Universidad Carlos III de Madrid
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# Aerospace Structures

## Chapter 2. Bending, shear and torsion of thin-walled beams

### Torsion on thin-walled beams

**Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero**

# CHAPTER 2. Bending, shear and torsion of thin-walled beams

## Torsion on thin walled beams

### OBJETIVES

- Knowledge of the hypothesis of torsion in thin-walled beams
- Calculate the shear stress distribution on open sections
- Calculate the shear stress distribution on single-cell closed sections
- Calculate the shear stress distribution on multiple cell sections

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## Torsion on thin walled beams

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## Torsion on thin walled beams

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## Previous knowledge

**Kinematic hypothesis**

$$u_x(x,y,z) = -y \cdot \phi_z(z)$$
$$u_y(x,y,z) = x \cdot \phi_z(z)$$
$$u_z(x,y,z) = w(x,y)$$

**Displacements field**

$$w(x,y) = \omega \cdot \Psi(x,y) \qquad \nabla^2 \Psi = 0$$

**Stresses field**

$$\tau_{xz} = \frac{\partial \varphi}{\partial y} \qquad \nabla^2 \varphi = -2 \cdot G \cdot \omega$$

$$\tau_{yz} = -\frac{\partial \varphi}{\partial x} \qquad M_T = 2 \cdot \int_A \varphi(x,y) \cdot dA$$

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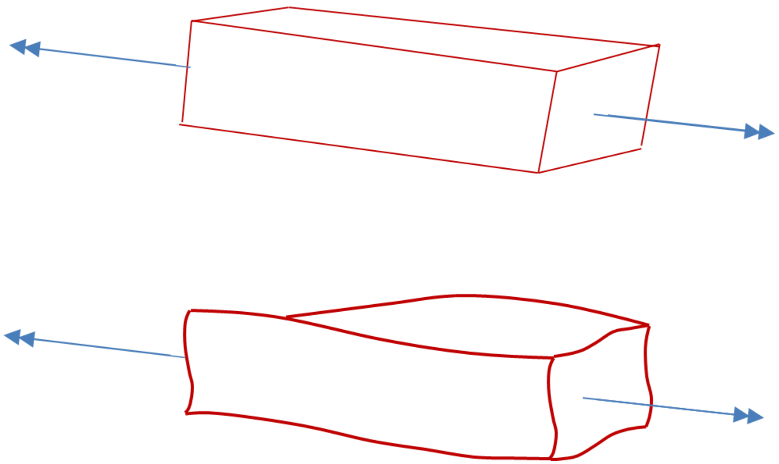
**Uniform torsion**

**The beam is subjected to a uniform torsional moment while section displacements are not constrained by boundary conditions**

**Saint-Venant hypothesis**

**Displacement field is produced by:**

- **A rotation of each section about a point. This point is the torsion centre**
- **Every section experience the same warping displacement**



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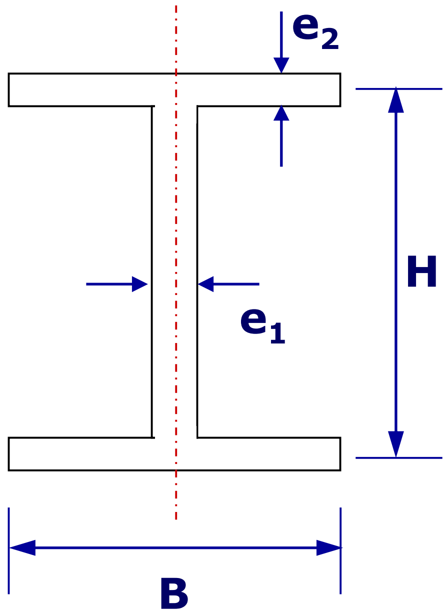
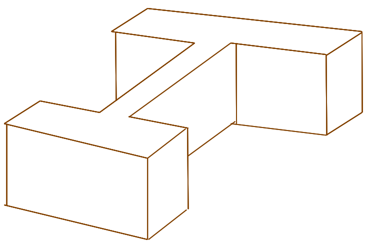
# Introduction

## Cross-section of a thin-walled beam



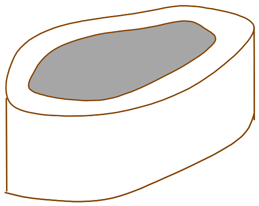

$$e_1 \ , \ e_2 \leq \frac{1}{10} \cdot \mathrm{Min}\{H, B\}$$

**OPEN**

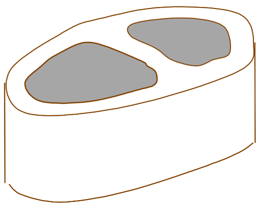


**CLOSED**

- **Single cell**



- **Multi cell**



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# CHAPTER 2. Bending, shear and torsion of thin-walled beams

## Session 4: Torsion on thin walled beams

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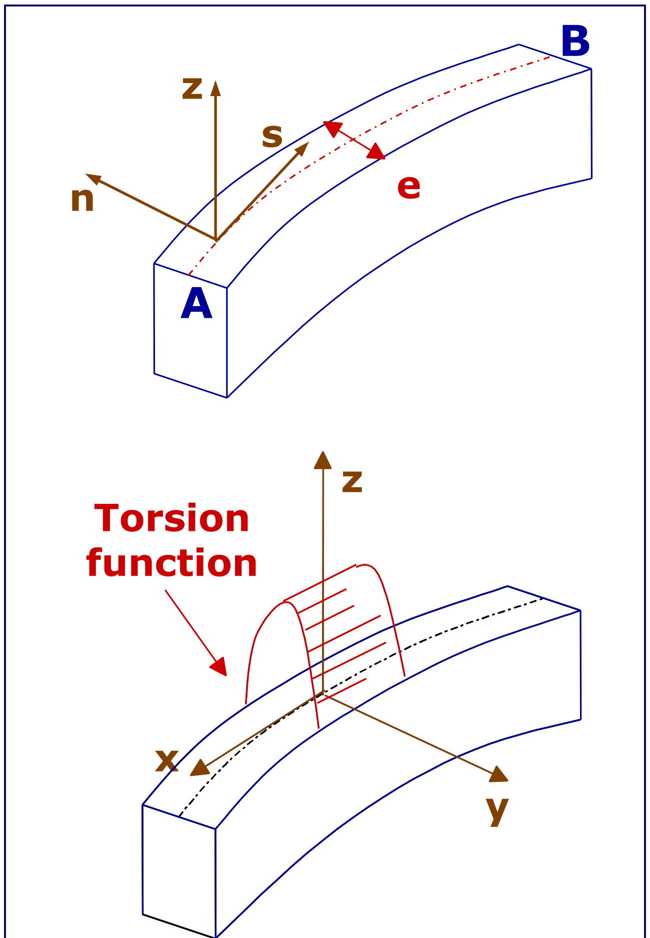
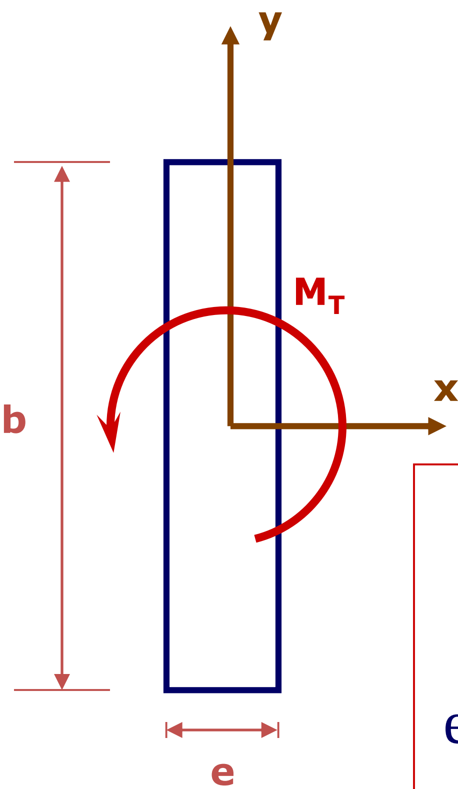
# Stress field on open sections

