Bachelor in Aerospace Engineering

Universidad Carlos III de Madrid
www.uc3m.es

Department of Continuum Mechanics and Structural Engineering

Aerospace Structures

Chapter 4. Laminate and sandwich structures

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Chapter 4. Laminate and Sandwich Structures

Laminate theory

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

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Laminate theory

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Material requirements in aerospace applications

- High strength and stiffness (stiffness is most important)
- Low thermal expansion coefficient (extreme temperatures -200/200ºC)
- Low density (essential)

Cascade phenomenon

1 Kg saving in orbital mass → 8 kg saving in fuel → Smaller tanks → Lower weight of the structure → 1 Kg saving in orbital mass

→ 40 kg saving in launch pad

20000 $ saving

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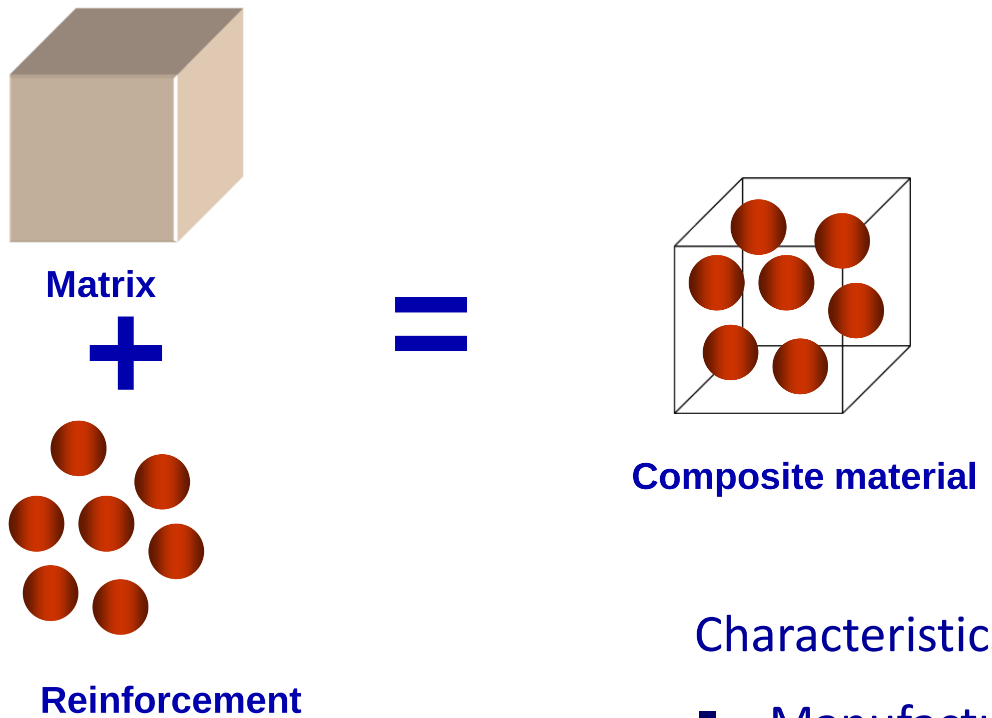
Composite definition



Characteristics:

- Manufactured by human being
- No chemical reactions
- Macroscopically heterogeneous
- Advanced material

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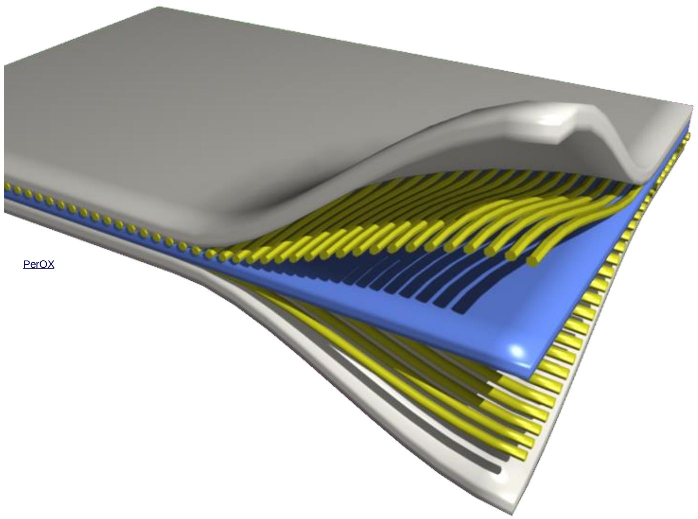
Laminate definition

http://commons.wikimedia.org/wiki/Category:Composite_materials#mediaviewer/File:Composite_3d.png



PerOX

Composite materials composed of several stacked thin layers. These layers are called laminas.

The lamina is the essential unit of composite materials. Typical lamina thickness is around 0.1-1 mm.

Each lamina can be composed of:

- Short fibres
- Unidirectional Long fibres
- Fabric fibres

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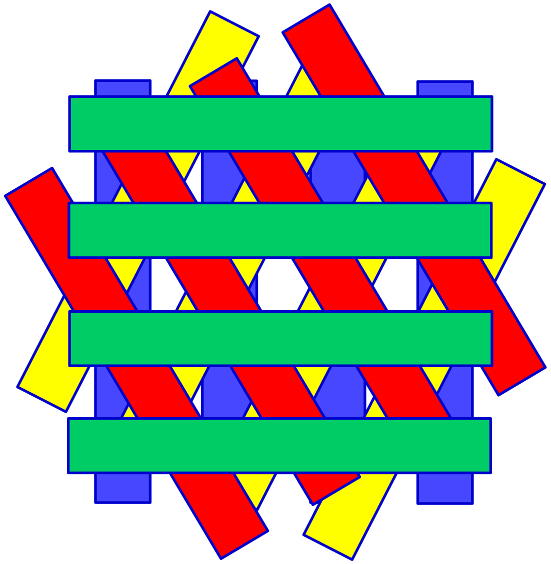
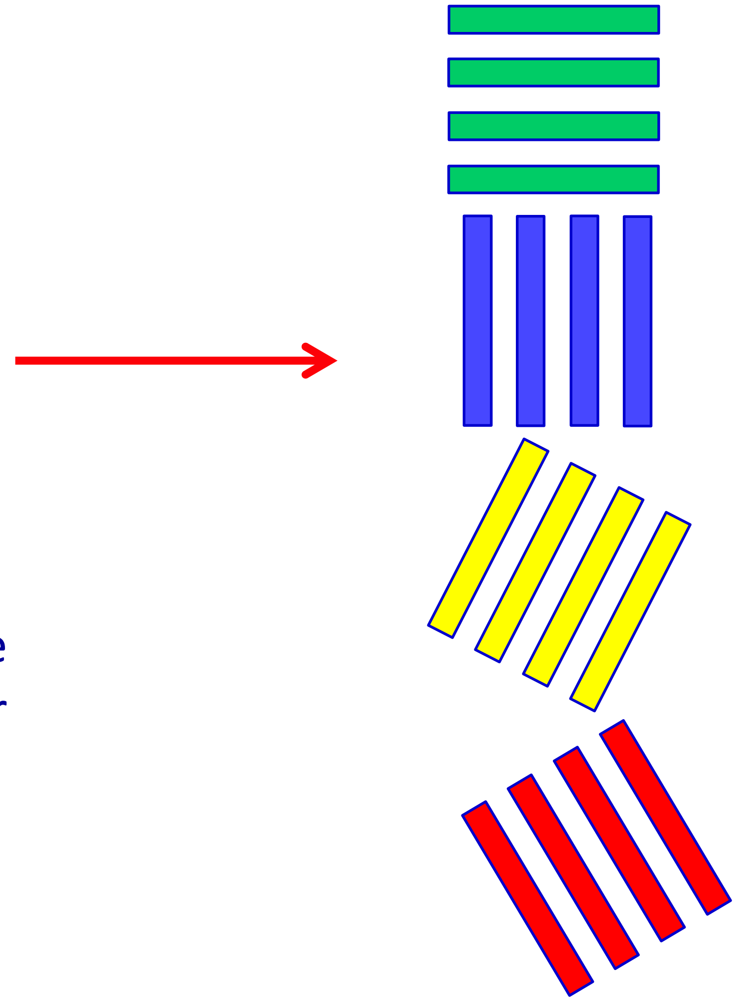
Laminate definition





Sublaminate: Laminates are made of the repetition of a finite number of lamina orientations

Example: [0 / 45 / -45 / 90]

Chapter 4. Laminate and Sandwich Structures

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Linear elastic material

Duhamel-Neumann formulation

$$\sigma_{ij} = \sum C_{ijkl} \cdot \varepsilon_{kl}$$

Matrix notation:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}$$

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Anisotropic behaviour

Linear elastic material

The tensor [C] is symmetric. For anisotropic materials **21** independent elastic constants are required

The inverse tensor can also be applied:

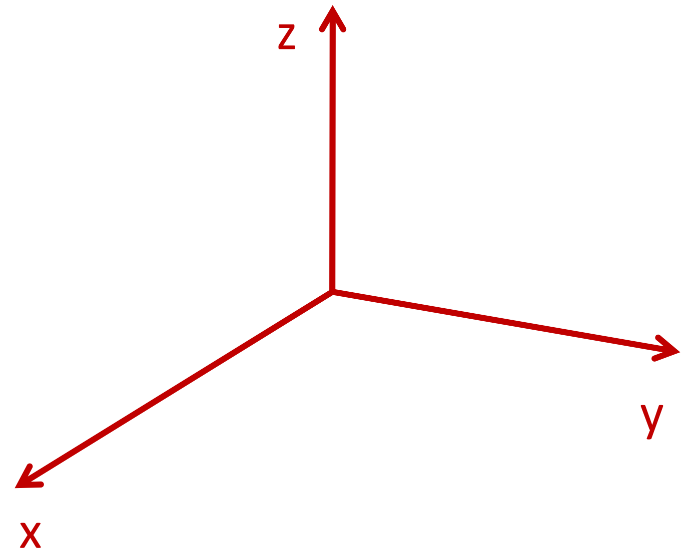
$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \cdot \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix}$$

Anisotropic behaviour

Orthotropic material

3 mutually perpendicular planes of elastic symmetry $\rightarrow$ 9 constants

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}$$



The constants have physical meaning

Planes of elastic symmetry xy xz yz

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Anisotropic behaviour

Orthotropic material

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}$$

$$s_{11} = \frac{1}{E_1} \qquad s_{12} = \frac{1}{E_2} \qquad s_{33} = \frac{1}{E_3}$$

Anisotropic behaviour

Orthotropic material

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}$$

$$s_{23} = \frac{1}{G_{23}} \qquad s_{55} = \frac{1}{G_{13}} \qquad s_{66} = \frac{1}{G_{12}}$$

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Orthotropic material

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}$$

$$s_{12} = -\frac{\nu_{12}}{E_2} \qquad s_{13} = -\frac{\nu_{13}}{E_3} \qquad s_{23} = -\frac{\nu_{23}}{E_3}$$

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Anisotropic behaviour

Orthotropic material

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}$$

$$s_{21} = -\frac{\nu_{21}}{E_1} \qquad s_{31} = -\frac{\nu_{31}}{E_1} \qquad s_{32} = -\frac{\nu_{32}}{E_2}$$

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Anisotropic behaviour

Orthotropic material

Restrictions to the values of the elastic constants

Symmetry of matrix [S]

$$\frac{\nu_{ji}}{E_i} = \frac{\nu_{ij}}{E_j} \quad , \quad \mathrm{i, j = 1...3} \quad , \quad \mathrm{i \neq j}$$

Terms of main diagonal of [S] and [C] are positives

$$0 < \nu_{ij} < \sqrt{\frac{E_i}{E_j}} \quad , \quad i, j = 1...3 \quad , \mathrm{i \neq j}$$

$$1 - \nu_{12} \cdot \nu_{21} - \nu_{23} \cdot \nu_{32} - \nu_{31} \cdot \nu_{13} -$$

$$- 2 \cdot \nu_{21} \cdot \nu_{32} \cdot \nu_{13} > 0$$

Chapter 4. Laminate and Sandwich Structures

Laminate theory

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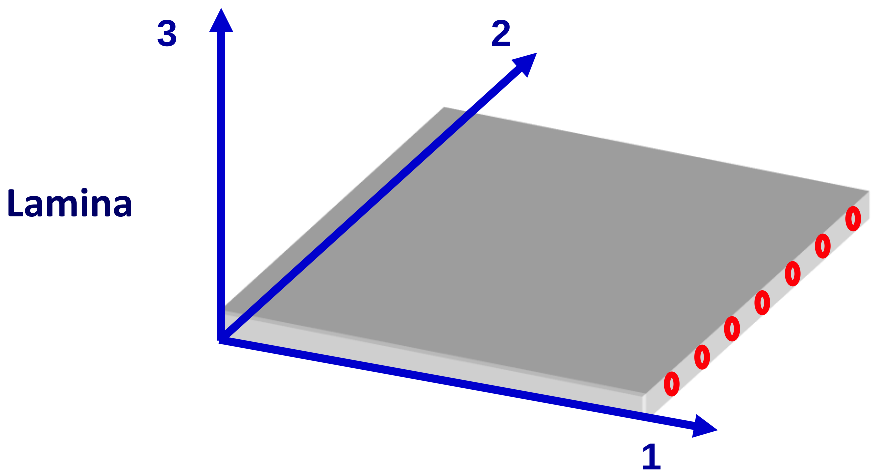
Lamina stiffness matrix

Stiffness matrix in lamina coordinate system (local system)



$$\sigma_{ij} = C_{ijkl} \cdot \varepsilon_{kl}$$

Plain stress state:

$$\sigma_3 = 0 \quad \tau_{13} = 0 \quad \tau_{23} = 0 \quad \rightarrow \quad [T] = \begin{bmatrix} \sigma_1 & \tau_{12} \\ \tau_{12} & \sigma_2 \end{bmatrix} \quad \rightarrow \quad \{\sigma\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

¡!

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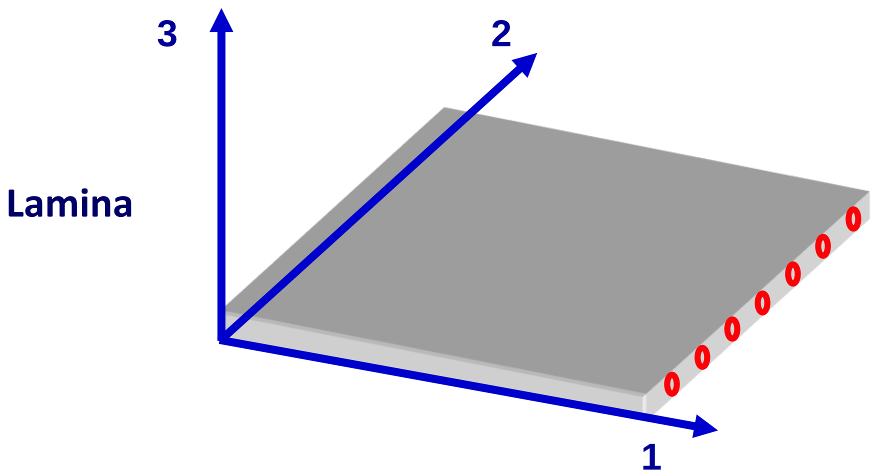
Lamina stiffness matrix

Stiffness matrix in lamina system



$$[Q]=\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{SS} \end{bmatrix}$$

$$[Q]=\begin{bmatrix} \dfrac{E_1}{1-\nu_{12}\cdot\nu_{21}} & \dfrac{\nu_{21}\cdot E_2}{1-\nu_{12}\cdot\nu_{21}} & 0 \\ \dfrac{\nu_{12}\cdot E_1}{1-\nu_{12}\cdot\nu_{21}} & \dfrac{E_2}{1-\nu_{12}\cdot\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

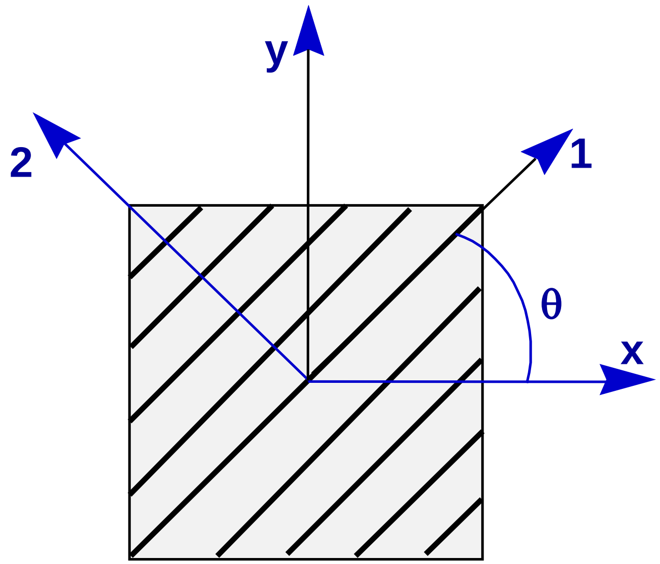
Plain stress state:

$$\{\sigma\}^{12}=[Q]\cdot\{\varepsilon\}^{12}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}=\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{SS} \end{bmatrix}\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

Lamina stiffness matrix

Stiffness matrix in global system



In axis 12: $\{\sigma\}^{12} = [Q]\cdot\{\varepsilon\}^{12}$

In axis xy: $\{\sigma\}^{xy} = [\overline{Q}]\cdot\{\varepsilon\}^{xy}$

$$[Q] \Rightarrow [\overline{Q}]$$

¿?

