Universidad Carlos III de Madrid
www.uc3m.es

Department of Continuum Mechanics and Structural Engineering

Aerospace Structures

Chapter 4. Plates and Shells

Circular plates

Index

Chapter 4. Plates and Shells

Circular plates

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Chapter 4. Plates and Shells

Circular plates

1. Equilibrium equations in polar coordinate system
2. Bending equations in polar coordinate system
3. Boundary conditions in polar coordinate system
4. Examples
5. References

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Equilibrium equations

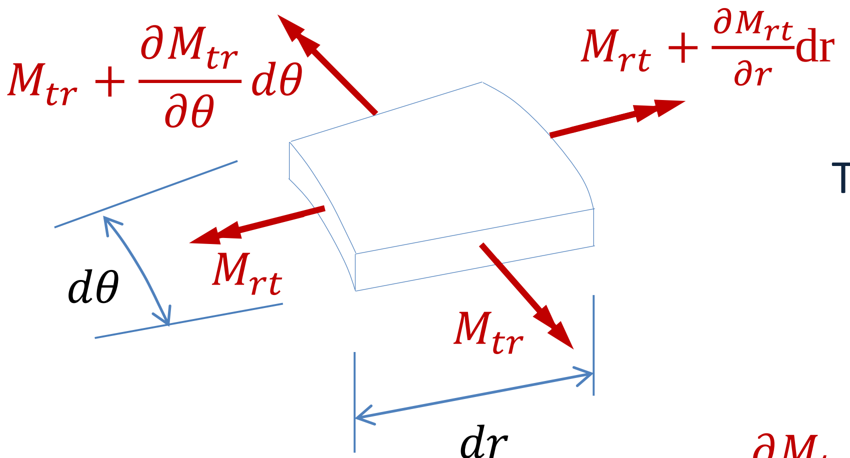
- Bending and torsion moments:

Defined per unit length

$$M_r = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{rr} \cdot z dz$$

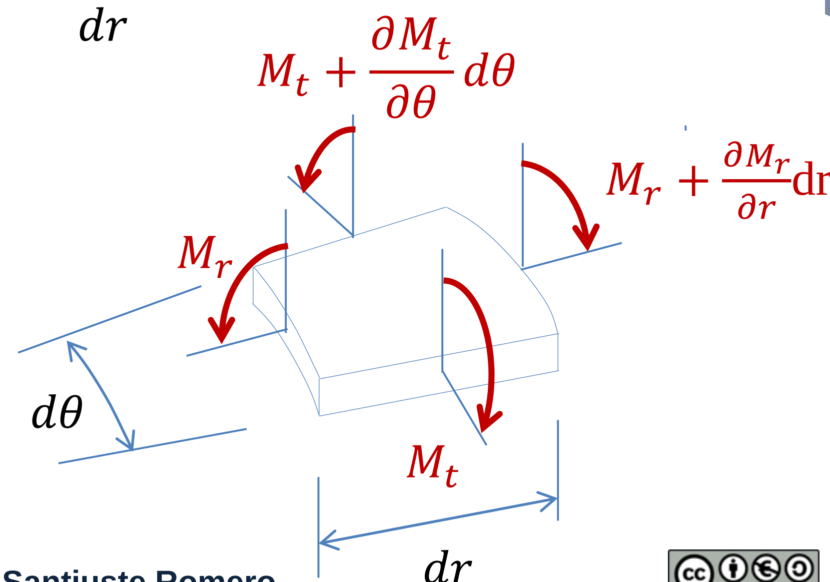
$$M_t = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{\theta\theta} \cdot z \cdot dz$$

$$M_{rt} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{r\theta} \cdot z \cdot dz$$



