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Department of Continuum Mechanics and Structural Engineering

Aerospace Structures

Chapter 4. Plates and Shells

Bending of thin plates

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Bending of thin plates

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Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Chapter 4. Plates and Shells

Bending of thin plates

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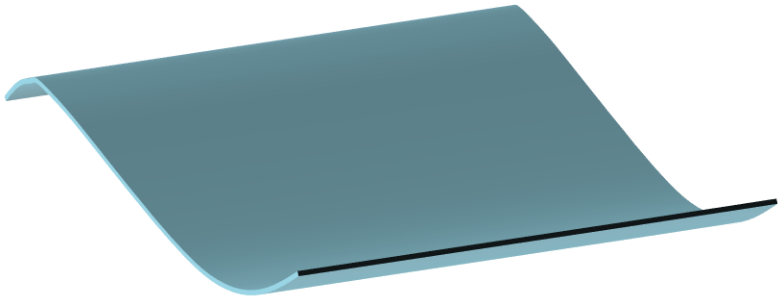
Introduction

Definition

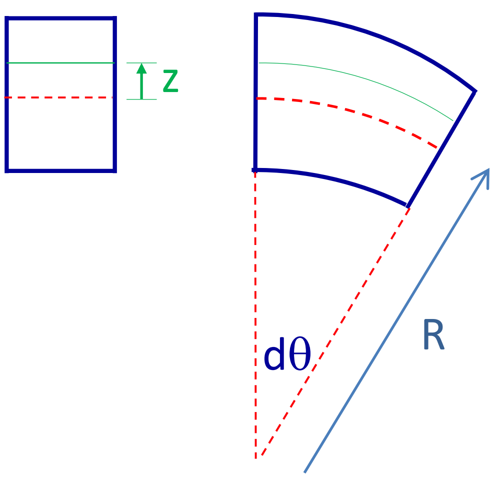
- Plate: 3D solid than be considered a 2D element because one dimension is smaller than the other two.



- Scientific literature use to classify plates as a function of the third dimension:
  - ✓ Thin plates (Hyp. Kirchhoff)
  - ✓ Thick plates (Hyp. Reissner Mindlin)

Introduction

- Thin plates (Hyp. Kirchhoff):
  - ✓ Straight lines normal to the mid-surface remain straight after deformation
  - ✓ straight lines normal to the mid-surface remain normal to the mid-surface after deformation
  - ✓ the thickness of the plate does not change during deformation



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Introduction

- Thin plates (Hyp. Kirchhoff):
  - ✓ According to these hypothesis, there is a neutral plane in the plate with no strains. Neutral plane is equivalent to neutral axis in beams
  - ✓ The strains field can be expressed as a function of a single variable, w (displacement of the mid-plane)
  - ✓ The problem has an exact solution in a limited number of cases. The partial differential equation must be degenerate in a ordinary differential equation
  - ✓ The problem is usually solved by numerical integration to find an approximated solution.

