Universidad Carlos III de Madrid
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Department of Continuum Mechanics and Structural Engineering

Aerospace Structures

Chapter 4. Plates and Shells

Shells of revolution

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

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1. Introduction
2. Membrane theory
3. Basic geometrical relations
4. Equilibrium equations for shells of revolution
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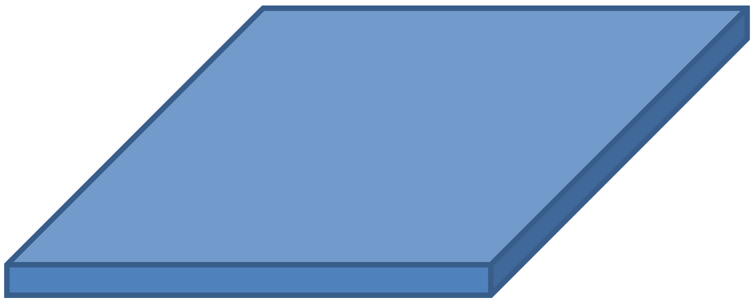
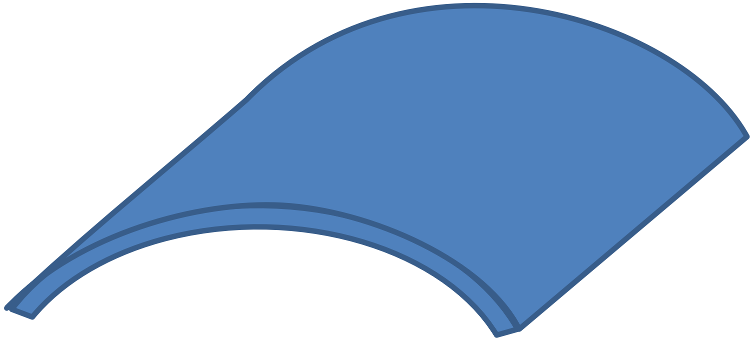
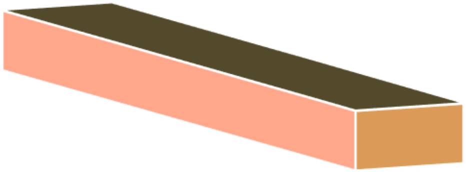
Differences between plates and shells?



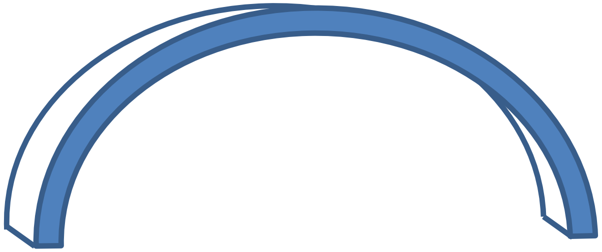
Plate



Shell



Beam



Arch

Differences between beams and arches?

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Introduction

General definition

A structural membrane or shell is a curved surface structure. Usually, it is capable of transmitting loads in more than two directions to supports. It is highly efficient structurally when it is so shaped, proportioned, and supported that it transmits the loads without bending or twisting.

A membrane or a shell is defined by its middle surface, halfway between its extrados, or outer surface and intrados, or inner surface. Thus, depending on the geometry of the middle surface, it might be a type of dome, barrel arch, cone, or hyperbolic paraboloid. Its thickness is the distance, normal to the middle surface, between extrados and intrados.

Advantages of shells structures

1. Efficiency of load-carrying behaviour
2. High degree of reserved strength and structural integrity
3. High specific strength (strength/ weight ratio)
4. Very high stiffness
5. Containment of space

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Introduction

Thin shells

A thin shell is a shell with a thickness relatively small compared with its other dimensions. But it should not be so thin that deformations would be large compared with the thickness.