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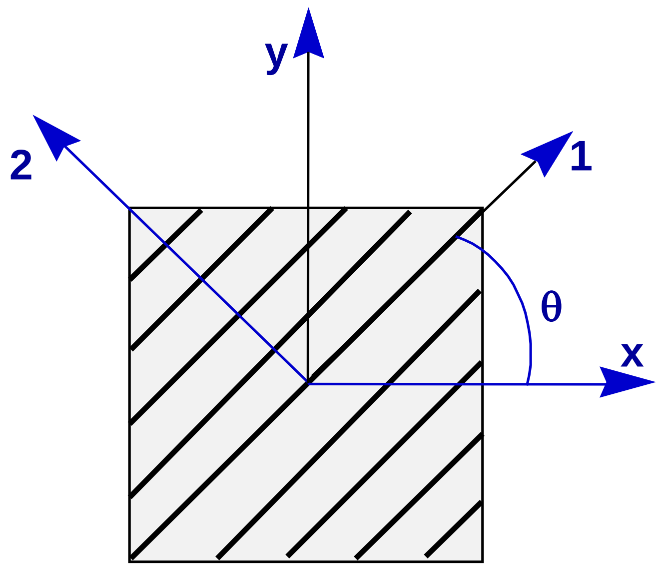
Lamina stiffness matrix

Stiffness matrix in global system



$m = \cos\theta \qquad n = \mathrm{sen}\theta$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [T] \cdot \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2} \cdot \gamma_{12} \end{Bmatrix} = [T] \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2} \cdot \gamma_{xy} \end{Bmatrix}$$

$$[T] = \begin{bmatrix} m^2 & n^2 & 2 \cdot m \cdot n \\ n^2 & m^2 & -2 \cdot m \cdot n \\ -m \cdot n & m \cdot n & (m^2 - n^2) \end{bmatrix}$$

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Lamina stiffness matrix

Stiffness matrix in global system

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix} = [Q]\cdot\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = [\overline{Q}]\cdot\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = [R]^{-1}\cdot\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \frac{1}{2}\cdot\gamma_{xy}\end{Bmatrix}$$

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix} = [R]^{-1}\cdot\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \frac{1}{2}\cdot\gamma_{12}\end{Bmatrix}$$

$$[\mathrm{R}] = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}$$

$$[\mathrm{R}]^{-1} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}$$

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Lamina stiffness matrix

Stiffness matrix in global system

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \cdot \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [Q] \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [R]^{-1} \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2} \cdot \gamma_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2} \cdot \gamma_{12} \end{Bmatrix} = [T] \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2} \cdot \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \cdot [Q] \cdot [R]^{-1} \cdot [T] \cdot [R] \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{xx} & \overline{Q}_{xy} & \overline{Q}_{xs} \\ \overline{Q}_{xy} & \overline{Q}_{yy} & \overline{Q}_{ys} \\ \overline{Q}_{xs} & \overline{Q}_{ys} & \overline{Q}_{ss} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

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Lamina stiffness matrix

Stiffness matrix in global system

$$\overline{Q}_{xx} = Q_{11} \bullet m^4 + 2 \bullet (Q_{12} + 2 \bullet Q_{SS}) \bullet n^2 \bullet m^2 + Q_{22} \bullet n^4$$

$$\overline{Q}_{yx} = (Q_{11} + Q_{22} - 4 \bullet Q_{SS}) \bullet n^2 \bullet m^2 + Q_{12} \bullet (n^4 + m^4)$$

$$\overline{Q}_{yy} = Q_{11} \bullet n^4 + 2 \bullet (Q_{12} + 2 \bullet Q_{SS}) \bullet n^2 \bullet m^2 + Q_{22} \bullet m^4$$

$$\overline{Q}_{xS} = (Q_{11} - Q_{12} - 2 \bullet Q_{SS}) \bullet n \bullet m^3 + (Q_{12} - Q_{22} + 2 \bullet Q_{SS}) \bullet m \bullet n^3$$

$$\overline{Q}_{yS} = (Q_{11} - Q_{12} - 2 \bullet Q_{SS}) \bullet n^3 \bullet m + (Q_{12} - Q_{22} + 2 \bullet Q_{SS}) \bullet n \bullet m^3$$

$$\overline{Q}_{SS} = (Q_{11} + Q_{22} - 2 \bullet Q_{12} - 2 \bullet Q_{SS}) \bullet n^2 \bullet m^2 + Q_{SS} \bullet (n^4 + m^4)$$

$m = \cos\theta \qquad n = \mathrm{sen}\theta$

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Chapter 4. Laminate and Sandwich Structures

Laminate theory

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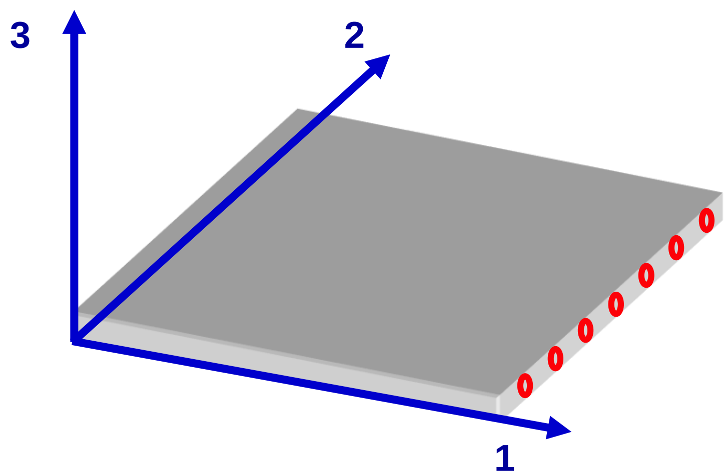
Hypotheses

- **Each lamina is consider as an orthotropic homogenous material**
- **The behaviour of lamina material is linear elastic up to failure**



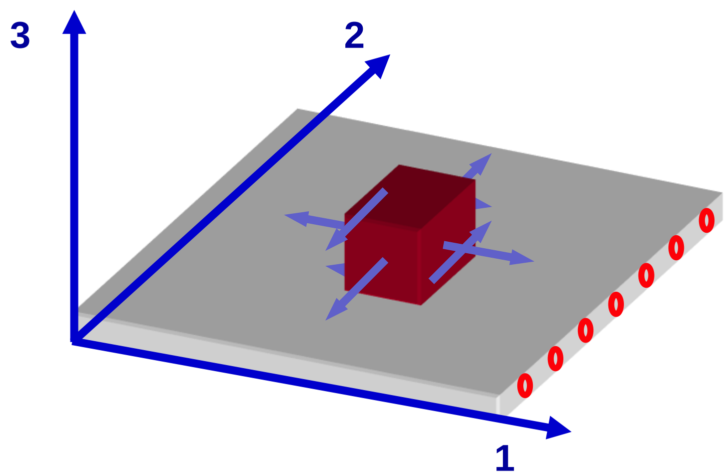
$$\{\sigma\}^{12} = [Q]\cdot\{\varepsilon\}^{12}$$

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{SS} \end{bmatrix}$$

$$[Q] = \begin{bmatrix} \dfrac{E_1}{1-\nu_{12}\cdot\nu_{21}} & \dfrac{\nu_{21}\cdot E_2}{1-\nu_{12}\cdot\nu_{21}} & 0 \\ \dfrac{\nu_{12}\cdot E_1}{1-\nu_{12}\cdot\nu_{21}} & \dfrac{E_2}{1-\nu_{12}\cdot\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

Hypotheses

- **Perfect bonding of laminas**
- **Plain stress state**



$$\sigma_x = 0 \qquad \sigma_x \neq 0$$

$$\tau_{yz} = 0 \qquad \sigma_y \neq 0$$

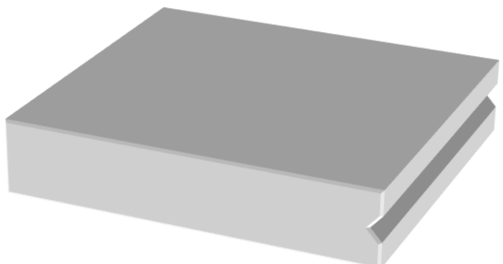
$$\tau_{xz} = 0 \qquad \tau_{yz} \neq 0$$

This hypothesis is not valid in some situations !!