## Single branch section




Torsion function is independent on coordinate s, except in extreme points




$$\nabla^2\varphi = -2\cdot G\cdot\omega$$

$$\varphi(x,y)\approx\varphi(x)$$

$$\frac{\partial^2\varphi}{\partial x^2} = -2\cdot G\cdot\omega$$

en $x=\pm\frac{e}{2}$ $\quad \varphi(x)=0$

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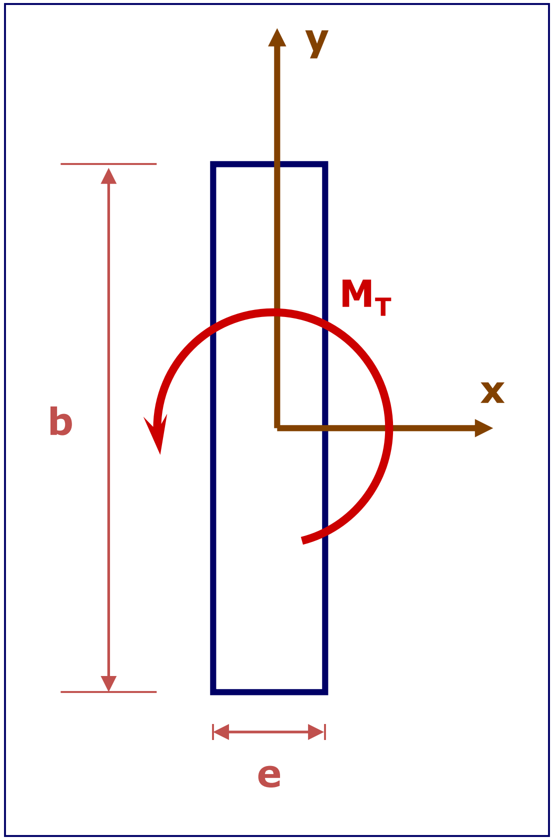
# Stress field on open sections

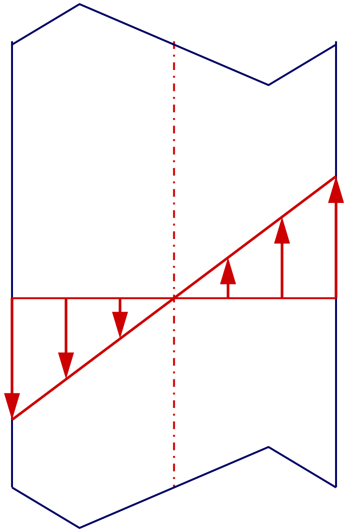
## Single branch section




$$\varphi(x) = -G\cdot\omega\cdot\left(x^2 - \frac{e^2}{4}\right)$$

$$\tau_{xz} = 0$$

$$\tau_{yz} = 2\cdot G\cdot\omega\cdot x$$



$$M_T = 2\cdot\int_A \varphi(x)\cdot dA$$

$$M_T = \frac{1}{3}\cdot G\cdot\omega\cdot b\cdot e^3$$

$$J = \frac{1}{3}\cdot b\cdot e^3$$

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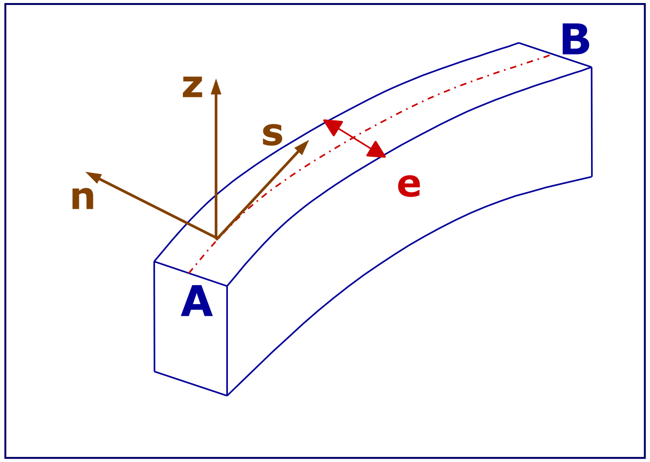
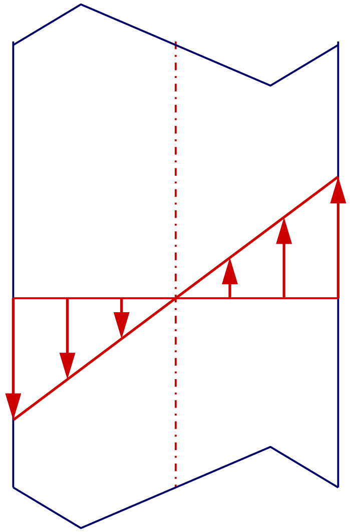
# Stress field on open sections