Calculation of the stresses in a thin shell generally is carried out in two major steps, both usually involving the solution of differential equations. In the first, bending and torsion are neglected (membrane theory). In the second step, corrections are made to the previous solution by superimposing the bending and shear stresses that are necessary to satisfy boundary conditions (bending theory)

Examples of thin shells

- Civil and architectural engineering
  - ✓ Large span-roofs
  - ✓ Liquid-retaining structures and water tanks
  - ✓ Concrete arch domes
- Mechanical engineering
  - ✓ Piping systems
  - ✓ Turbine disks
  - ✓ Pressure vessels
- Biomechanics

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Kirchhoff-Love hypothesis

- Thickness is small comparing to the curvature radius of the mid-surface
- Small displacements and strains (equilibrium is verified in the undeformed shape)
- Straight lines normal to the mid-surface remain straight and normal after deformation
- Stresses in perpendicular direction to the mid-surface are neglected

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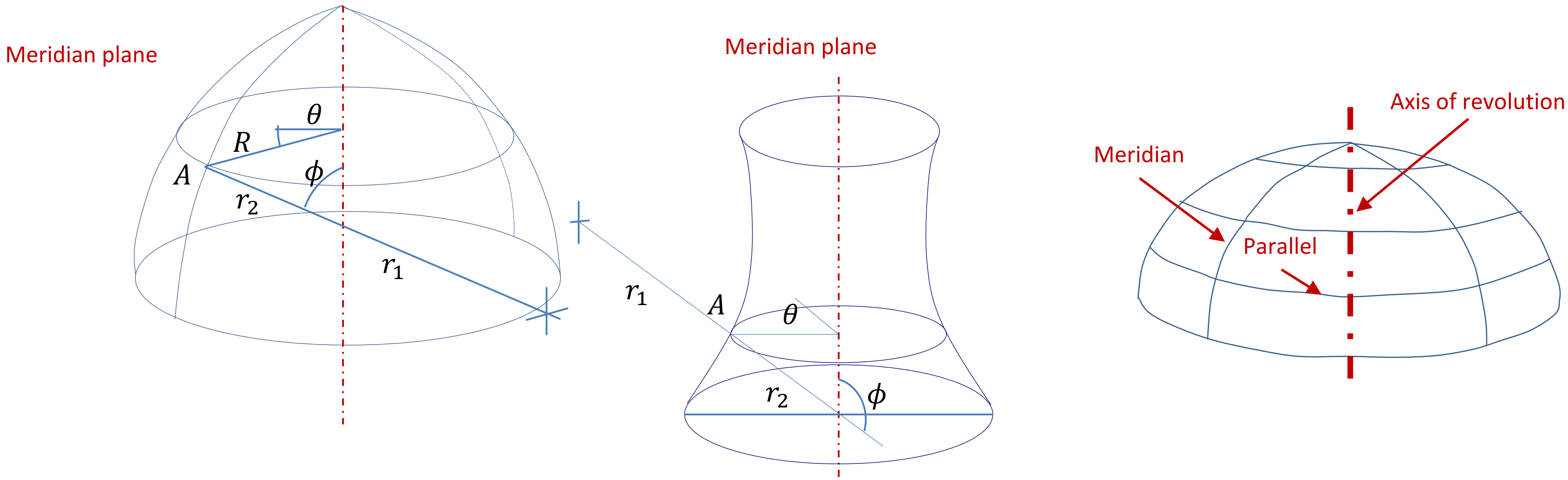
Geometrical relations



ϕ: meridian angle

r_1: first main radius. It is the curvature radius of the meridian

θ: circumferential angle

r_2: second main radius. It is the distance to the revolution axis in normal direction to the shell surface

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Shells of revolution

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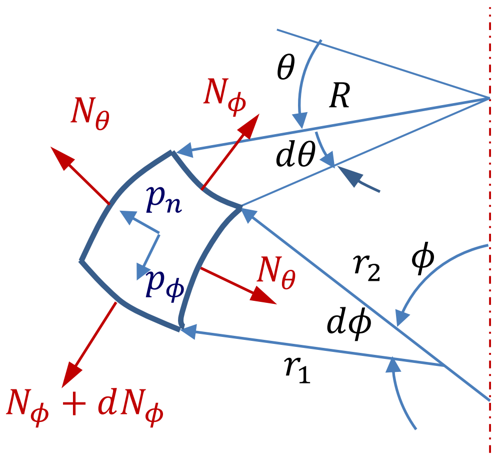
Equilibrium equations



- Bending and torsion moments are neglected (Membrane theory)
- Shear forces are neglected in mid-surface (Axial-symmetry)

Internal forces:

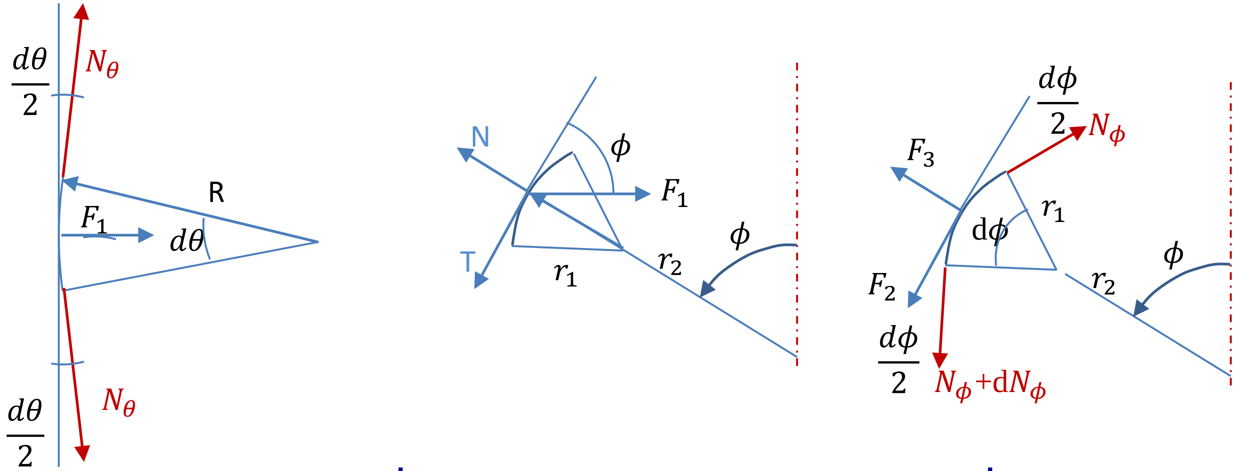
- N_ϕ: per unit length in meridian direction
- N_θ: per unit length in circumferential direction (independent on θ)

External loads per unit surface

- p_n: in normal direction
- p_ϕ: in meridian direction
- p_θ: does not exist (axial-symmetrical load)

Equilibrium equations

Equilibrium in meridian direction



$$F_1 = 2N_\theta r_1 d\phi sen\frac{d\theta}{2} \approx N_\theta r_1 d\phi d\theta$$

Tangential component

$$-F_1\cos\phi = -N_\theta r_1 \cos\phi d\phi d\theta$$

$$F_2 \approx N_\phi dR d\theta + R dN_\phi d\theta$$

$$F_2 \approx d\left(RN_\phi\right)d\theta$$

Tangential component of the external load:

$$p_\phi r_1 R d\phi d\theta$$

$$d\left(RN_\phi\right) - N_\theta r_1 \cos\phi d\phi + p_\phi r_1 R d\phi = 0$$

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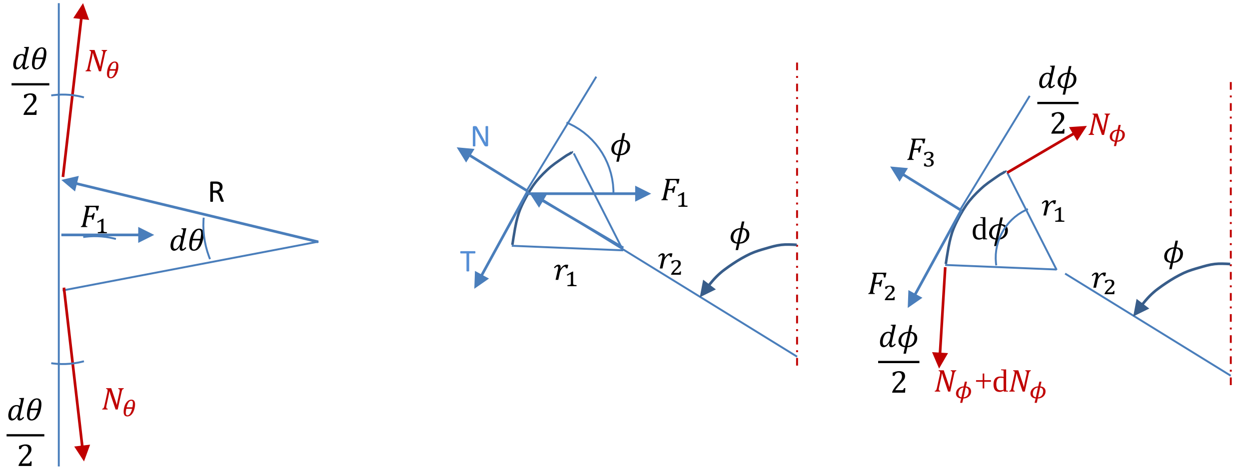
Equilibrium equations