Torsion moment

Bending moments



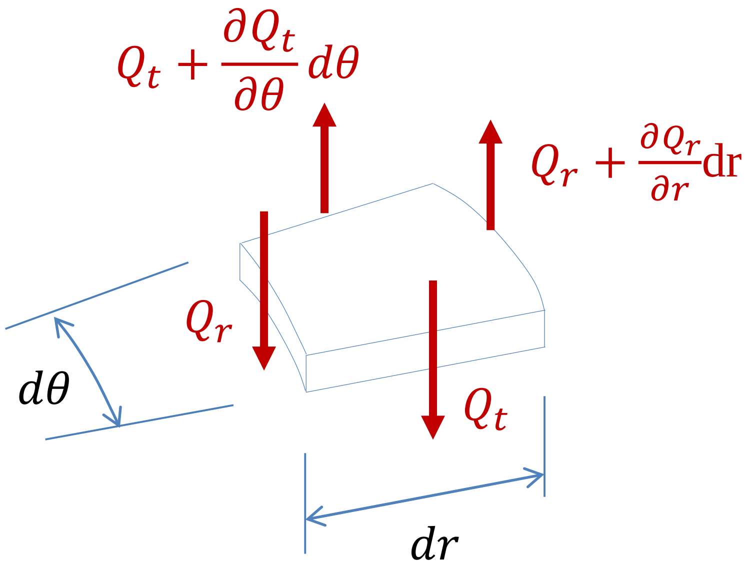
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Equilibrium equations

- Shear force

$$Q_r = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{rz} \cdot z dz$$

$$Q_t = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{\theta z} \cdot z dz$$



Defined per unit length

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Equilibrium equations

Internal forces can be expressed in terms of vertical displacement (same procedure than in Cartesian coordinate system)

$$M_r = -D\cdot\left(\frac{\partial^2 w}{\partial r^2} + \nu\cdot\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right)$$

$$M_t = -D\cdot\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial r^2}\right)$$

$$M_{rt} = -D\cdot(1-\nu)\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right)$$

$$Q_r = -D\cdot\frac{\partial}{\partial r}\left(\nabla^2 w\right)$$

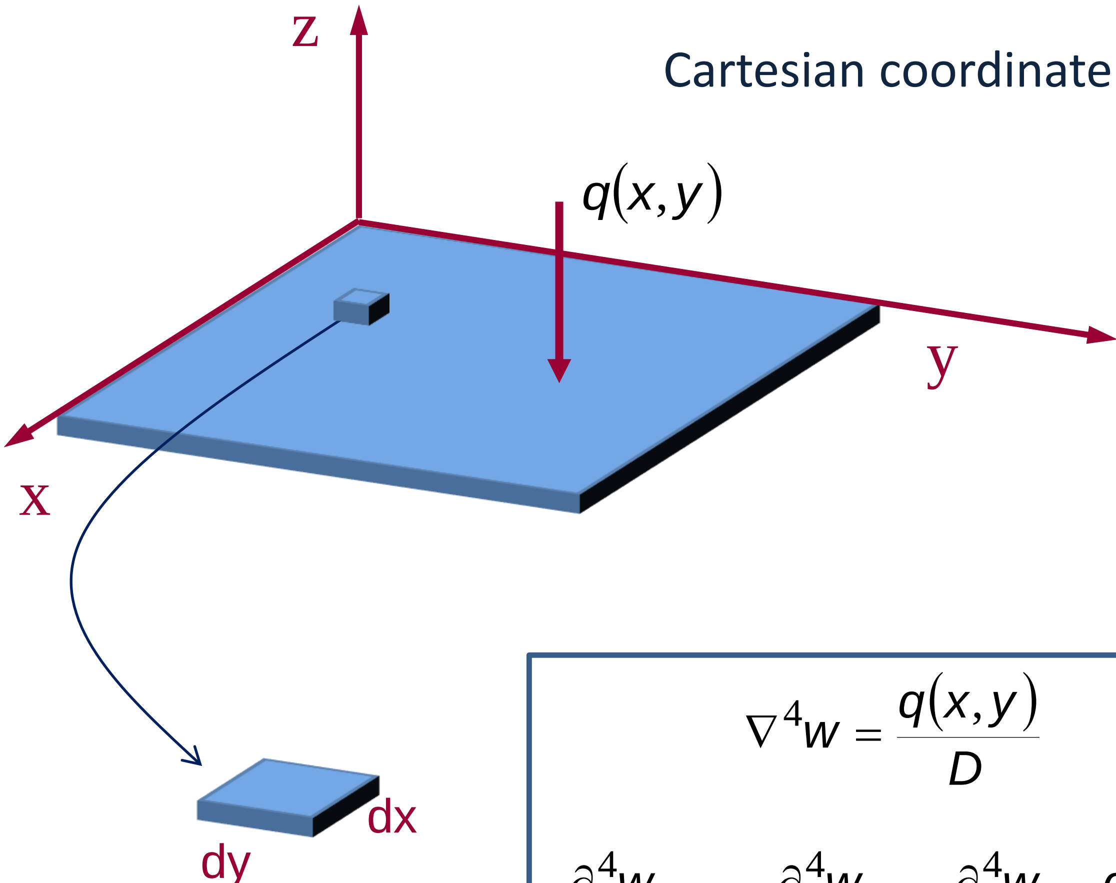
$$Q_t = -\frac{D}{r}\cdot\frac{\partial}{\partial \theta}\left(\nabla^2 w\right)$$