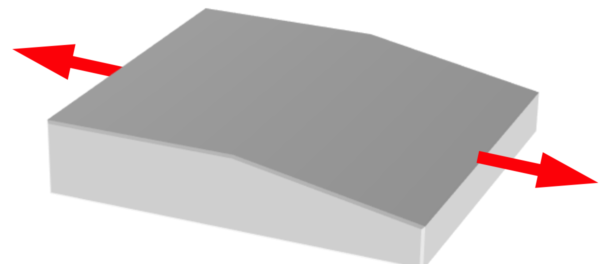
Hypotheses

Situations where plain stress is not a valid hypothesis

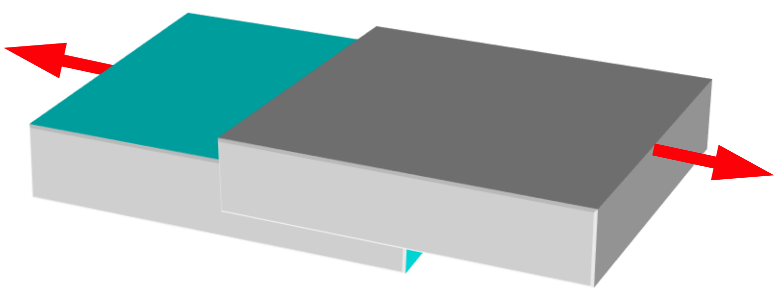
Free end



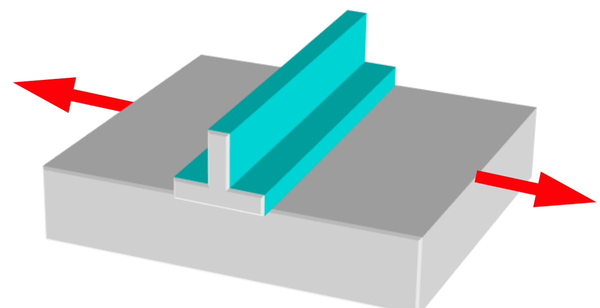
Variable thickness



Bonded joints



Stringers



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Hypotheses

- **Small displacements**
- **Kirchhoff hypotheses**

A straight line perpendicular to the mid plane remains straight an perpendicular to the mid surface after deformation

$$\gamma_{yz} = 0$$

$$\gamma_{xz} = 0$$

No changes in laminate thickness

$$\varepsilon_z = 0$$

The displacement of each point can be expressed as a function of the displacements and the rotations of the mid-plane

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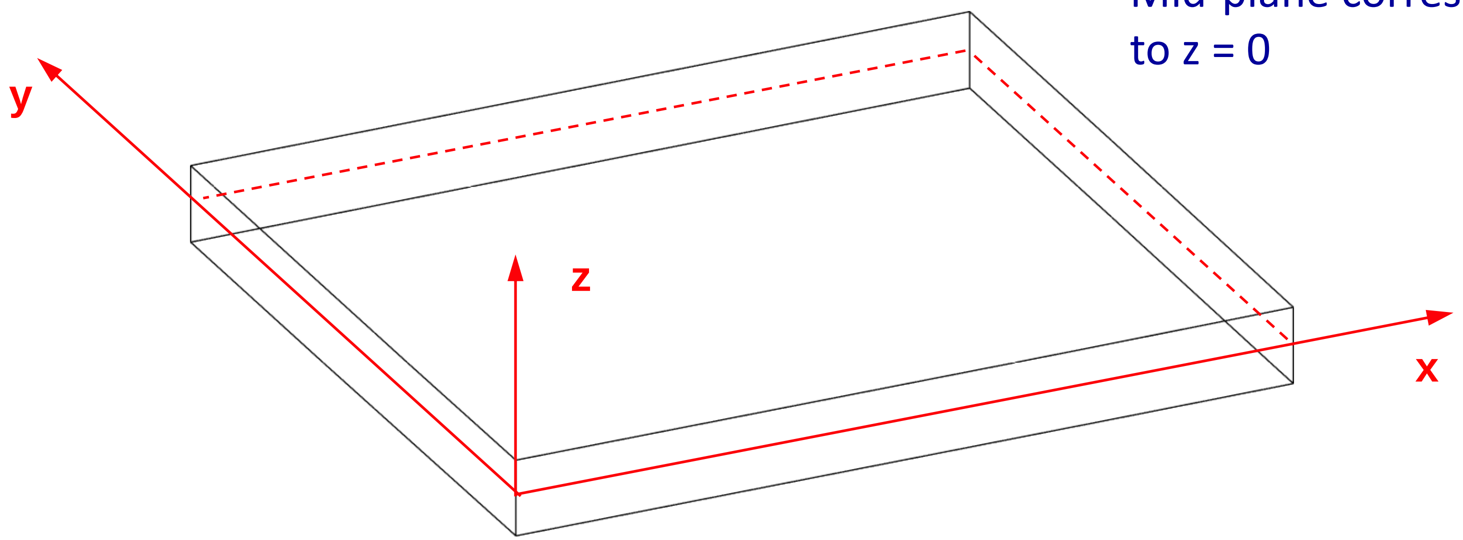
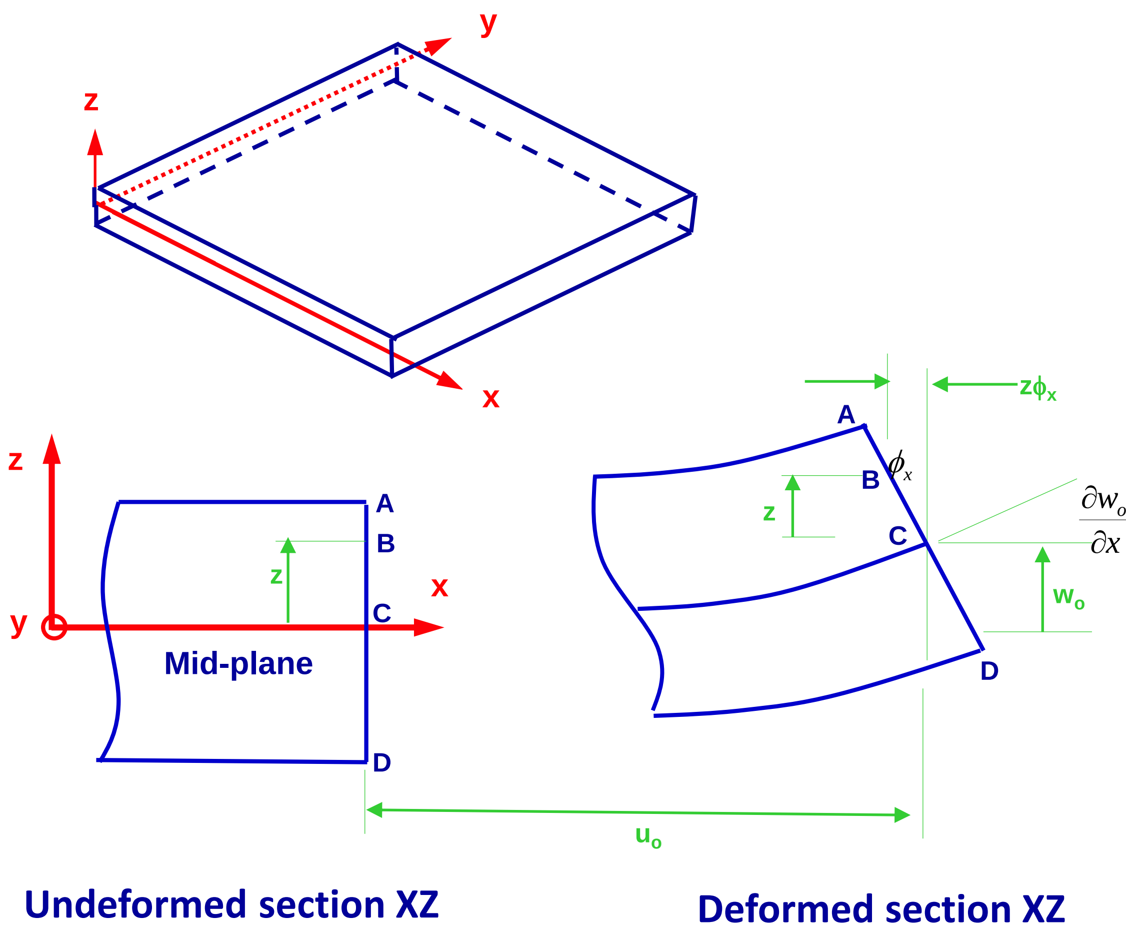
Definition of laminate axes



Mid-plane corresponds to z = 0

Laminate kinematics



$$\phi_x = \frac{\partial w_o}{\partial x}$$

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Laminate kinematics

Displacements

$$u(x,y,z) = u_o(x,y) - \phi_x \cdot z$$
$$v(x,y,z) = v_o(x,y) - \phi_y \cdot z$$
$$w(x,y,z) = w_o(x,y)$$

Laminate theory

$$\phi_x = \frac{\partial w_o}{\partial x}$$
$$\phi_y = \frac{\partial w_o}{\partial y}$$

Strains

$$\varepsilon_x = \frac{\partial u_o}{\partial x} - z \cdot \frac{\partial^2 w_o}{\partial x^2}$$
$$\varepsilon_y = \frac{\partial v_o}{\partial y} - z \cdot \frac{\partial^2 w_o}{\partial y^2}$$
$$\varepsilon_y = 0$$

$$\gamma_{xy} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial y} - z \cdot 2 \cdot \frac{\partial^2 w_o}{\partial x \cdot \partial y}$$
$$\gamma_{xz} = 0$$
$$\gamma_{yz} = 0$$

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Laminate kinematics

Strains in mid-plane

$$\begin{cases} \varepsilon^o_x = \dfrac{\partial u_o}{\partial x} \\ \varepsilon^o_y = \dfrac{\partial v_o}{\partial y} \\ \gamma^o_{xy} = \dfrac{\partial u_o}{\partial y} + \dfrac{\partial v_o}{\partial x} \end{cases}$$

Curvatures

$$\begin{cases} k_x = -\dfrac{\partial^2 w_o}{\partial x^2} \\ k_y = -\dfrac{\partial^2 w_o}{\partial y^2} \\ k_{xy} = -2\dfrac{\partial^2 w_o}{\partial x \cdot \partial y} \end{cases}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^o_x \\ \varepsilon^o_y \\ \gamma^o_{xy} \end{Bmatrix} + z \cdot \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}$$

Linear variation of strains along thickness direction!!