## Single branch section




$$M_T = G \cdot J \cdot \omega$$



$$\tau_{sz} = \frac{2 \cdot M_T \cdot n}{J}$$

$$\tau_{sz,\max} = \frac{e \cdot M_T}{J}$$

Uniform thickness:

$$J = \frac{1}{3} \cdot b \cdot e^3$$

Non-uniform thickness:

$$J = \frac{1}{3} \int_0^{S_p} e^3(s) \cdot ds$$

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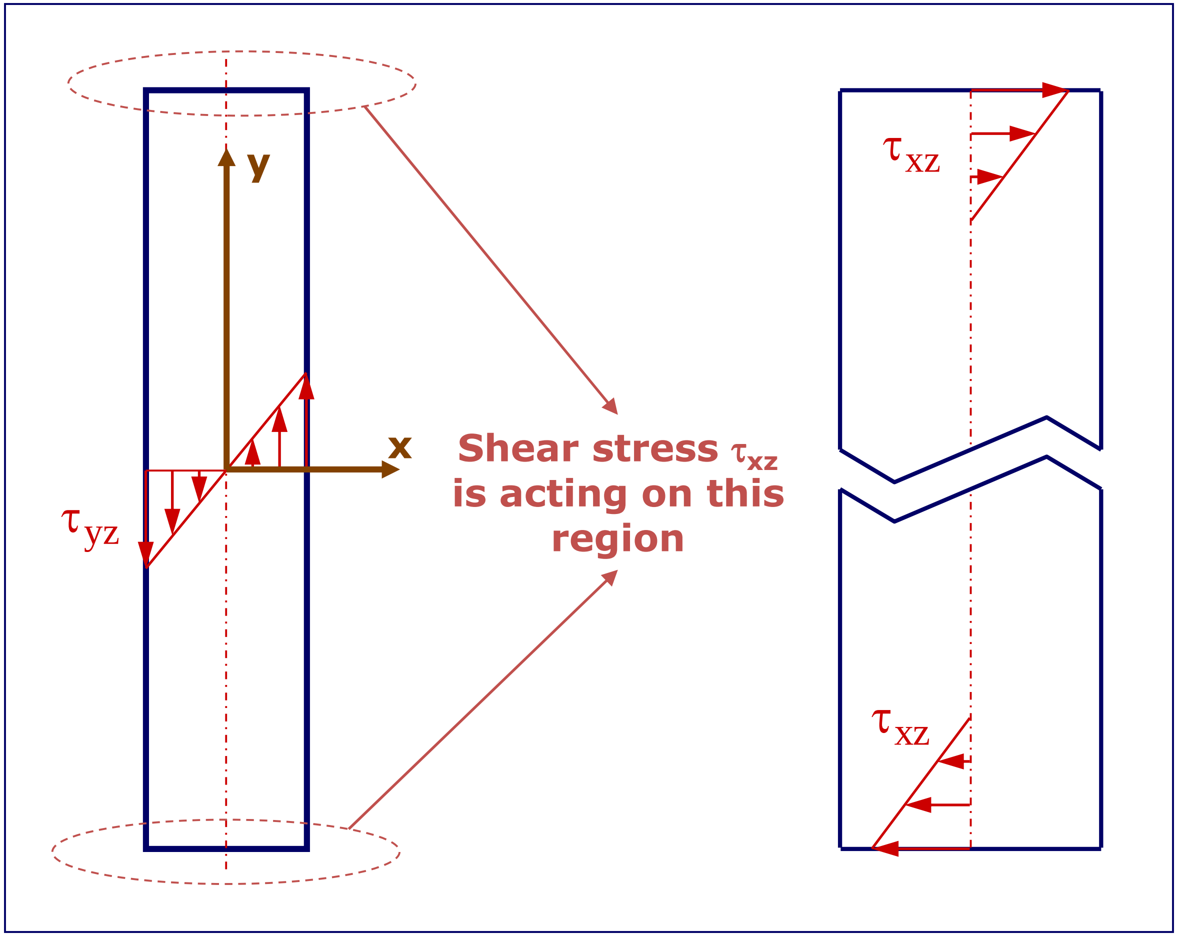
# Stress field on open sections

## Single branch section

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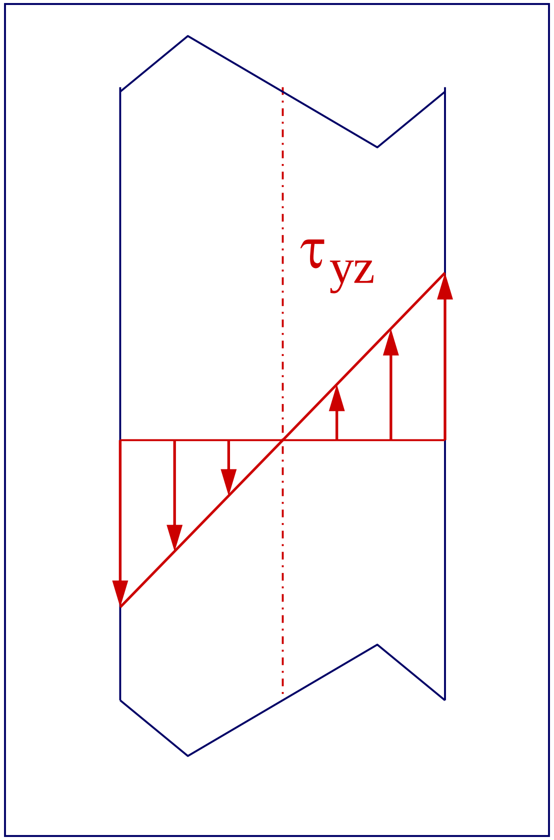
# Stress field on open sections

## Single branch section




Calculating the resultant force of shear stresses:

$$\int_{-e/2}^{e/2} \tau_{yz} \cdot x \cdot b \cdot dx = \frac{1}{6} \cdot G \cdot \omega \cdot b \cdot e^3$$

Since:

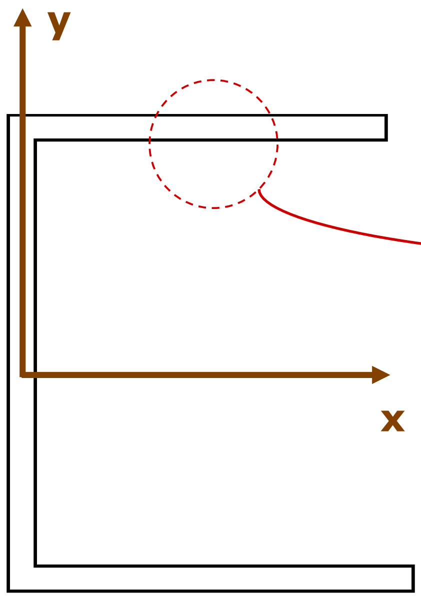
$$M_T = \frac{1}{3} \cdot G \cdot \omega \cdot b \cdot e^3$$