Chapter 4. Plates and Shells

Circular plates

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Bending equations



Cartesian coordinate system:

Del operator

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Laplace operator (in-plane)

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^4 w = \frac{q(x,y)}{D}$$

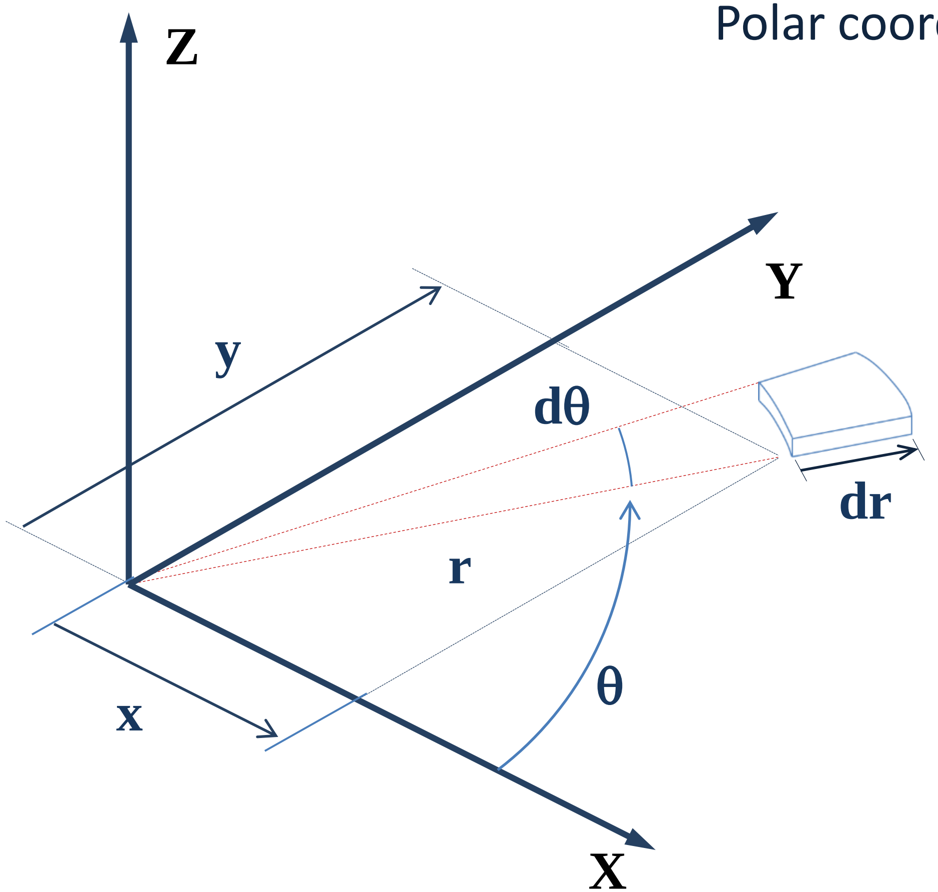
$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D}$$

$$D = \frac{E \cdot h^3}{12 \cdot (1 - \nu^2)}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Bending equations

Polar coordinate system:



Del operator

$$\nabla = \frac{\partial}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{u}_\theta + \frac{\partial}{\partial z}\vec{u}_z$$