Chapter 4. Laminate and Sandwich Structures

Laminate theory

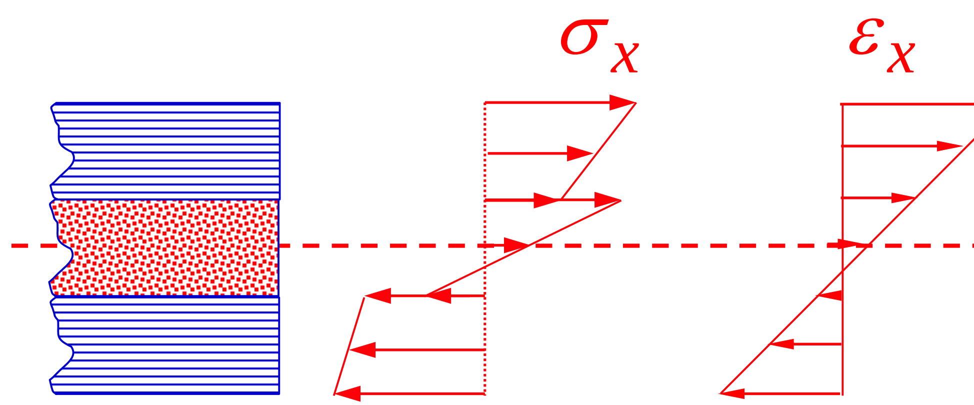
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Stress distribution

$$\{\sigma\}_i = [\bar{Q}]_i \cdot \{\varepsilon\}$$

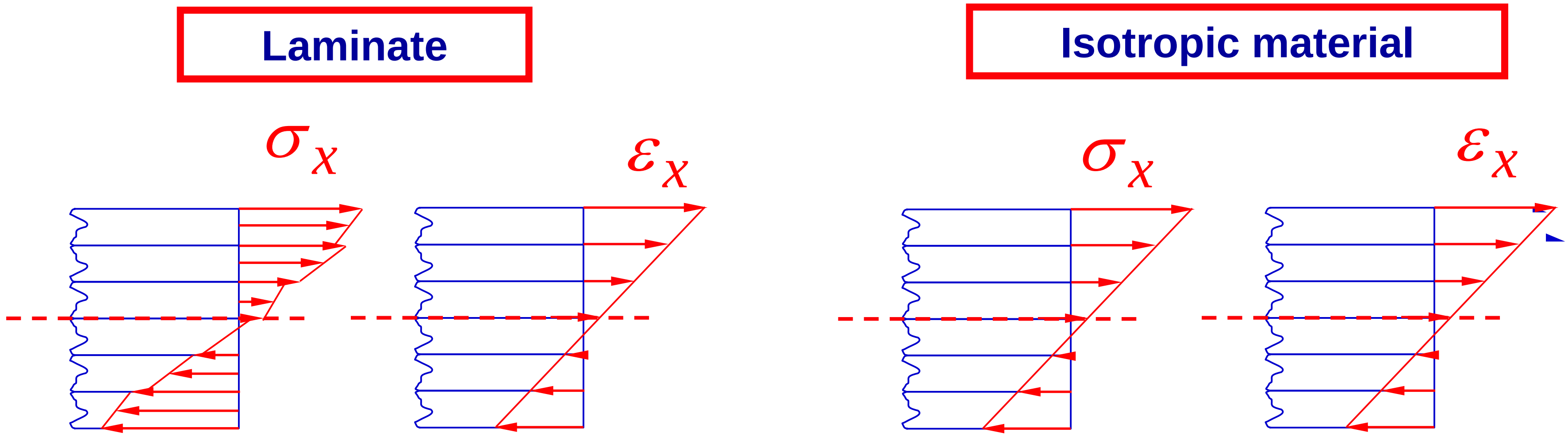
$$\{\varepsilon\} = \{\varepsilon^o\} + z \cdot \{k\}$$

Stress is a discontinuous function of z



Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

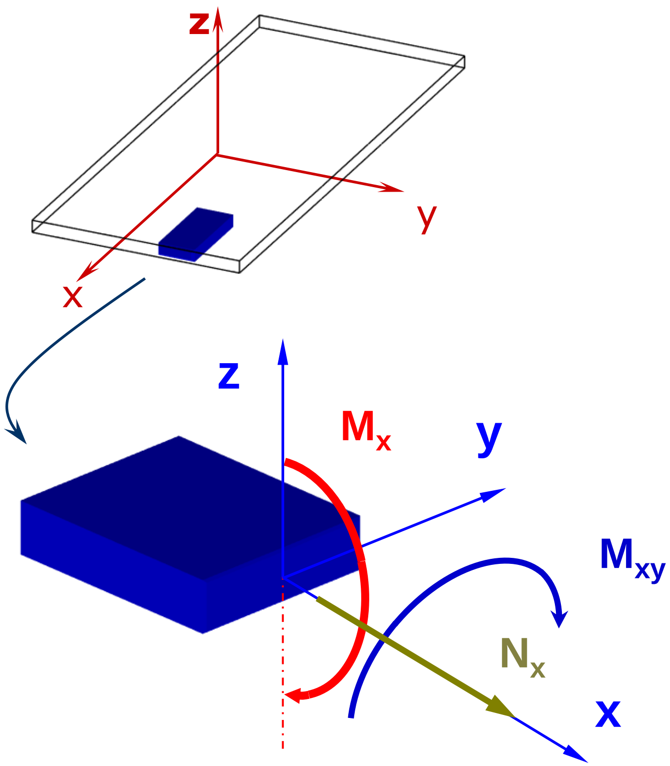
Stress distribution

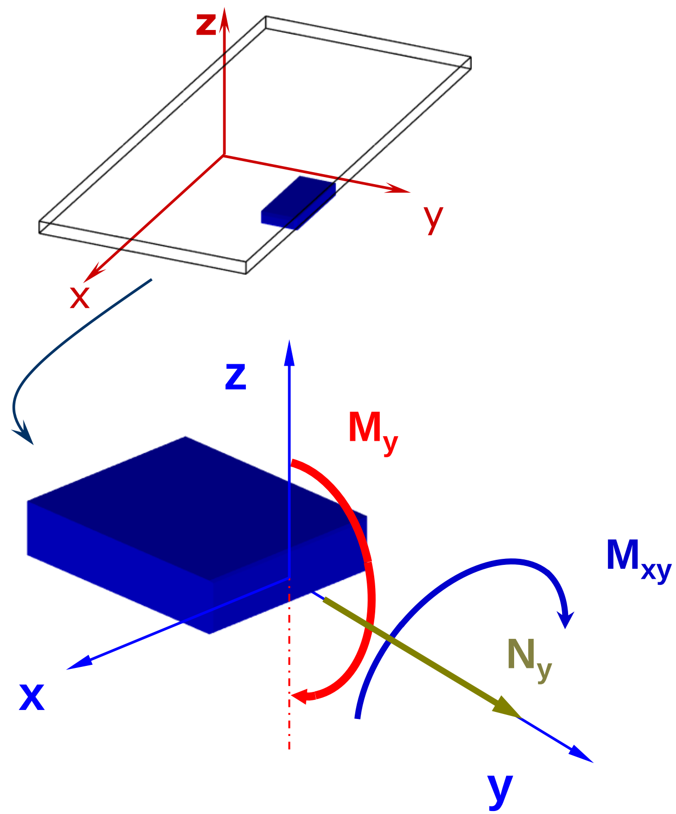


Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Internal forces

Internal forces defined per unit length

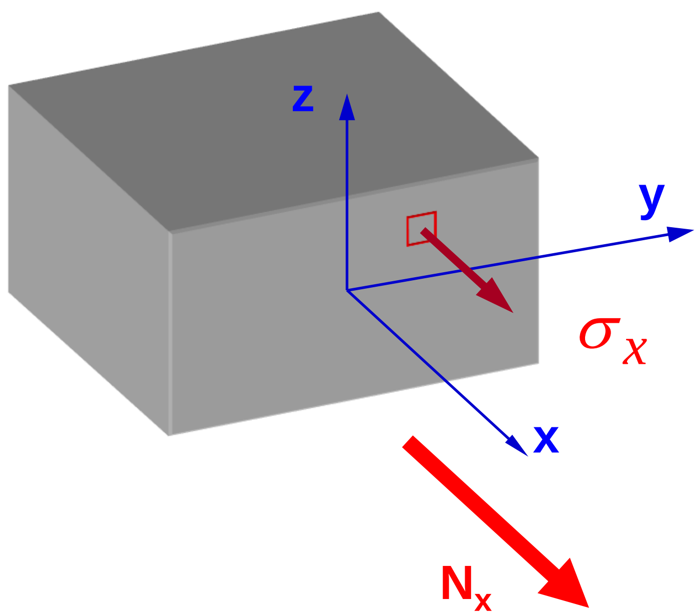




Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Internal forces

Internal forces defined per unit length



$$N_x = \int_{-h/2}^{h/2} \sigma_x \cdot dz$$

$$N_y = \int_{-h/2}^{h/2} \sigma_y \cdot dz \qquad \{N\} = \int_{-h/2}^{h/2} \{\sigma\} \cdot dz$$