Shear stress $\tau_{yz}$ is in equilibrium with $M_T/2$

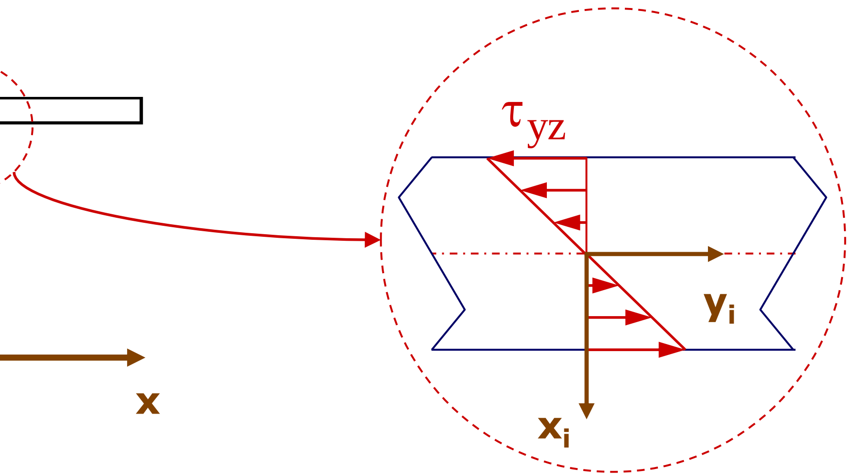
$\tau_{xz}$ equilibrates the other $M_T/2$

# Stress field on open sections

## Branched sections




**For each branch**




$$\tau_{yz_i} = 2 \cdot G \cdot \omega_i \cdot x$$

$$J_i = \frac{1}{3} \cdot b_i \cdot e_i^3$$

$$M_{T_i} = G \cdot J_i \cdot \omega_i$$

$$M_T = \sum_i M_{T_i}$$

$$\omega = \omega_i$$

$$J_{global} = \sum_i \frac{1}{3} \cdot b_i \cdot e^3$$

Hypothesis: The stress concentration at the corners is neglected

$$\frac{M_{T_1}}{G \cdot J_1} = \frac{M_{T_2}}{G \cdot J_2} = \cdots = \frac{M_{T_n}}{G \cdot J_n} = \frac{\sum M_{T_i}}{\sum G \cdot J_i} = \frac{M_T}{G \cdot J_{global}}$$

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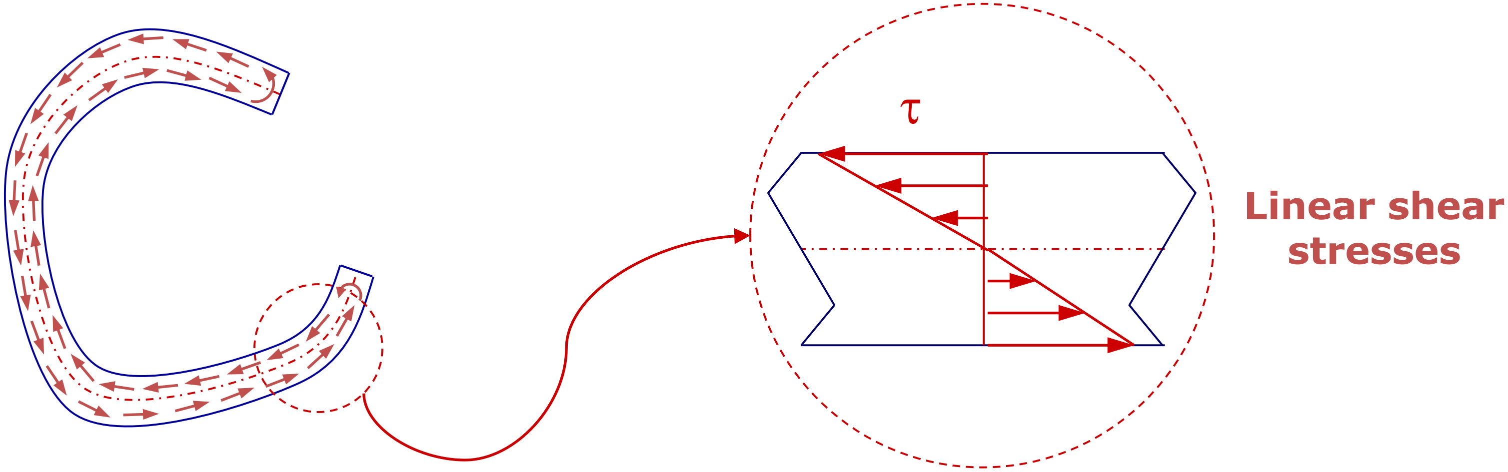
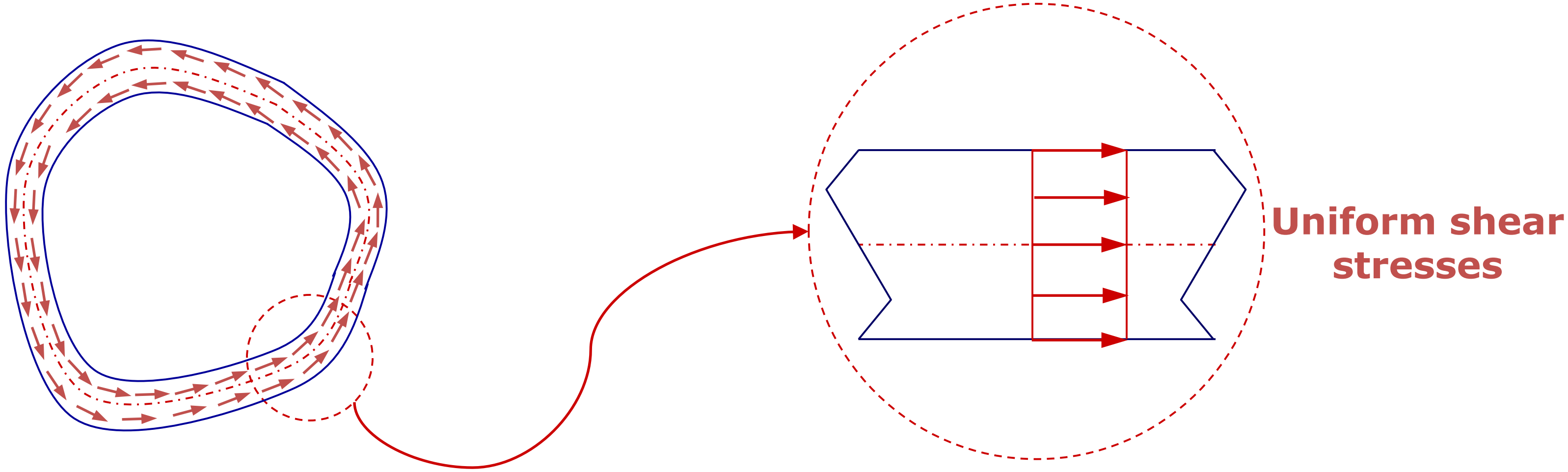
# Stress field on closed sections