Equilibrium in normal direction



$$F_1 \approx N_\theta r_1 d\phi d\theta$$

Normal component:

$$-F_1 sen\phi = -N_\theta r_1 sen\phi d\phi d\theta$$

Normal component of the external load:

$$p_n r_1 R d\phi d\theta$$

$$F_3 = -N_\phi R d\theta sen\frac{d\phi}{2} - \left(N_\phi + dN_\phi\right)\left(R + dr\right) d\theta sen\frac{d\phi}{2}$$

$$F_3 \approx -RN_\phi d\theta d\phi$$

$$\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} = p_n$$

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Equilibrium equations

$$d\left(RN_\phi\right) - N_\theta r_1 \cos\phi d\phi + p_\phi r_1 R d\phi = 0$$

$$\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} = p_n$$

$$d\left(RN_\phi\right) - r_1\left(r_2 p_n - \frac{r_2}{r_1} N_\phi\right)\cos\phi d\phi = -r_1 R p_\phi d\phi$$

Multiplying by $\mathrm{sen}\phi / d\phi$ and considering $R = r_2 \mathrm{sen}\phi$

$$\frac{d\left(r_2 \mathrm{sen}^2\phi N_\phi\right)}{d\phi} = r_1 r_2 p_n \cos\phi \,\mathrm{sen}\phi - r_1 r_2 \mathrm{sen}^2\phi\, p_\phi d\phi$$

Integrating over ϕ

$$N_\phi = \frac{\int r_1 r_2 \mathrm{sen}\phi\left(p_n \cos\phi - p_\phi \mathrm{sen}\phi\right) d\phi + k}{r_2 \mathrm{sen}^2\phi}$$

k is obtained from boundary conditions

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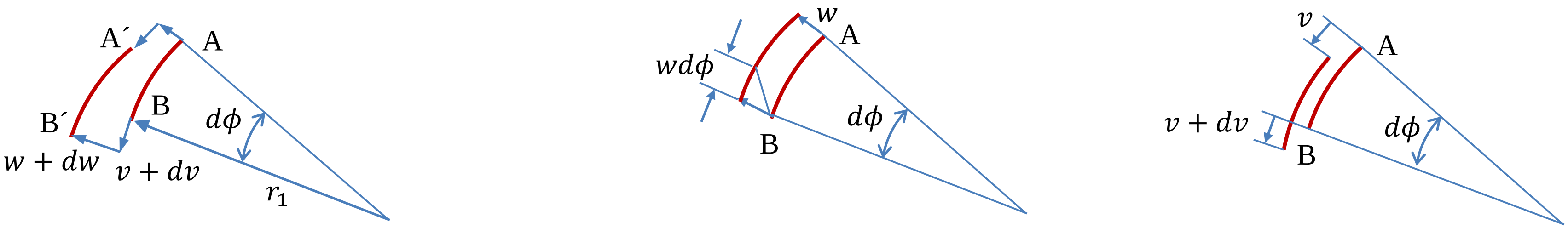
Displacement equations

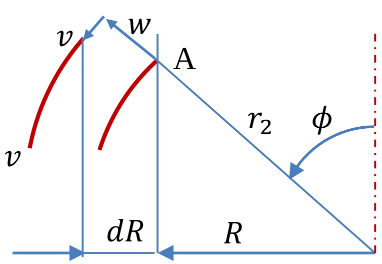
Displacements: v, tangent to meridian; w, in normal direction

Strains



$$\varepsilon_\phi = \frac{A'B' - AB}{AB} = \frac{dv + wd\phi}{r_1 d\phi} = \frac{1}{r_1}\left(\frac{dv}{d\phi} + w\right)$$

v



$$\varepsilon_\theta = \frac{2\pi(R + dR) - 2\pi R}{2\pi R} = \frac{dR}{R} \qquad dR = v\cos\phi + wsen\phi$$

$$\varepsilon_\theta = \frac{v\cos\phi + wsen\phi}{R} = \frac{1}{r_2}(v\cot\phi + w)$$ with $R = r_2 sen\phi$