Introduction

- Additional hypothesis:
  - ✓ Elastic-linear behaviour
  - ✓ Small deformations

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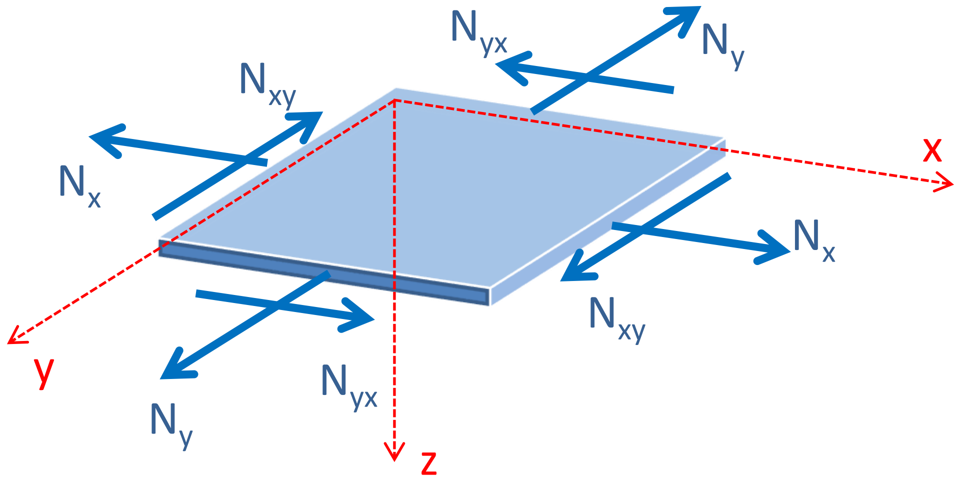
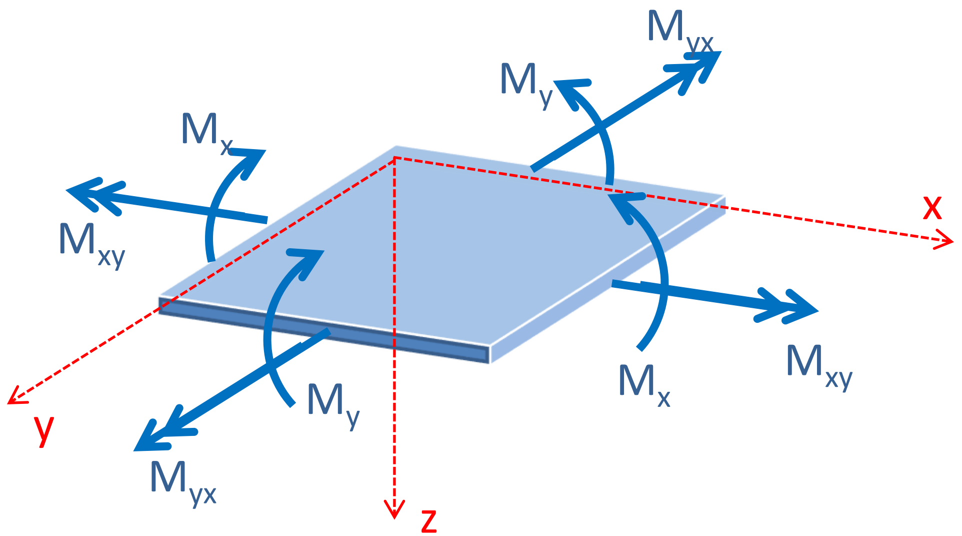
Equilibrium equations

- External forces:



- External moments:



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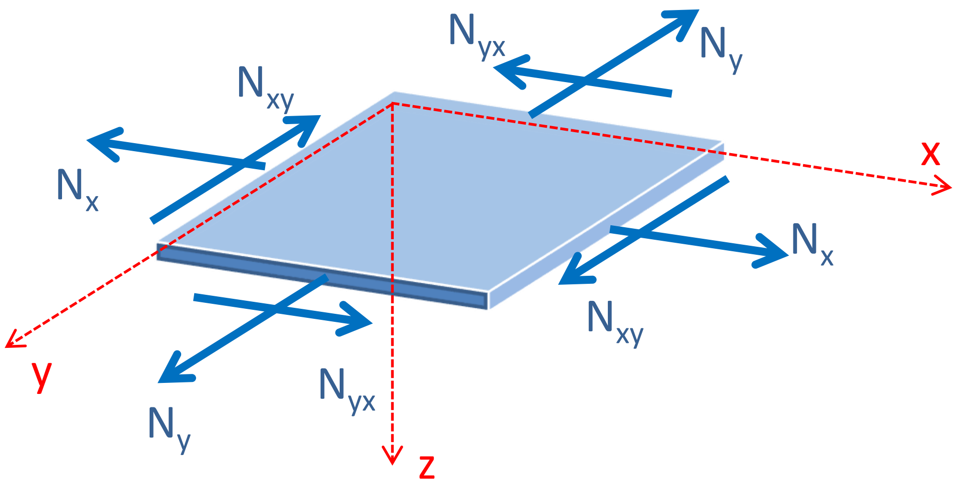
Equilibrium equations