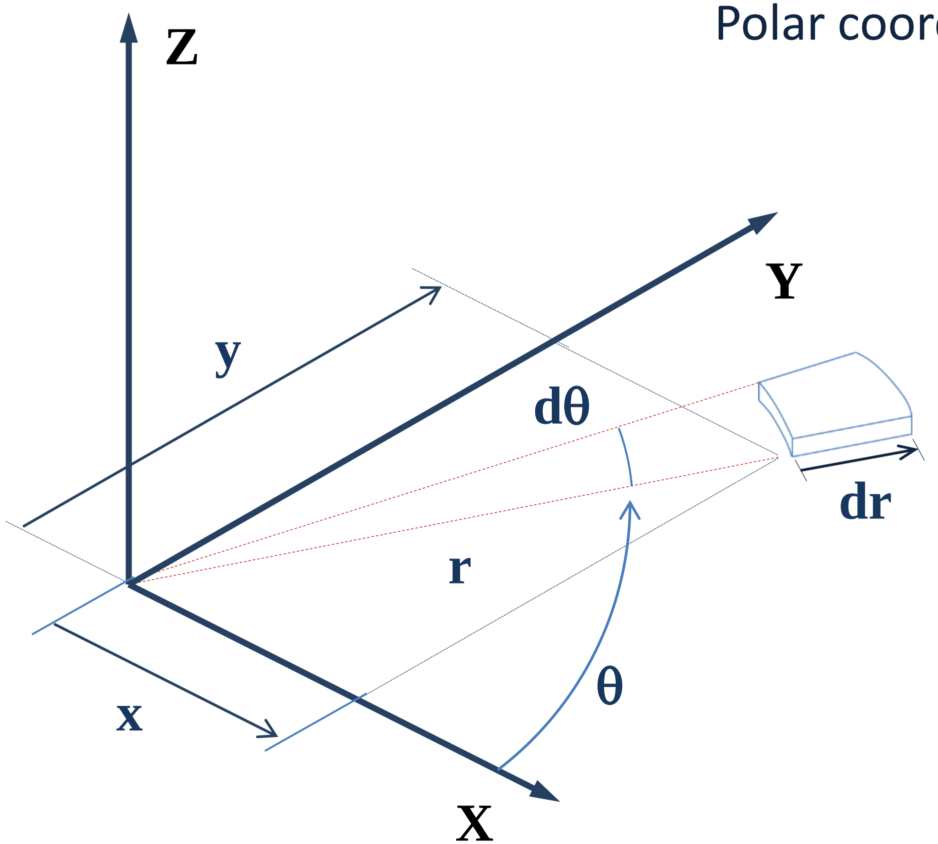
Laplace operator (in-plane)

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Bending equations

Polar coordinate system:



$$\nabla^4 w = \frac{q}{D}$$

$$\nabla^4 w(r,\theta) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) w = \frac{q(r,\theta)}{D}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Bending equations

Solution of the differential equation

$$w = w_h + w_p$$

$$\nabla^4 w_h = 0 \qquad \nabla^4 w_p = \frac{q}{D}$$

The solution of the homogeneous equation can be expressed as the next series:

$$w_h(r,\theta) = \sum_{m=0}^{\infty} R_m(r)\cdot\cos(m\cdot\theta) + \sum_{m=1}^{\infty} S_m(r)\cdot sen(m\cdot\theta)$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)R_m = 0$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)S_m = 0$$

Bending equations

The solutions to these equations are:

$$R_o = A_o + B_o \cdot \ln r + C_o \cdot r^2 + D_o \cdot r^2 \cdot \ln r$$

$$R_1, S_1 = A_1 \cdot r + B_1 \cdot \frac{1}{r} + C_1 \cdot r^3 + D_1 \cdot r \cdot \ln r$$

$$R_m, S_m = A_m \cdot r^m + B_m \cdot \frac{1}{r^m} + C_m \cdot r^{m+2} + D_m \cdot \frac{1}{r^{m-2}}$$

The coefficient of these equations can be found imposing boundary conditions

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Chapter 4. Plates and Shells