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Displacement equations

Strains

$$\frac{dv}{d\phi}+w=r_1\varepsilon_\phi$$

$$v\cot\phi+w=r_2\varepsilon_\theta$$

$$\frac{dv}{d\phi}-v\cot\phi=r_1\varepsilon_\phi-r_2\varepsilon_\theta$$

Hook laws

$$\varepsilon_\phi==\frac{1}{E}\left(\sigma_\phi-\nu\sigma_\theta\right)=\frac{1}{Eh}\left(N_\phi-\nu N_\theta\right)$$

$$\varepsilon_\phi=\frac{1}{E}\left(\sigma_\theta-\nu\sigma_\phi\right)=\frac{1}{Eh}\left(N_\theta-\nu N_\phi\right)$$

$$\frac{dv}{d\phi}-v\cot\phi=\frac{1}{Eh}\left[\left(r_1+\nu r_2\right)N_\phi-\left(r_2+\nu r_1\right)N_\theta\right]$$

$$sen\phi\frac{d}{d\phi}\left(\frac{v}{sen\phi}\right)$$

$$v=sen\phi\left\{\frac{1}{Eh}\int\frac{1}{sen\phi}\left[\left(r_1+\nu r_2\right)N_\phi-\left(r_2+\nu r_1\right)N_\theta\right]d\phi+k\right\}$$

$$w=r_2\varepsilon_\theta-v\cot\phi=\frac{r_2}{Eh}\left(N_\theta-\nu N_\phi\right)-v\cot\phi$$

k is obtained from boundary conditions

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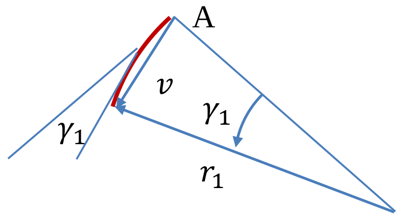
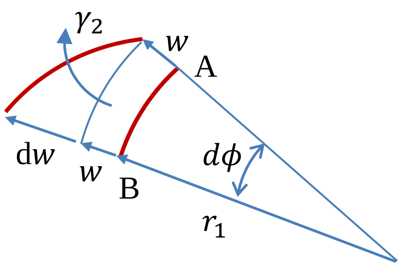
Displacement equations

Meridian rotation





γ_1: rotation motivated by the displacement v of point A

γ_2: rotation motivated by the difference between displacements w of points A and B

γ : meridian rotation

$$\gamma = \gamma_1 - \gamma_2 = \frac{v}{r_1} - \frac{dw}{r_1 d\phi}$$

δ : horizontal displacement

$$\delta = R\varepsilon_\theta = \frac{1}{Eh} r_2 sen\phi \left(N_\theta - \nu N_\phi \right)$$

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Displacement equations

Meridian rotation

$$\gamma = \gamma_1 - \gamma_2 = \frac{v}{r_1} - \frac{dw}{r_1 d\phi}$$

$$\frac{d}{d\phi}\left(v\cot\phi + w\right) = \frac{dv}{d\phi}\cot\phi - \frac{v}{sen^2\phi} = r_2\frac{d\varepsilon_\theta}{d\phi}$$

On the other hand

$$\left(\frac{dv}{d\phi} - v\cot\phi\right)\cot\phi = \frac{dv}{d\phi}\cot\phi - v\frac{\cos^2\phi}{sen^2\phi} = \frac{\cot\phi}{Eh}\left[\left(r_1 + \nu r_2\right)N_\phi - \left(r_2 + \nu r_1\right)N_\theta\right]$$

Subtracting the above equations and dividing by r_1

$$\gamma = \frac{1}{r_1}\left\{\frac{\cot\phi}{Eh}\left[\left(r_1 + \nu r_2\right)N_\phi - \left(r_2 + \nu r_1\right)N_\theta\right] - \frac{d}{d\phi}\left[\frac{1}{Eh}\left(N_\theta - \nu N_\phi\right)\right]\right\}$$

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