- External forces are obtained by the through-the-thickness integration of normal and shear stressess:

$$N_x = \int_z \sigma_{xx} dz$$

$$N_y = \int_z \sigma_{yy} dz$$

$$N_{xy} = \int_z \sigma_{xy} dz$$



External forces and external moments are defined by unit length

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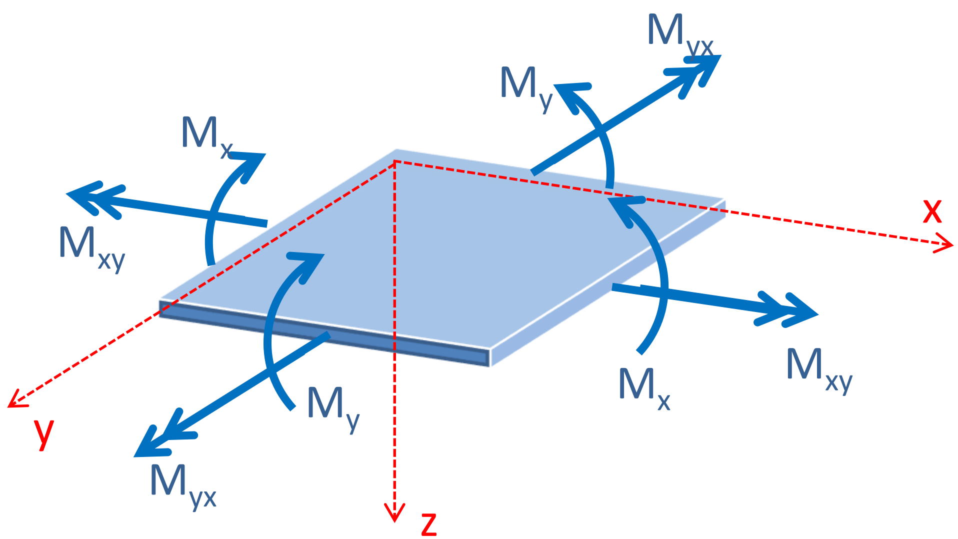
Equilibrium equations

- External bending and torsion moments:

$$M_x = \int_z \sigma_{xx} z dz$$

$$M_y = \int_z \sigma_{yy} z dz$$

$$M_{xy} = \int_z \sigma_{xy} z dz$$



External forces and external moments are defined by unit length

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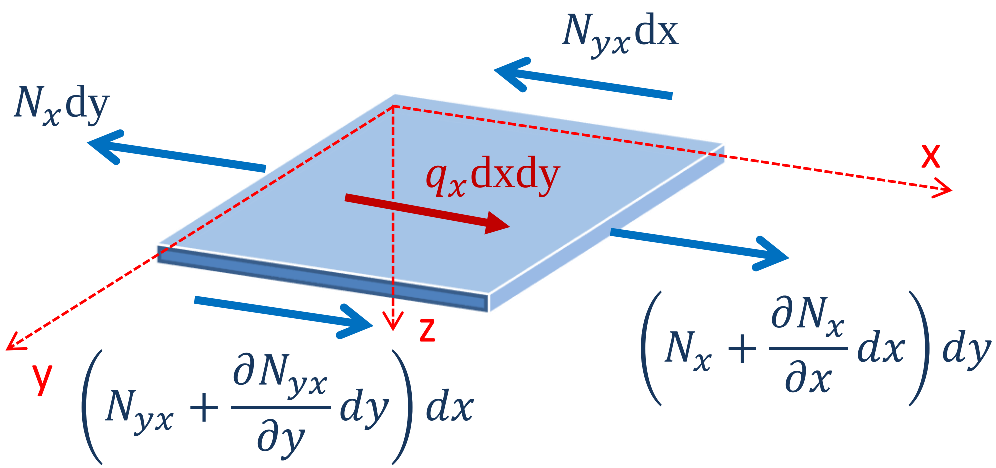
Equilibrium equations

- Equilibrium in a plate differential element with dimensions: dx, dy

Equilibrium in x direction:



$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = q_x$$

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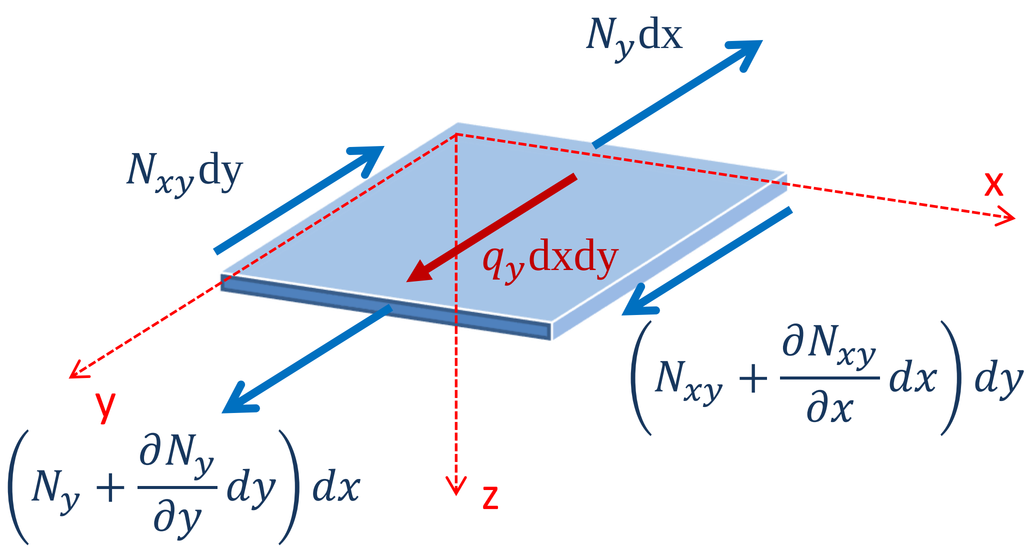
Equilibrium equations

- Equilibrium in a plate differential element with dimensions: dx, dy

Equilibrium in y direction:



$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = q_y$$

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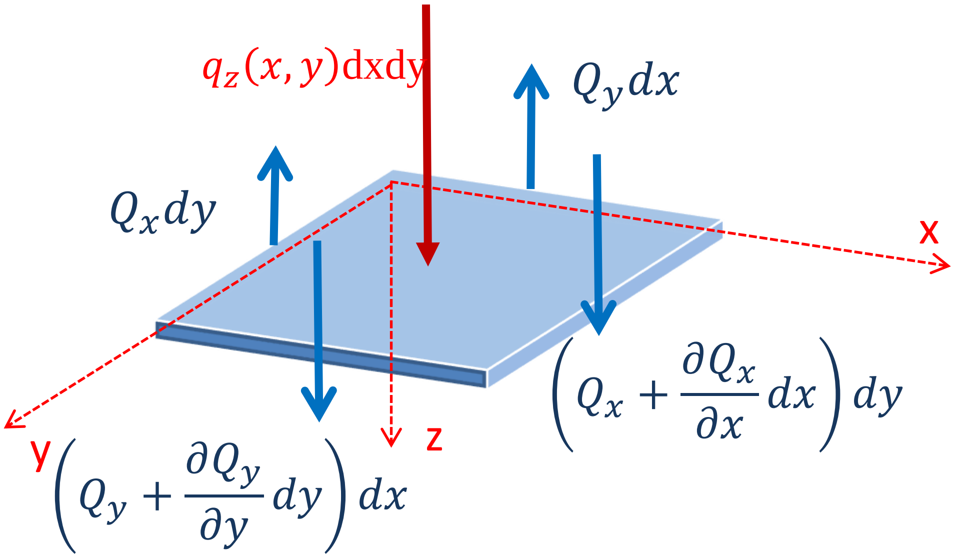
Equilibrium equations

- Equilibrium in a plate differential element with dimensions: dx, dy

Equilibrium in z direction:

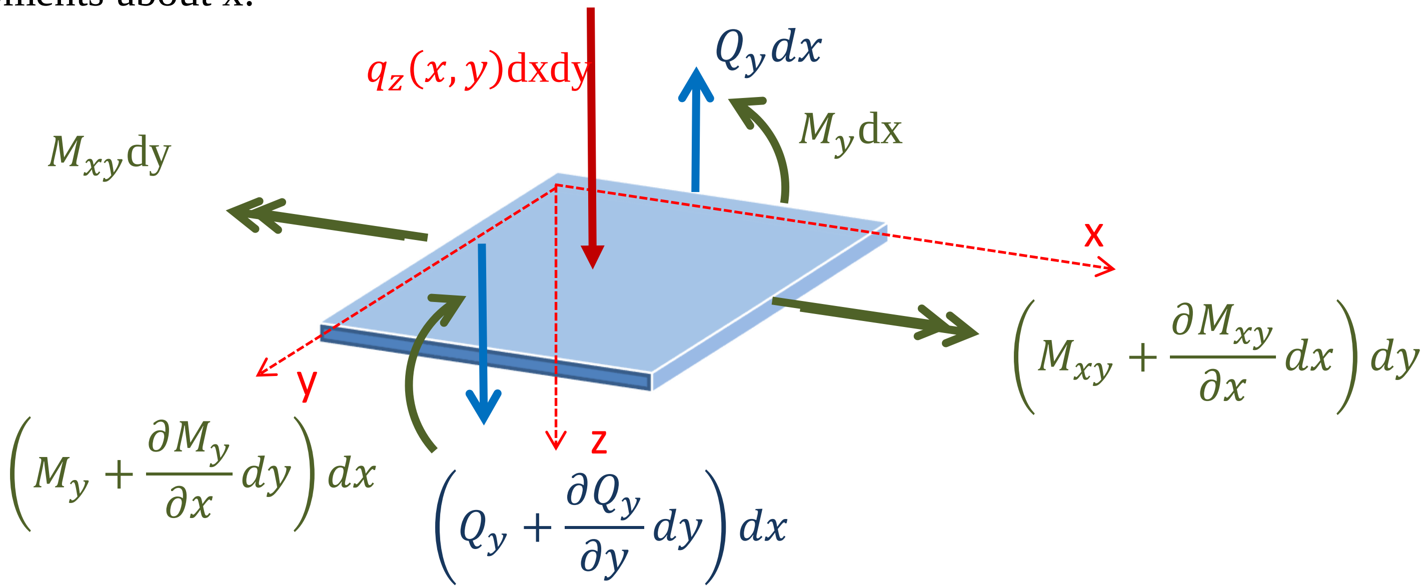


$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q$$

Equilibrium equations

- Equilibrium in a plate differential element with dimensions: dx, dy

Equilibrium of moments about x:

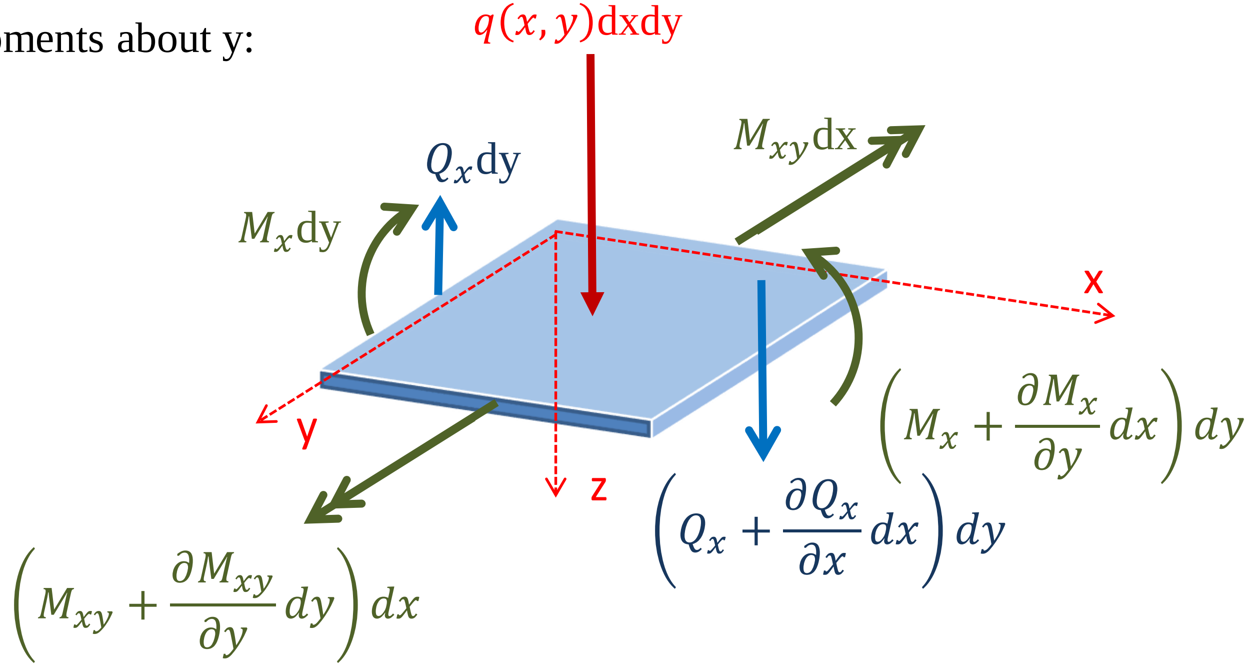


$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}=Q_y$$

Equilibrium equations

- Equilibrium in a plate differential element with dimensions: dx, dy

Equilibrium of moments about y:



$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x$$

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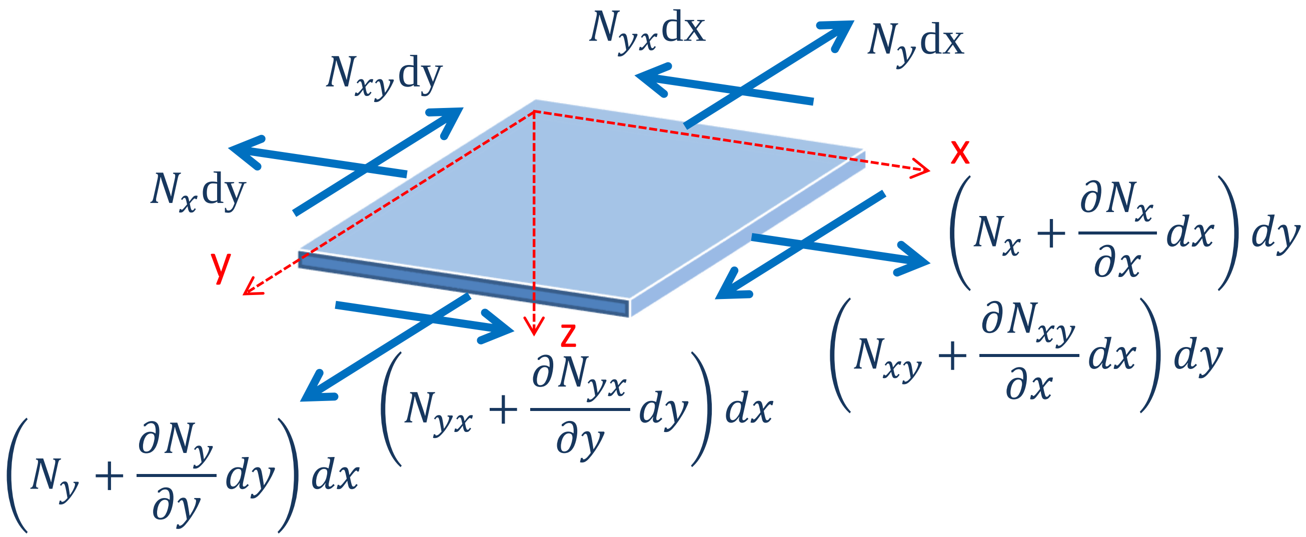
Equilibrium equations

- Equilibrium in a plate differential element with dimensions: dx, dy

Equilibrium of moments about z:



$$\sum M_z = 0 \longrightarrow \text{It is automatically satisfied}$$

Equilibrium equations

- Equilibrium in a plate differential element. Summary:

$$\sum F_x = 0 \longrightarrow \frac{\partial N_y}{\partial x} + \frac{\partial N_{yx}}{\partial y} = q_x$$

$$\sum F_y = 0 \longrightarrow \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = q_y$$

$$\sum F_z = 0 \longrightarrow \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q$$

$$\sum M_x = 0 \longrightarrow \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y$$

$$\sum M_y = 0 \longrightarrow \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x$$

$$\sum M_z = 0 \longrightarrow \text{Identity}$$

Equilibrium equations

- The problem of in-plane loads is mathematically uncoupled of the problem of out-of-plane loads.
- The combination of out-of-plane expression yields:

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{y}}{\partial y} = Q_y \rightarrow \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q \leftarrow \frac{\partial M_{x}}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x$$

$$\downarrow$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

Second order differential equation obtained from equilibrium equations

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Session 5. Bending of thin plates