$$N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \cdot dz$$

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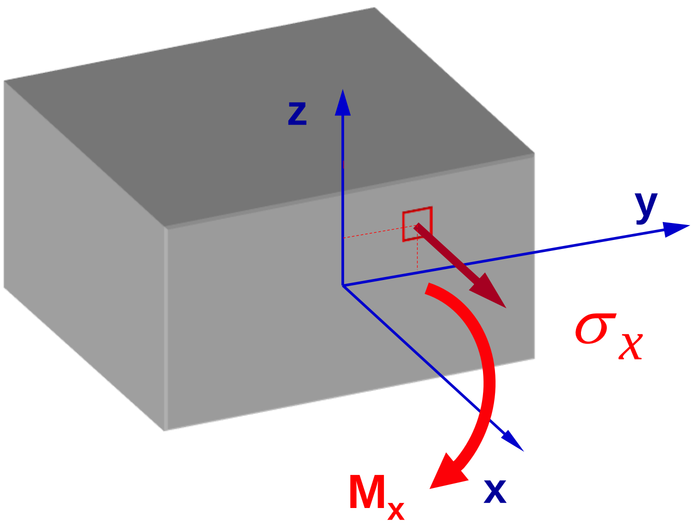
Internal forces

Internal forces defined per unit length



$$M_x = \int_{-h/2}^{h/2} \sigma_x \cdot z \cdot dz$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y \cdot z \cdot dz \qquad \{M\} = \int_{-h/2}^{h/2} \{\sigma\} \cdot z \cdot dz$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \cdot z \cdot dz$$

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Chapter 4. Laminate and Sandwich Structures

Laminate theory

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Laminate stiffness matrix

$$\{\varepsilon\} = \{\varepsilon^o\} + z \cdot \{k\}$$

$$\{\sigma\}_i = [\bar{Q}]_i \cdot \{\varepsilon\}$$

$$\{N\} = \int_{-h/2}^{h/2} \{\sigma\} \cdot dz$$

$$\{N\} = [A] \cdot \{\varepsilon^o\} + [B] \cdot \{k\}$$

$$[A] = \int_{-h/2}^{h/2} [\bar{Q}]_i \cdot dz$$

$$[B] = \int_{-h/2}^{h/2} [\bar{Q}]_i \cdot z \cdot dz$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Laminate stiffness matrix

$$\{\varepsilon\} = \{\varepsilon^{o}\} + z \cdot \{k\}$$

$$\{\sigma\}_i = [\bar{Q}]_i \cdot \{\varepsilon\}$$

$$\{M\} = \int_{-h/2}^{h/2} \{\sigma\} \cdot z \cdot dz$$

$$\{M\} = [B] \cdot \{\varepsilon^{o}\} + [D] \cdot \{k\}$$

$$[D] = \int_{-h/2}^{h/2} [\bar{Q}]_i \cdot z^2 \cdot dz$$

$$[B] = \int_{-h/2}^{h/2} [\bar{Q}]_i \cdot z \cdot dz$$

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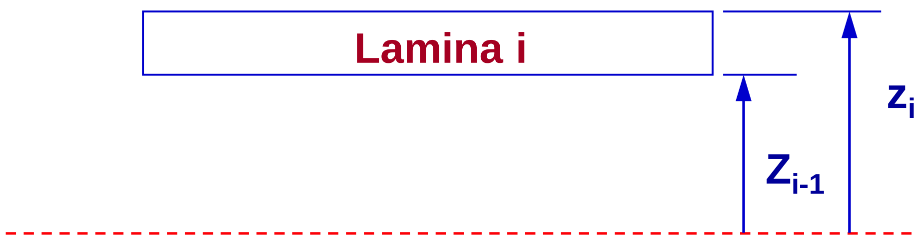
Laminate stiffness matrix

Plain stiffness matrix [A]

$$[A] = \int_{-h/2}^{h/2} [\bar{Q}]_i \cdot dz \quad \Rightarrow \quad [A] = \sum_{i}^{N} \int_{z_{i-1}}^{z_i} [\bar{Q}]_i \cdot dz$$



$$[A] = \sum_{i}^{N} [\bar{Q}]_i \cdot h_i$$

Laminate stiffness matrix

$$[A]=\int_{-h/2}^{h/2}[\bar{Q}]_i\cdot dz$$

$$[D]=\int_{-h/2}^{h/2}[\bar{Q}]_i\cdot z^2\cdot dz$$

$$[B]=\int_{-h/2}^{h/2}[\bar{Q}]_i\cdot z\cdot dz$$

Plain stiffness matrix

$$[A]=\sum_i^N[\bar{Q}]_i\cdot h_i$$

Bending stiffness matrix

$$[D]=\frac{1}{3}\cdot\sum_i^N[\bar{Q}]_i\cdot\left(Z_i^3-Z_{i-1}^3\right)$$

Coupling stiffness matrix

$$[B]=\frac{1}{2}\cdot\sum_i^N[\bar{Q}]_i\cdot\left(Z_i^2-Z_{i-1}^2\right)$$

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Laminate stiffness matrix

$$\{N\} = [A]\cdot\{\varepsilon^o\} + [B]\cdot\{k\}$$

$$\{M\} = [B]\cdot\{\varepsilon^o\} + [D]\cdot\{k\}$$

$$\{\varepsilon^o\} = [a]\cdot\{N\} + [b]\cdot\{M\}$$

$$\{k\} = [c]\cdot\{N\} + [d]\cdot\{M\}$$

$$[a] = [A]^{-1} - \left\{[B^c]\cdot[D^c]^{-1}\right\}\cdot[C^c]$$

$$[b] = [B^c]\cdot[D^c]^{-1}$$

$$[c] = -[D^c]^{-1}\cdot[C^c]$$

$$[d] = [D^c]^{-1}$$

$$[B^c] = -[A]^{-1}\cdot[B]$$

$$[C^c] = [B]\cdot[A]^{-1}$$

$$[D^c] = [D] - \left\{[B]\cdot[A]^{-1}\right\}\cdot[B]$$

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Laminate stiffness matrix

Plain stiffness matrix [A]

$$[A]=\sum_{i}^{N}[\bar{Q}]_i \cdot h_i$$

- Relates axial forces (membrane forces) with plain strains
- Independent on stacking sequence

Laminate $[0/90]_S$ ⟺ Laminate $[90/0]_S$

Same in plane behaviour

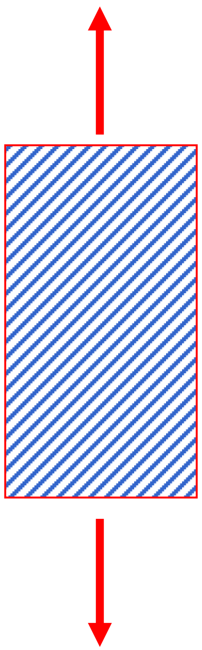
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

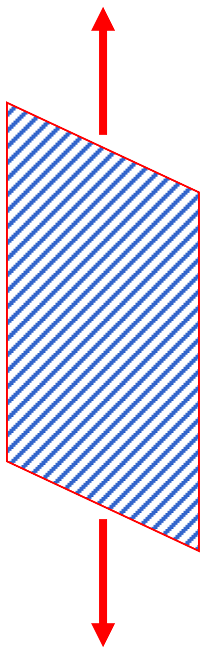
Laminate stiffness matrix

Plain stiffness matrix [A]

$$[A] = \sum_{i}^{N} [\bar{Q}]_i \cdot h_i$$

- Relates axial forces (membrane forces) with plain strains
- Independent on stacking sequence
- If $A_{1S} \neq 0$ and $A_{2S} \neq 0$ there is a coupling effect between axial and shear forces





$$\begin{Bmatrix} N_x \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{1s} \\ A_{12} & A_{22} & A_{2s} \\ A_{1s} & A_{2s} & A_{ss} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Laminate stiffness matrix