Circular plates

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Boundary conditions

Clamped in r = a

$$w\big|_{r=a} = 0$$

$$\frac{\partial w}{\partial r}\Big|_{r=a} = 0$$

Simply supported in r = a

$$w\big|_{r=a} = 0$$

$$M_r\big|_{r=a} = 0$$

Free end in r = a

$$\left(Q_r + \frac{1}{r}\cdot\frac{\partial M_{tr}}{\partial \theta}\right)\Big|_{r=a} = 0$$

$$M_r\big|_{r=a} = 0$$

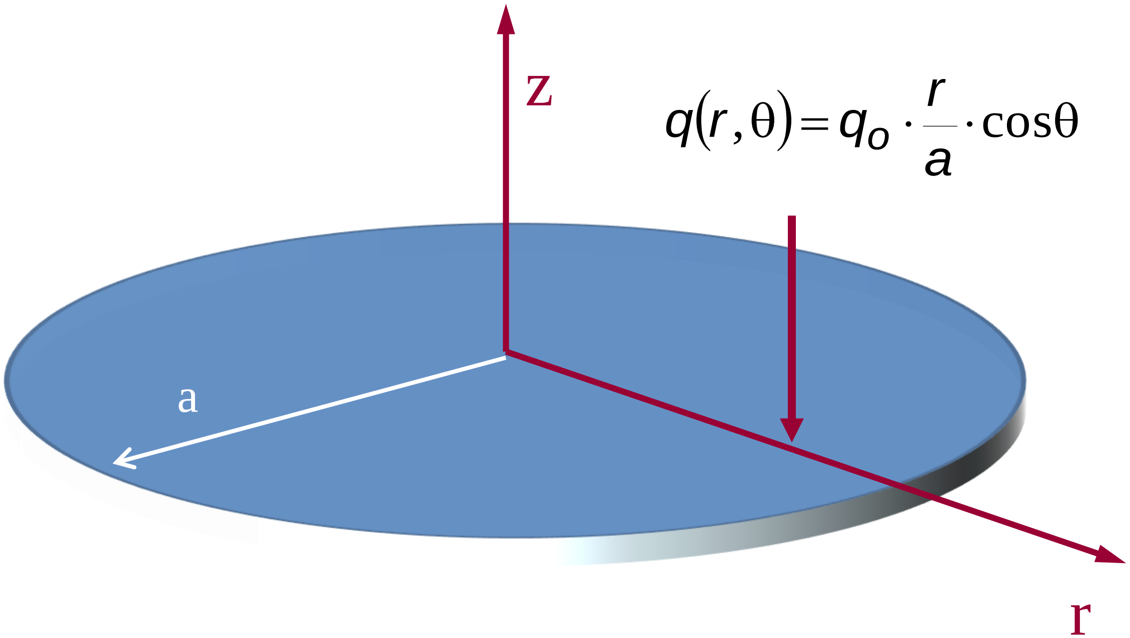
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Chapter 4. Plates and Shells

Circular plates

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Example: Load directly related to radius



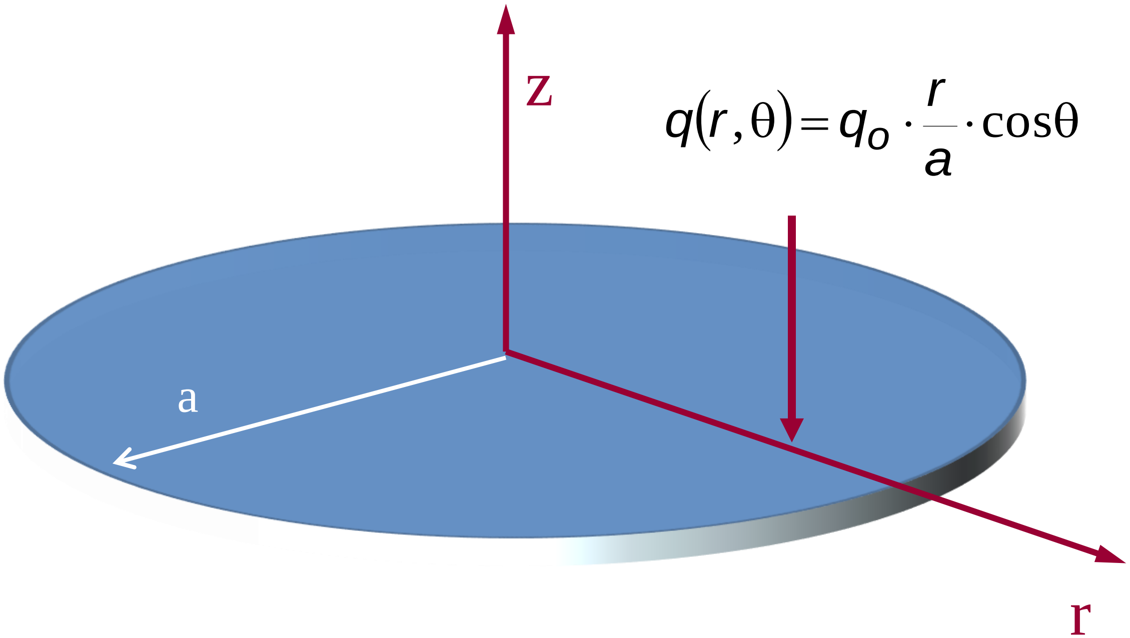
$$q(r,\theta) = q_o \cdot \frac{r}{a} \cdot \cos\theta$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) w = \frac{q_o \cdot r \cdot \cos\theta}{a \cdot D}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Examples

