2} \cdot M_y$$

Shear stresses

$$q(s) = q(0) - K_y \cdot Q_y + -K_x \cdot Q_x$$



Open section

$$q(s) = q(0) - K_y \cdot Q_y - K_x \cdot Q_x$$

Closed sections

$$q(s) = q(s)_{open} + q(0)$$

$$q(0) = - \frac{\int_0^s \frac{q(s)_{open}}{G \cdot e(s)} \cdot ds}{\int_0^s \frac{1}{G \cdot e(s)} \cdot ds}$$

- 1. Cervera Ruiz, M. y Blanco Diaz, E. “Mecánica de Estructuras: Libro 2. Métodos de análisis”**
Ed. UPC, 2001
Cap.6 Esfuerzo cortante

- 2. Garrido, J.A. Y Foces, A. “Resistencia de Materiales”.**
Secretariado de Publicaciones. Universidad de Valladolid. 1994
Cap.4 Tracción (compresión)-flexión en barras esbeltas de sección de pared delgada

- 3. Ortiz Berrocal, L. “Resistencia de Materiales”.**
Ed. McGraw Hill. 2002
Cap.4 Teoría general de la flexión. Análisis de tensiones



- 1. Gere, J.M. “Resistencia de Materiales”
Thomson. 2002**
Cap.1 Tracción, compresión y cortante
Cap.4 Esfuerzos cortantes y momentos flexionantes
Cap.5 Tensiones en vigas (temas básicos)
Cap.6 Tensiones en vigas (temas avanzados)

- 2. Benham, P.P. Y Crawford. “Mechanics of engineering materials”
Logman Scientific & Technical, 1987**
Cap.6 Bending: Stress

Mechanical Design Group
University of Massachusetts Amherst
<http://www.ecs.umass.edu/mie/labs/mda/dlib/peter/statload.htm>