## Comparison between open and closed sections

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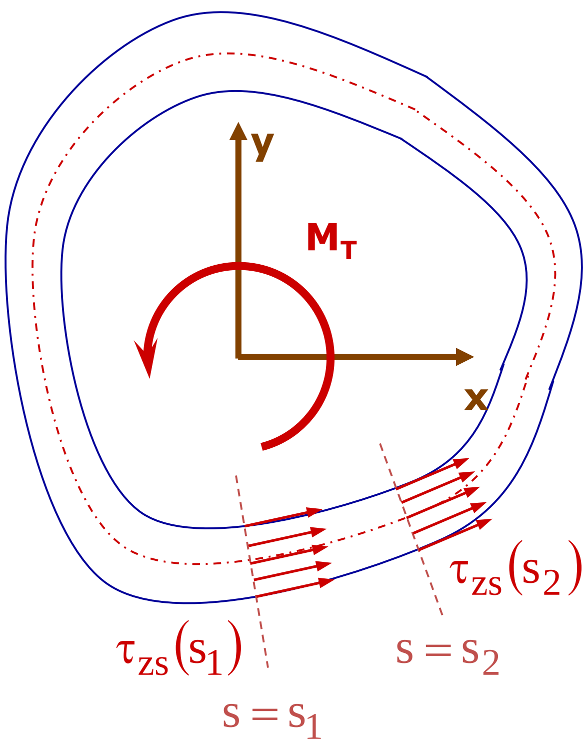
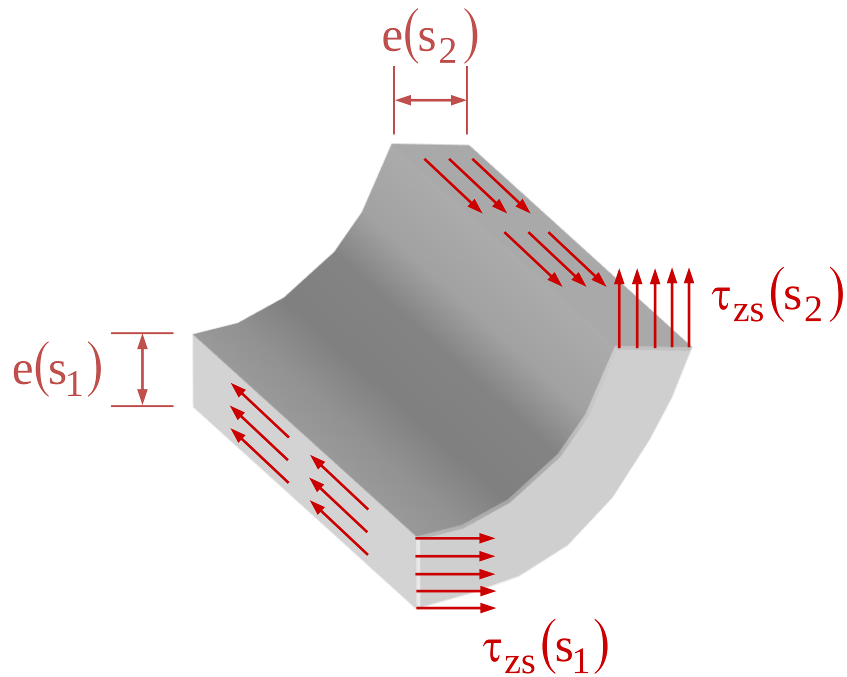
# Stress field on closed sections




$$\tau_{zs}(s_1)\cdot e(s_1)\cdot dz = \tau_{zs}(s_2)\cdot e(s_2)\cdot dz$$

$$q_T(s) = cte \Leftarrow \begin{cases} q_T(s_1) = \tau_{zs}(s_1)\cdot e(s_1) \\ q_T(s_2) = \tau_{zs}(s_2)\cdot e(s_2) \end{cases}$$

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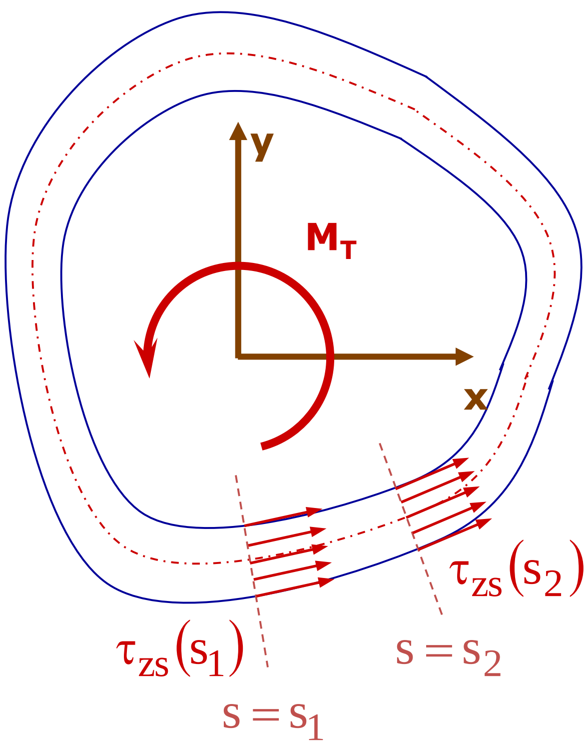
# Stress field on closed sections




$$M_T = \oint \tau_{zs}(s)\cdot e(s)\cdot r(s)\cdot ds$$

Using circular sector concept

$$2\cdot\Omega = \oint r(s)\cdot ds$$

$$M_T = q_T \cdot 2\cdot\Omega$$

$$\oint \gamma_{zs}(s)\cdot ds = 2\cdot\omega\cdot\Omega$$

$$M_T = G\cdot\omega\cdot\frac{4\cdot\Omega^2}{\oint \frac{ds}{e(s)}}$$

$$J = \frac{4\cdot\Omega^2}{\oint \frac{ds}{e(s)}}$$

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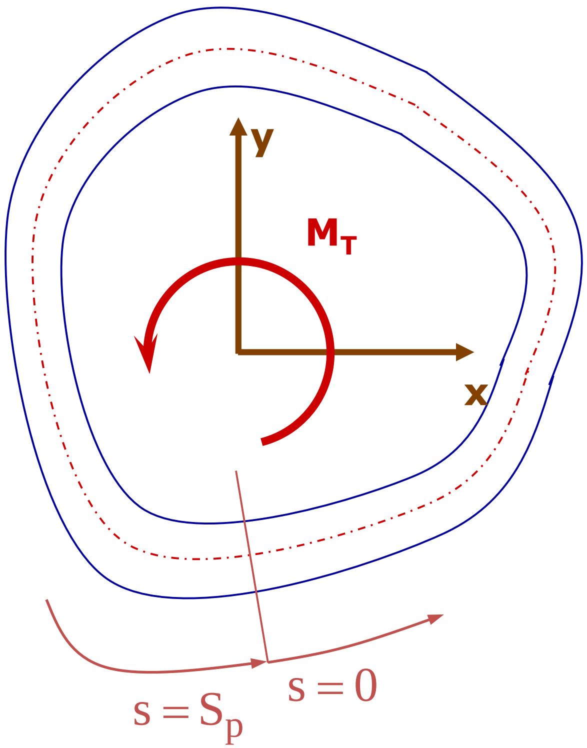
# Stress field on closed sections




$$u_z(s) = u_z(s+S_p)$$

$$u_z(s) = u_z(0) + \int_0^s \gamma_{zs} \cdot ds - \int_0^s \frac{\partial \phi_z}{\partial z} \cdot r(s) \cdot ds$$

$$u_z(s+S_p) = u_z(0) + \int_0^{s+S_p} \gamma_{zs} \cdot ds - \int_0^{s+S_p} \frac{\partial \phi_z}{\partial z} \cdot r(s) \cdot ds$$