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Bending equations

A first order analysis of the strains yields:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{xz} = 2\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = 2\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

Considering Kirchhoff hypothesis: → $\varepsilon_{zz} = 0$

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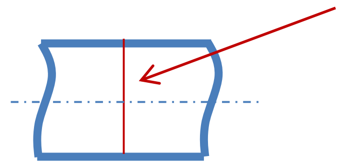
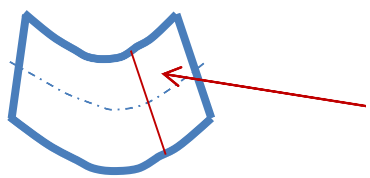
Bending equations

Non deformed Section

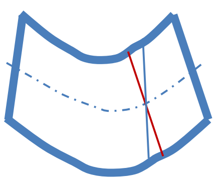


Kirchhoff–Love plate theory



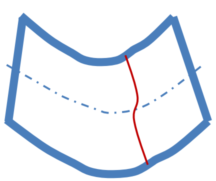
The section remains straight and perpendicular to the mid-plane

Mindlin–Reissner theory(One-order theory)



The section remains straight but not perpendicular to the mid-plane

The section does not remain straight



Higher-order theories

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Bending equations

Kirchhoff hypothesis:

$$\varepsilon_{xx} = \frac{\partial u^0}{\partial x} + z\frac{\partial \theta_1}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v^0}{\partial y} + z\frac{\partial \theta_2}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} + z\left(\frac{\partial \theta_1}{\partial y} + \frac{\partial \theta_2}{\partial x}\right)$$

$$\gamma_{xz} = \theta_1 + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \theta_2 + \frac{\partial w}{\partial y}$$

Neglecting displacement produced by shear strains (thin plate)

$$\theta_1 = -\frac{\partial w}{\partial x} \qquad \theta_2 = -\frac{\partial w}{\partial y}$$

Bending equations

Relationship between stresses and strains:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} \dfrac{E}{1-\nu^2} & \dfrac{\nu E}{1-\nu^2} & 0 & 0 & 0 \\ \dfrac{\nu E}{1-\nu^2} & \dfrac{E}{1-\nu^2} & 0 & 0 & 0 \\ 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

Bending equations

The previous equation is multiplied by coordinate z and integrated through the thickness to obtain the relationship between moments and displacements

$$\begin{Bmatrix} \int \sigma_{xx} zdz \\ \int \sigma_{yy} zdz \\ \int \tau_{xy} zdz \\ \int \tau_{xz} zdz \\ \int \tau_{yz} zdz \end{Bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 & 0 & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 & 0 & 0 \\ 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \int \varepsilon_{xx} zdz \\ \int \varepsilon_{yy} zdz \\ \int \gamma_{xy} zdz \\ \int \gamma_{xz} zdz \\ \int \gamma_{yz} zdz \end{Bmatrix}$$

↓

$$M_x = \frac{Et^3}{12\left(1-\nu^2\right)}\left(\frac{\partial \theta_1}{\partial x} + \nu \frac{\partial \theta_2}{\partial y}\right)$$

$$M_y = \frac{Et^3}{12\left(1-\nu^2\right)}\left(\nu \frac{\partial \theta_1}{\partial x} + \frac{\partial \theta_2}{\partial y}\right)$$

$$M_x = \frac{Gt^3}{12}\left(\frac{\partial \theta_1}{\partial y} + \nu \frac{\partial \theta_2}{\partial x}\right)$$

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Bending equations

Neglecting the displacement produced by shear force (thin plate) and using the equilibrium equations it yields:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D\left(\nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_x = -D(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q(x,y)$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

Fourth-order partial differential equation for a plate subjected to a transverse distributed load.