Example: Load directly related to radius



$$q(r,\theta) = q_o \cdot \frac{r}{a} \cdot \cos\theta$$

Particular solution

$$\nabla^4 w_p = \frac{q_o \cdot r \cdot \cos\theta}{a \cdot D}$$

There are fourth derivatives in the equation, thus this is a possible solution:

Where A is a constant

$$w_p(r,\theta) = A \cdot r^5 \cdot \cos\theta$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Example: Load directly related to radius

$$w_p(r,\theta) = A \cdot r^5 \cdot \cos\theta$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) w = \frac{q_o \cdot r \cdot \cos\theta}{a \cdot D}$$

$$w_p(r,\theta) = 5.2 \cdot 10^{-3} \cdot \frac{q_o \cdot r^5 \cdot \cos\theta}{a \cdot D}$$

Particular solution

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Example: Load directly related to radius

Homogenous solution

$$w_h(r,\theta)=\sum_{m=0}^{\infty}R_m(r)\cdot\cos(m\cdot\theta)+\sum_{m=1}^{\infty}S_m(r)\cdot sen(m\cdot\theta)$$

Since load only has terms in cosine, the solution is:

$$w_h(r,\theta)=R_1(r)\cdot\cos\theta$$

$$R_1(r)=A_1\cdot r+B_1\cdot\frac{1}{r}+C_1\cdot r^3+D_1\cdot r\cdot\ln r$$

Example: Load directly related to radius

Final solution:

$$w(r,\theta)=\frac{q_o\cdot a^4}{192\cdot D}\left(\rho^5+A^*\cdot\rho+B^*\cdot\frac{1}{\rho}+C^*\cdot\rho^3+D^*\cdot\rho\cdot\ln\rho\right)\cdot\cos\theta \qquad \rho=\frac{r}{a}$$

Constant are calculated imposing boundary conditions

- Vertical displacement is null in the midpoint:

$$B^*=0$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Examples

Example: Load directly related to radius

$$M_r = -D \cdot \left(\frac{\partial^2 w}{\partial r^2} + \nu \cdot \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right)$$

$$M_t = -D \cdot \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \nu \frac{\partial^2 w}{\partial r^2} \right)$$

$$M_{rt} = -D \cdot (1-\nu) \left(\frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} \right)$$

⇓

$$M_r = -D \cdot \frac{q_o \cdot a^2}{192} \left[(20 + 4 \cdot \nu) \cdot \rho^3 + (6 + 2 \cdot \nu) \cdot C^* \cdot \rho + \frac{D^* \cdot (1+\nu)}{\rho} \right] \cdot \cos\theta$$

$$D^* = 0$$

Otherwise bending moment is infinite in the midpoint

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Example: Load directly related to radius

$$w(r,\theta)=\frac{q_o \cdot a^4}{192 \cdot D}\left(\frac{r}{a^5}^5+A^* \cdot \frac{r}{a}+C^* \cdot \frac{r^3}{a^3}\right) \cdot \cos\theta$$