Using circular sector concept $\omega_o(s) = \int_0^s \frac{1}{2} \cdot r(s) \cdot ds$

$$u_z(s) = u_z(0) + \int_0^s \gamma_{zs} \cdot ds - 2 \cdot \frac{\partial \phi_z}{\partial z} \cdot \omega_o(s)$$

$$u_z(s+S_p) = u_z(0) + \int_0^{s+S_p} \gamma_{zs} \cdot ds - 2 \cdot \frac{\partial \phi_z}{\partial z} \cdot \omega_o(s+S_p)$$

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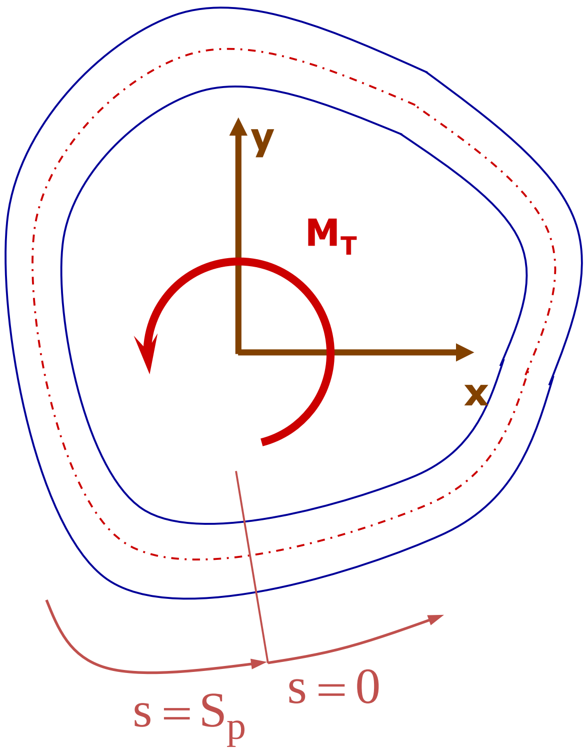
# Stress field on closed sections




$$u_z(s) = u_z(s + S_p)$$

$$\int_0^{s} + \int_s^{s+S_p} = \int_0^{s+S_p} \longrightarrow \int_s^{s+S_p} = \int_0^{s+S_p} - \int_0^{s} = \oint$$

$$0 = \int_s^{s+S_p} \gamma_{zs} \cdot ds - 2 \cdot \frac{\partial \phi_z}{\partial z} \cdot \left( \omega_o(s + S_p) - \omega_o(s) \right)$$

$$\omega_o(s + S_p) - \omega_o(s) = \Omega$$

$$\int_0^{S_p} \gamma_{zs} \cdot ds = 2 \cdot \frac{\partial \phi_z}{\partial z} \cdot \Omega$$

$$\tau(s) = \frac{M_T}{2 \cdot \Omega \cdot e(s)}$$

$$\tau_{max} = \frac{M_T}{2 \cdot \Omega \cdot e_{min}}$$

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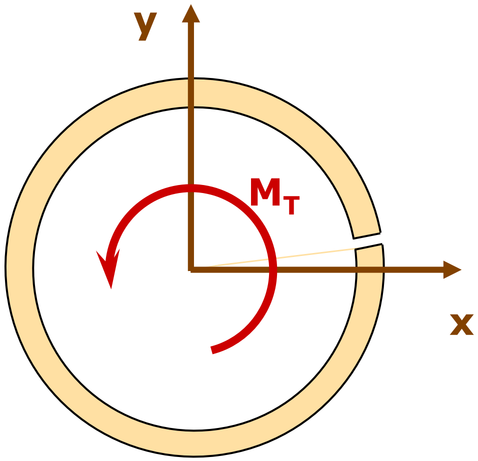
# Stress field on closed sections

## Comparison between open and closed sections

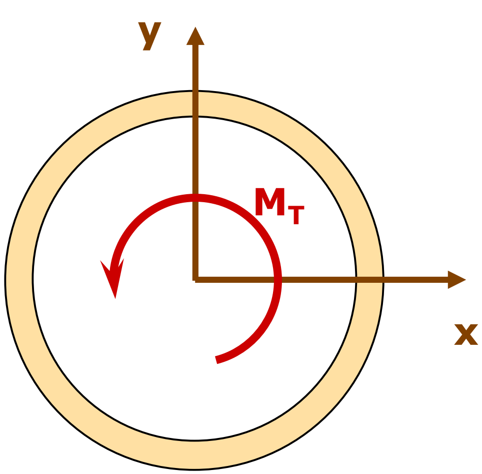



Closed sections are more effective to resist torsional moments

Example: circular sections

$$J_O = \frac{1}{3} \cdot (2 \cdot \pi \cdot R) \cdot e^3 \qquad J_O = \frac{2}{3} \cdot \pi \cdot R \cdot e^3$$

$$J_C = \frac{4 \cdot (\pi \cdot R^2)^2}{\oint \frac{ds}{e}} \qquad J_C = 2 \cdot \pi \cdot R^3 \cdot e$$




$$\frac{J_O}{J_C} = \frac{1}{3} \cdot \left(\frac{e}{R}\right)^2 << 1 \longrightarrow \frac{\omega_C}{\omega_O} = \frac{1}{3} \cdot \left(\frac{e}{R}\right)^2 << 1$$

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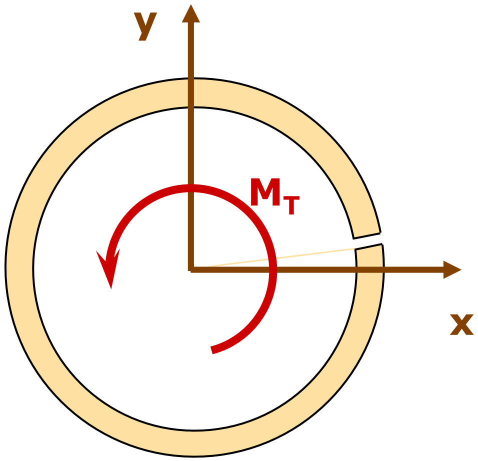
# Stress field on closed sections