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Bending equations

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

The parameter D is the equivalent bending stiffness of the plate

$$D = \frac{Eh^3}{12\left(1 - \nu^2\right)}$$

- Boundary conditions are necessary to solve the differential equation
- There are different solutions for the displacement field, w(x,y), depending on boundary conditions

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Boundary conditions

- Clamped end: Displacements and rotations are restrained.

$$w(x_0) = 0$$
$$\frac{\partial w}{\partial x}(x_0) = 0$$
$$\frac{\partial w}{\partial y}(x_0) = 0$$

$$\rightarrow \quad \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

- Pinned end: Displacements and bending moments are restrained.

$$w(x_0) = 0$$
$$M_{nn}(x_0) = 0$$

$$\rightarrow \quad \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

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Boundary conditions

- Free end: Bending moment, torsion moment and shear force are restrained.

$$M_{nn}(x_0) = 0$$
$$M_{nt}(x_0) = 0$$
$$Q_n(x_0) = 0$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

- Symmetrical axis: This is not a boundary condition but it is usual in the analysis of plates. Supposing that y is a symmetrical axis:

$$M_{xy}(x_0) = 0$$
$$Q_y(x_0) = 0$$
$$\frac{\partial w}{\partial y}(x_0) = 0$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

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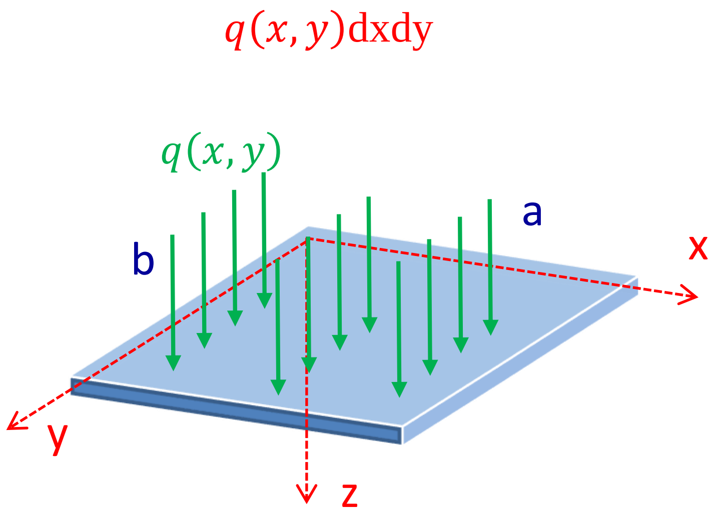
Example



- Plate dimensions: a·b
- Pinned at all the ends

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

Navier solution:

$$w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

A_{mn} are the coefficients that must satisfy the differential equation

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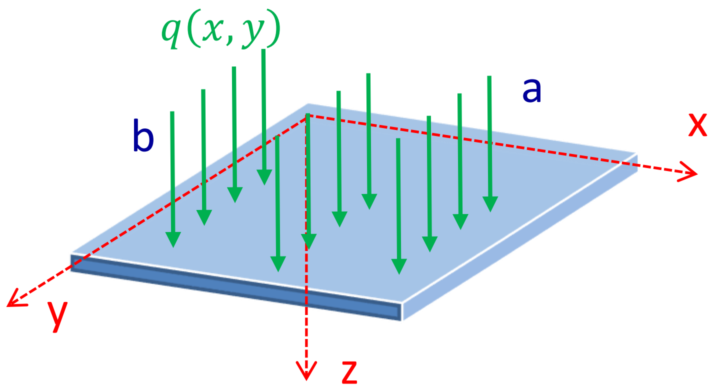
Example



- Plate dimensions: a·b
- Pinned at all the ends

$$\frac{\partial^4 w}{dx^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

External load q(x,y) can also be expressed as a function of Fourier series:

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \longrightarrow A_{mn} = \frac{1}{\pi^4 D}\frac{a_{mn}}{\left[\left(m/a\right)^2 + \left(n/b\right)^2\right]^2}$$

$$w(x,y) = \frac{1}{\pi^4 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{mn}}{\left[\left(m/a\right)^2 + \left(n/b\right)^2\right]^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

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References

1. **Megson, T.H.G. “Aircraft Structures”**
   Elsevier. 2007
   Cap.7 Bending of thin plates

2. **Timoshenko, Stephen “Theory of plates and shells “**
   McGraw-Hill, 1959

1. **Monleón Cremades, Salvador “Análisis de vigas, arcos, placas y láminas : una presentación unificada”**
   Universidad Politécnica de Valencia. 1999

2. **J. A. Jurado Albarracín-Martinón y S. Hernández Ibáñez, “Análisis estructural de placas y láminas”. Tercera Edición. Andavira editora 2014**

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