Bending stiffness matrix [D]

$$[D] = \frac{1}{3} \cdot \sum_{i}^{N} [\bar{Q}]_i \cdot \left(z_i^3 - z_{i-1}^3 \right)$$

- Relates bending moments with curvatures
- Depends on stacking sequence

Laminado $[90/0]_S$ ⟺ Laminado $[0/90]_S$

Different bending behaviour

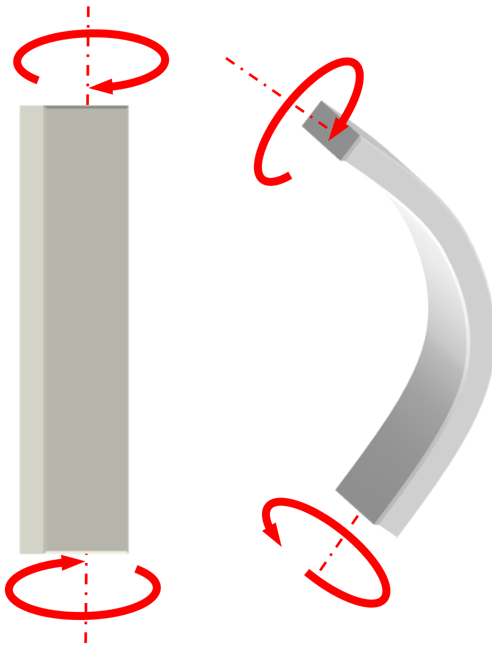
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Laminate stiffness matrix

Bending stiffness matrix [D]

$$[D] = \frac{1}{3} \cdot \sum_{i}^{N} [\bar{Q}]_i \cdot \left(z_i^3 - z_{i-1}^3 \right)$$

- Relates bending moments with curvatures
- Depends on stacking sequence
- If $D_{1S} \neq 0$ and $D_{2S} \neq 0$ there is a coupling effect between bending and torsion moments



$$\begin{Bmatrix} 0 \\ 0 \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{1s} \\ D_{12} & D_{22} & D_{2s} \\ D_{1s} & D_{2s} & D_{ss} \end{bmatrix} \cdot \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

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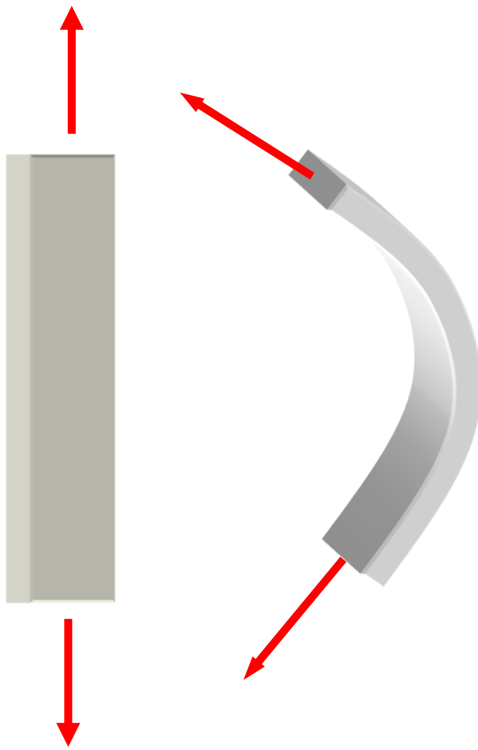
Laminate stiffness matrix

Coupling stiffness matrix [B]

$$[B]=\frac{1}{2}\cdot\sum_{i}^{N}[\bar{Q}]_i\cdot\left(z_i^2-z_{i-1}^2\right)$$

- Relates membrane forces with curvatures and moments with plain strains
- Depends on stacking sequence



$$\begin{Bmatrix} N_x \\ 0 \\ 0 \end{Bmatrix}=\begin{bmatrix} B_{11} & B_{12} & B_{1s} \\ B_{12} & B_{22} & B_{2s} \\ B_{1s} & B_{2s} & B_{ss} \end{bmatrix}\cdot\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

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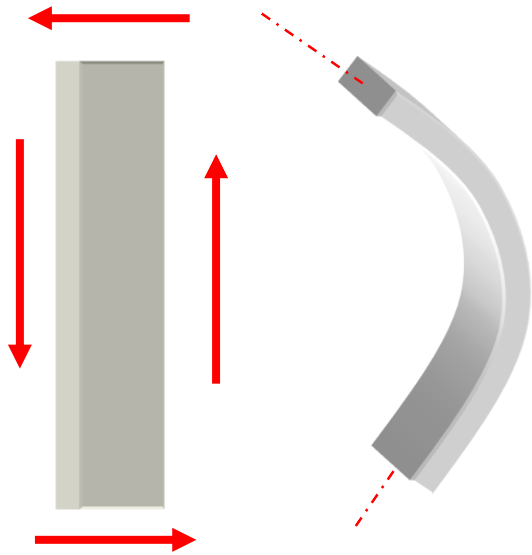
Laminate stiffness matrix

Coupling stiffness matrix [B]

$$[B]=\frac{1}{2}\cdot\sum_{i}^{N}[\bar{Q}]_i\cdot\left(z_i^2-z_{i-1}^2\right)$$

- Relates membrane forces with curvatures and moments with plain strains
- Depends on stacking sequence
- If $B_{1S}\neq 0$ and $B_{2S}\neq 0$ there is a coupling effect between axial forces and torsion moment



$$\begin{Bmatrix}0\\0\\N_{xy}\end{Bmatrix}=\begin{bmatrix}B_{11} & B_{12} & B_{1s}\\B_{12} & B_{22} & B_{2s}\\B_{1s} & B_{2s} & B_{ss}\end{bmatrix}\cdot\begin{Bmatrix}K_x\\K_y\\K_{xy}\end{Bmatrix}$$

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Chapter 4. Laminate and Sandwich Structures

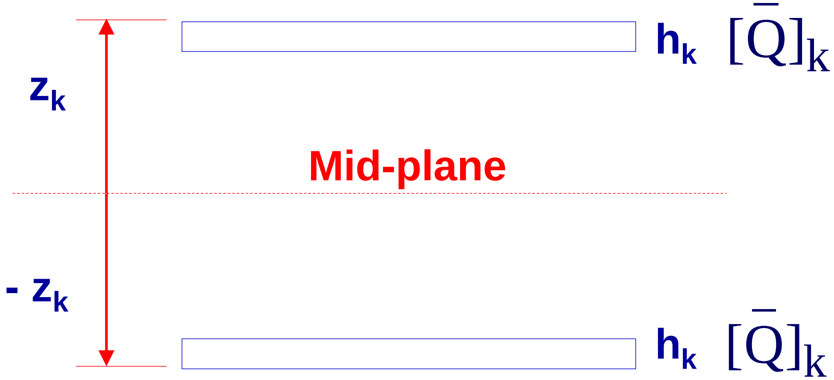
Laminate theory

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Laminate configurations

Symmetric laminate

The stacking sequence is symmetrical with respect to the laminate mid-plane



$$[B] = \frac{1}{2} \cdot \sum_{i}^{N} [\bar{Q}]_i \cdot \left(Z_i^2 - Z_{i-1}^2 \right)$$

⇓

$$[B] = [0]$$

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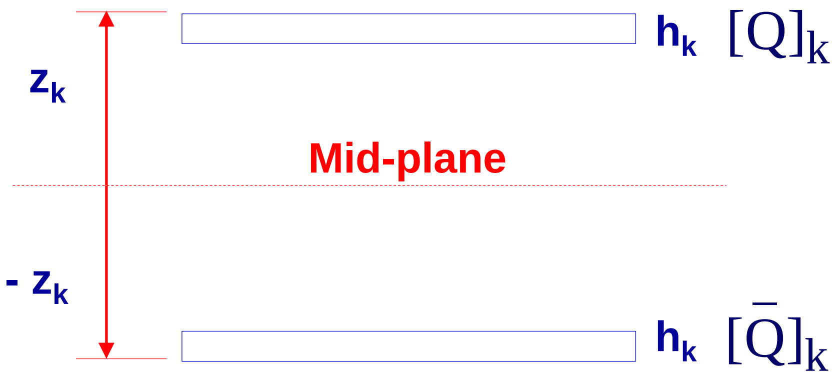
Laminate configurations