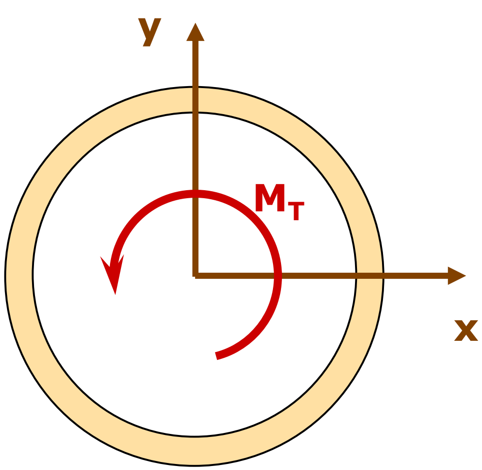
## Comparison between open and closed sections




Closed sections are more effective to resist torsional moments

Example: circular sections

$$\tau_{O\max} = \frac{M_T \cdot e}{J_O} \qquad \tau_{O\max} = \frac{3 \cdot M_T}{2 \cdot \pi \cdot R \cdot e^2}$$




$$\tau_C = \frac{M_T}{2 \cdot \Omega \cdot e} \qquad \tau_C = \frac{M_T}{2 \cdot \pi \cdot R^2 \cdot e}$$

$$\frac{\tau_{O_{\max}}}{\tau_C} = \frac{3 \cdot R}{e} >> 1$$

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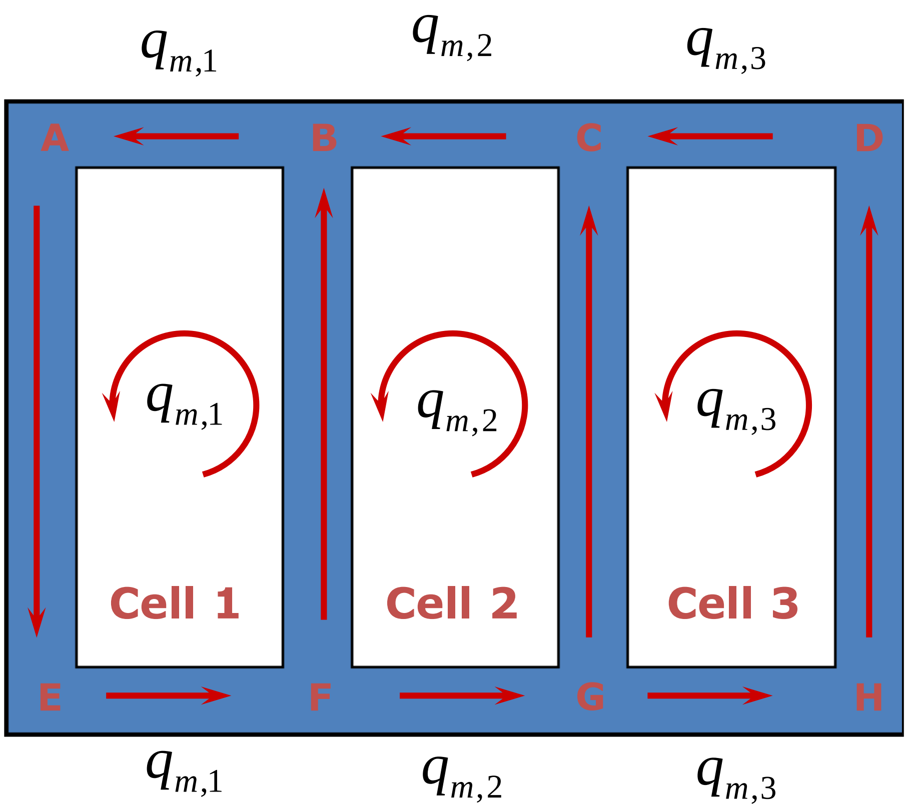
# Multi-cell closed sections

## TORSIONAL MOMENT




$$\delta_{ii} = \frac{1}{G} \oint_i \frac{ds}{t(s)}$$

$$\delta_{ij} = -\frac{1}{G} \oint_{ij} \frac{ds}{t(s)}$$

$$M_T = \sum_i 2 \cdot \Omega_i \cdot q_i$$

**Compatibility equations:**

$$\delta_{11} \cdot q_{m,1} + \delta_{12} \cdot q_{m,2} - 2 \cdot \Omega_1 \cdot w = 0$$

$$\delta_{22} \cdot q_{m,2} + \delta_{12} \cdot q_{m,1} + \delta_{23} \cdot q_{m31} - 2 \cdot \Omega_2 \cdot w = 0$$

$$\delta_{33} \cdot q_{m,3} + \delta_{23} \cdot q_{m,2} - 2 \cdot \Omega_3 \cdot w = 0$$

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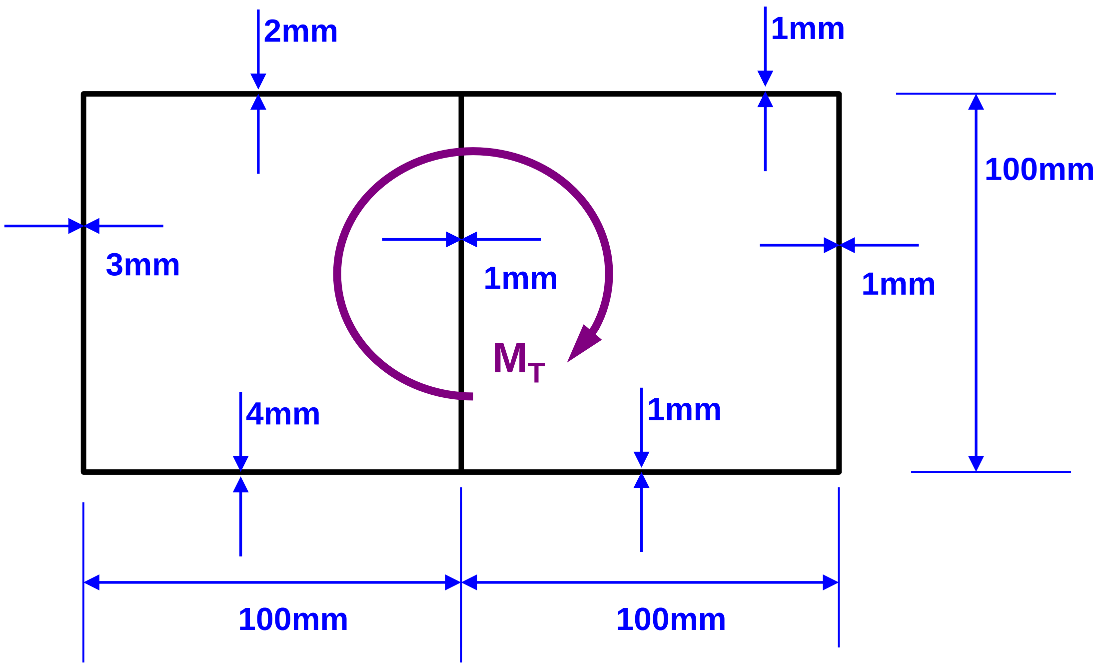
# Multi-cell closed sections