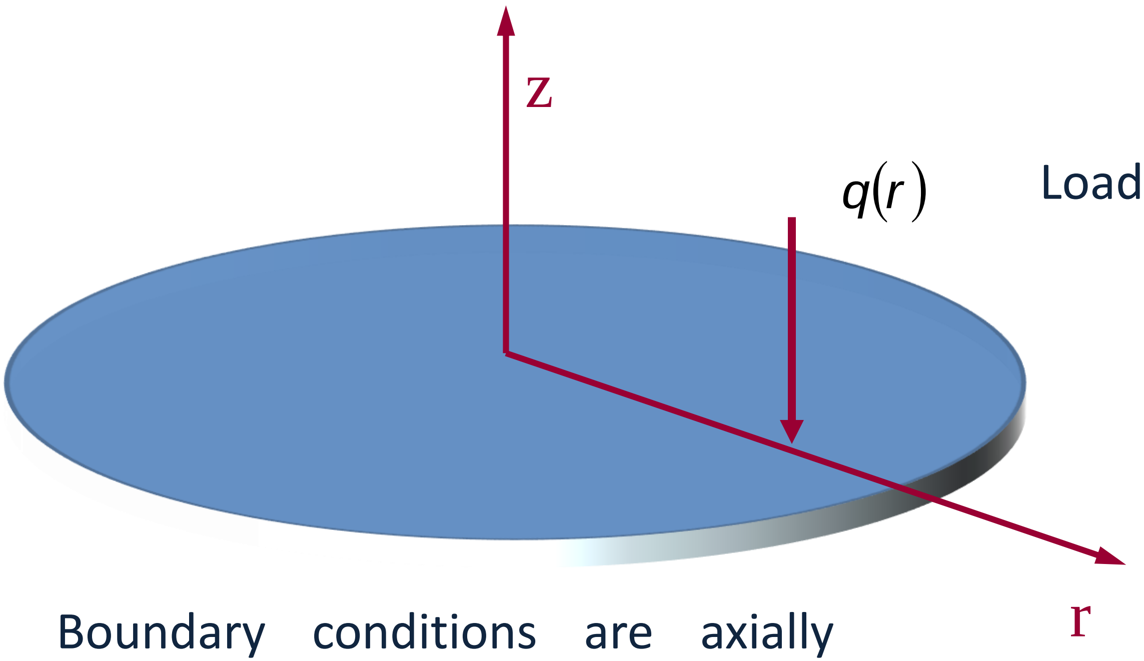
The other constants are found imposing boundary conditions at the external edge r=a

Example, simply supported plate:

$$w\big|_{r=a}=0 \qquad M_r\big|_{r=a}=0 \quad \Rightarrow \quad A^*=\frac{7+\nu}{3+\nu} \qquad C^*=-2\frac{5+\nu}{3+\nu}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Example: Axial symmetry



Loads are angle independent

Boundary conditions are axially symmetrical

$$\nabla^4 w = \frac{q}{D}$$

$$\nabla^4 w(r) = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) = \frac{q(r)}{D}$$

Example: Axial symmetry

Internal forces equations considering angle independency

$$M_r = -D \cdot \left(\frac{d^2 w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} \right)$$

$$M_t = -D \cdot \left(\frac{1}{r} \frac{dw}{dr} + \nu \frac{d^2 w}{dr^2} \right)$$

$$M_{rt} = 0$$

$$Q_r = -D \cdot \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} \left(r \frac{dw}{dr} \right) \right)$$

$$Q_t = 0$$

Example: Axial symmetry

Differential equation can be integrated when load is uniform:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{\partial}{dr}\right)=\frac{q_o}{D}$$

$$w_h(r)=A_o+B_o\cdot\ln r+C_o\cdot r^2+D_o\cdot r^2\cdot\ln r$$

$$w_p(r)=\frac{q_o\cdot r^4}{64D}$$

Coefficients are found imposing boundary conditions

References

1. **Megson, T.H.G. "Aircraft Structures"**
   Elsevier. 2007
   Cap.7 Bending of thin plates

2. **Timoshenko, Stephen "Theory of plates and shells "**
   McGraw-Hill, 1959

1. **Monleón Cremades, Salvador "Análisis de vigas, arcos, placas y láminas : una presentación unificada"**
   Universidad Politécnica de Valencia. 1999

2. **J. A. Jurado Albarracín-Martinón y S. Hernández Ibáñez, "Análisis estructural de placas y láminas". Tercera Edición. Andavira editora 2014**