Symmetric laminate

The stacking sequence is symmetrical with respect to the laminate mid-plane



$$\{N\} = [A] \cdot \{\varepsilon^o\}$$

$$\{M\} = [D] \cdot \{k\}$$

$$[a] = [A]^{-1}$$

$$[d] = [D]^{-1}$$

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Laminate configurations

Equilibrated laminate

Same number of laminas oriented at +θ and a -θ

An equilibrated laminate can be symmetric, asymmetric or antisymmetric

Symmetric:	$[\pm\theta_1/ \pm\theta_2]_S$	(8 plies)
Antisymmetric:	$[\theta_1/ \theta_2/ -\theta_1/ -\theta_2]$	(4 plies)
Asymmetric:	$[\theta_1/ -\theta_2/ -\theta_1/ \theta_2]$	(4 plies)

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Laminate configurations

Orthotropic laminate

Orthotropic lamina axes correspond to laminate axis

In orthotropic laminates: $A_{xs} = A_{ys} = 0$

In orthotropic symmetric laminates: $D_{xs} = D_{ys} = 0$

Example:

$[0/90/90/0]_S$

Quasi-isotropic laminate

A laminate is quasi-isotropic if:

$$A_{xx} = A_{yy}$$

$$A_{xx} - A_{xy} = 2 \cdot A_{ss}$$

$$A_{xs} = A_{ys} = 0$$

Example:

$[0/90/+45/-45]_S$

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Chapter 4. Laminate and Sandwich Structures

Laminate theory

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Laminate equivalent constants

Membrane forces (symmetric laminates)

Average stresses in laminate

$$\sigma_x^o = \frac{N_x}{H}$$

$$\sigma_y^o = \frac{N_y}{H}$$

$$\tau_{xy}^o = \frac{N_{xy}}{H}$$

$$H = \sum_i^N h_i$$

$$\{N\} = [A] \cdot \{\varepsilon^o\}$$

$$\{\sigma^o\} = [A^*] \cdot \{\varepsilon^o\}$$

Normalized plain stiffness matrix $$[A^*] = \frac{[A]}{H}$$

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Laminate equivalent constants

Membrane forces (symmetric laminates)

$$\{\sigma^o\} = |A^*| \cdot \{\varepsilon^o\}$$

⇩

$$\{\varepsilon^o\} = |a^*| \cdot \{\sigma^o\}$$

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = \begin{bmatrix} a_{11}^* & a_{12}^* & a_{1s}^* \\ a_{21}^* & a_{22}^* & a_{2s}^* \\ a_{s1}^* & a_{s2}^* & a_{ss}^* \end{bmatrix} \cdot \begin{Bmatrix} \sigma_x^o \\ \sigma_y^o \\ \tau_{xy}^o \end{Bmatrix}$$

⇕

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_2 & 0 \\ -\nu_{21}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

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Laminate equivalent constants

Membrane forces (symmetric laminates)

Engineering plain constants

$$E_1^o = \frac{1}{\overset{*}{a}_{11}} \qquad \nu_{21}^o = -\frac{\overset{*}{a}_{21}}{\overset{*}{a}_{11}}$$

$$E_2^o = \frac{1}{\overset{*}{a}_{22}} \qquad \nu_{12}^o = -\frac{\overset{*}{a}_{12}}{\overset{*}{a}_{22}}$$

$$G_{12}^o = \frac{1}{\overset{*}{a}_{ss}}$$

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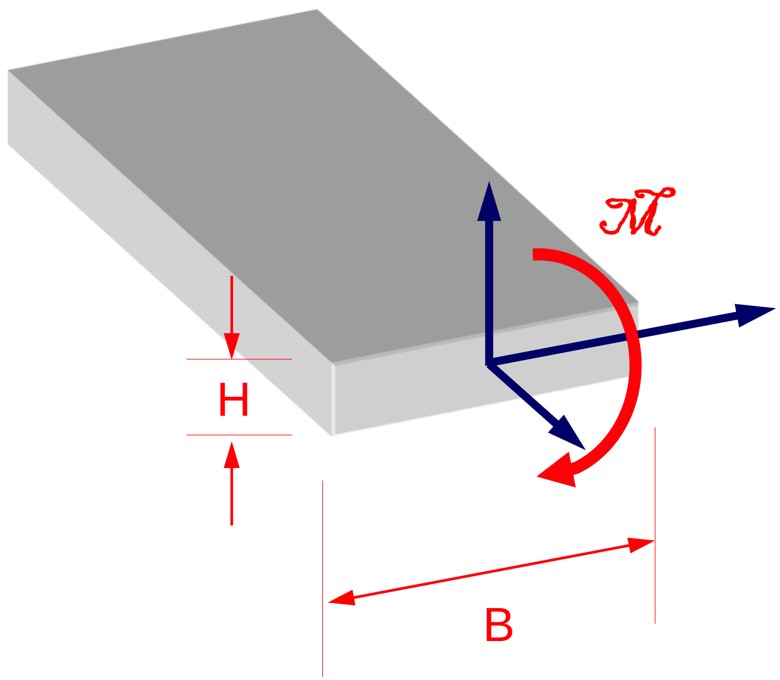
Laminate equivalent constants

Bending and torsion moments (symmetric laminates)

Isotropic material:

$$\sigma_x = \frac{\mathcal{M} \cdot z}{I} \qquad I = \frac{1}{12} \cdot B \cdot H^3$$



$$\varepsilon_x\Big|_{max.} = \frac{H}{2} \cdot k_x$$

$$\sigma_x\Big|_{max.} = \frac{6}{B \cdot H^2} \cdot \mathcal{M}$$

$$M_x = \frac{\mathcal{M}}{B}$$

$$\sigma_x\Big|_{max.} = \frac{6}{H^2} \cdot M_x$$

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Laminate equivalent constants

Bending and torsion moments (symmetric laminates)

Maximum (average) stresses in laminate

$$\sigma_x^f = \frac{6 \cdot M_x}{H^2}$$

$$\sigma_y^f = \frac{6 \cdot M_y}{H^2}$$

$$\tau_{xy}^f = \frac{6 \cdot M_{xy}}{H^2}$$

Maximum strains in laminate

$$\{\varepsilon\}\big|_{max.} = z_{max.} \cdot \{k\}\big|$$

$$\varepsilon_x^f = \frac{H \cdot k_x}{2}$$

$$\varepsilon_y^f = \frac{H \cdot k_y}{2}$$

$$\gamma_{xy}^f = \frac{H \cdot k_{xy}}{2}$$

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Laminate equivalent constants

Bending and torsion moments (symmetric laminates)

$$\{M\} = [D] \cdot \{k\}$$

$$\{\sigma^f\} = [D^*] \cdot \{\varepsilon^f\}$$

Normalized bending stiffness matrix $$[D^*] = \frac{12 \cdot [D]}{H^3}$$

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Laminate equivalent constants

Bending and torsion moments (symmetric laminates)

$$\{\sigma^f\} = |D^*| \cdot \{\varepsilon^f\}$$

⇩

$$\{\varepsilon^f\} = |d^*| \cdot \{\sigma^o\}$$

$$\begin{Bmatrix} \varepsilon_x^f \\ \varepsilon_y^f \\ \gamma_{xy}^f \end{Bmatrix} = \begin{bmatrix} d_{11}^* & d_{12}^* & d_{1s}^* \\ d_{21}^* & d_{22}^* & d_{2s}^* \\ d_{s1}^* & d_{s2}^* & d_{ss}^* \end{bmatrix} \cdot \begin{Bmatrix} \sigma_x^f \\ \sigma_y^f \\ \tau_{xy}^f \end{Bmatrix}$$

⇕

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_2 & 0 \\ -\nu_{21}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Laminate equivalent constants

Bending and torsion moments (symmetric laminates)

Engineering bending constants

$$E_1^f = \frac{1}{d_{11}^*} \qquad \nu_{21}^f = -\frac{d_{21}^*}{d_{11}^*}$$

$$E_2^f = \frac{1}{d_{22}^*} \qquad \nu_{12}^f = -\frac{d_{12}^*}{d_{22}^*}$$

$$G_{12}^f = \frac{1}{d_{ss}^*}$$

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Laminate equivalent constants

Asymmetric laminate

$$\{N\}=[A]\cdot\{\varepsilon^o\}+[B]\cdot\{k\}$$

$$\{M\}=[B]\cdot\{\varepsilon^o\}+[D]\cdot\{k\}$$

$$\{\sigma^o\}=[A^*]\cdot\{\varepsilon^o\}+[B^*]\cdot\{\varepsilon^f\}$$

$$\{\sigma^f\}=3\cdot[B^*]\cdot\{\varepsilon^o\}+[D^*]\cdot\{\varepsilon^f\}$$

$$\{\sigma^o\}=\frac{\{N\}}{H}$$

$$\{\sigma^f\}=\frac{6\cdot\{M\}}{H^2}$$

$$\{\varepsilon^f\}=\frac{H\cdot\{k\}}{2}$$

$$[A^*]=\frac{[A]}{H} \qquad [D^*]=\frac{12\cdot[D]}{H^3}$$

Normalized coupling stiffness matrix

$$[B^*]=\frac{2\cdot[B]}{H^2}$$

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Chapter 4. Laminate and Sandwich Structures

Laminate theory

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