## Example




**MT= 5 kNm**
**G= 25000 MPa**

**Find the maximum shear stress**

$\tau_{max}$= **36 MPa**

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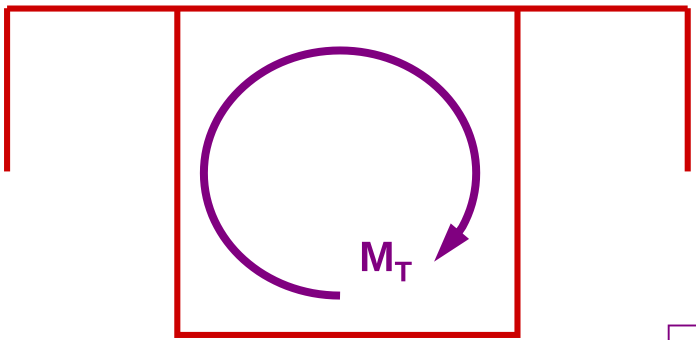
# Multi-cell closed sections

## Combined open and closed sections




$$M_T = Q_y \cdot \frac{L}{2}$$

$$M_T = G \cdot J \cdot \omega$$

$$J = J_{cl} + J_{op}$$

$$J_{cl} = \frac{4 \cdot \Omega^2}{\oint \frac{ds}{t(s)}} = \frac{4 \cdot 200^2 \cdot 2}{200 \cdot 4} = 16 \cdot 10^6 mm^4$$

$$J_{op} = \sum \frac{1}{3} s \cdot t^3 = 4 \frac{1}{3} 100 \cdot 2^3 = 1066.7 mm^4$$

$$J = J_{cl} + J_{op} = 16.001 \cdot 10^6 mm^4 \approx 16 \cdot 10^6 mm^4$$

$$\omega = \frac{M_T}{GJ} = \frac{100 \cdot 10^3 \cdot 100}{25000 \cdot 16 \cdot 10^6} = 25 \cdot 10^{-6} rad/mm$$

**Closed cell**

$$q_{cl} = \frac{G \cdot J_{cl} \cdot \omega}{2 \cdot \Omega} = 125 N/mm$$

$$\tau_{cl} = \frac{q_{cl}}{t} = 62.5 MPa$$

**Open branches**

$$\tau_{op,\max} = t \cdot G \cdot \omega = 2 \cdot 25000 \cdot 25 \cdot 10^{-6} = 1.25 MPa$$

**Maximum shear stress ($Q_y$ +$M_T$): 344.2 MPa**

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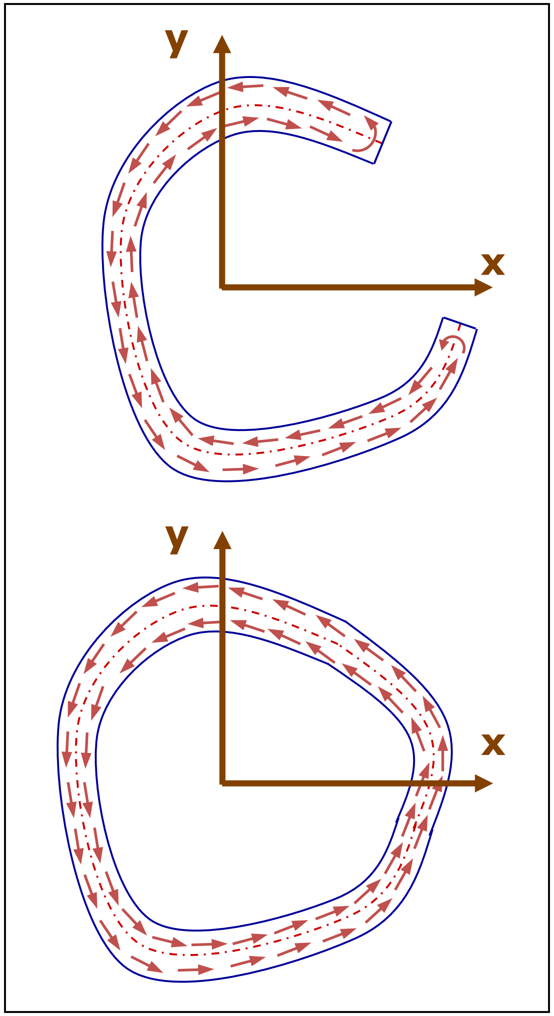
# Summary




**Single branched open sections**

$$M_T = G \cdot J \cdot \omega \qquad J = \frac{1}{3} \cdot b \cdot e^3$$

$$\tau_{max} = G \cdot \omega \cdot e$$

**Closed sections**

$$M_T = G \cdot J \cdot \omega \qquad J = \frac{4 \cdot \Omega^2}{\oint \frac{ds}{e(s)}}$$

$$\tau = \frac{M_T}{2 \cdot \Omega \cdot e}$$

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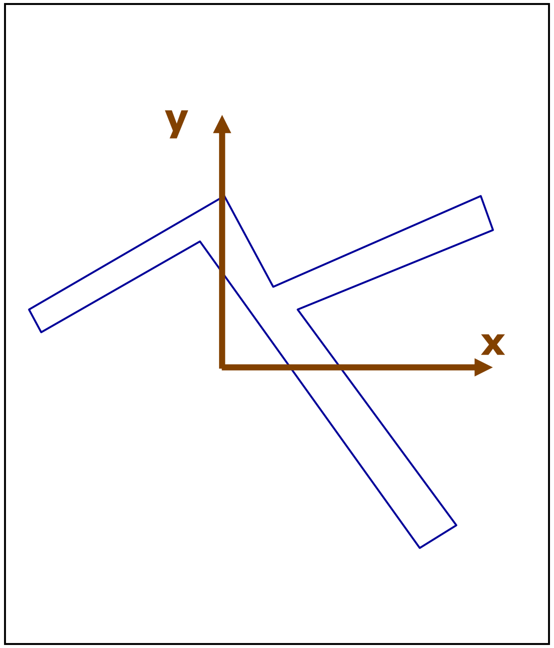
# Summary




**Multiple branched sections**

$$M_T = \sum_i M_{Ti}$$

$$\omega = \omega_i$$

$$\tau_{yz_i} = 2 \cdot G \cdot \omega_i \cdot x$$

$$J_i = \frac{1}{3} \cdot b_i \cdot e_i^3$$

$$M_{Ti} = G \cdot J_i \cdot \omega_i$$

$$J_{global} = \sum_i \frac{1}{3} \cdot b_i \cdot e^3$$

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3. Stress field on single-cell closed sections
4. Stress field on multi-cell sections
5. Summary
6. References

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