Bachelor in Aerospace Engineering

Universidad Carlos III de Madrid
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Department of Continuum Mechanics and Structural Engineering

Aerospace Structures

Chapter 2. Bending, shear and torsion of thin-walled beams

Bending and shear of open and closed, thin-walled beams (II)

CHAPTER 2. Bending, shear and torsion of thin-walled beams

Bending and shear of open and closed, thin-walled beams (II)

OBJETIVES

- Define and determine the shear centre
- Estimate the shear stress distribution by a alternate method to the proposed in section 2
- Calculate the shear stress distribution on multiple cell sections

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

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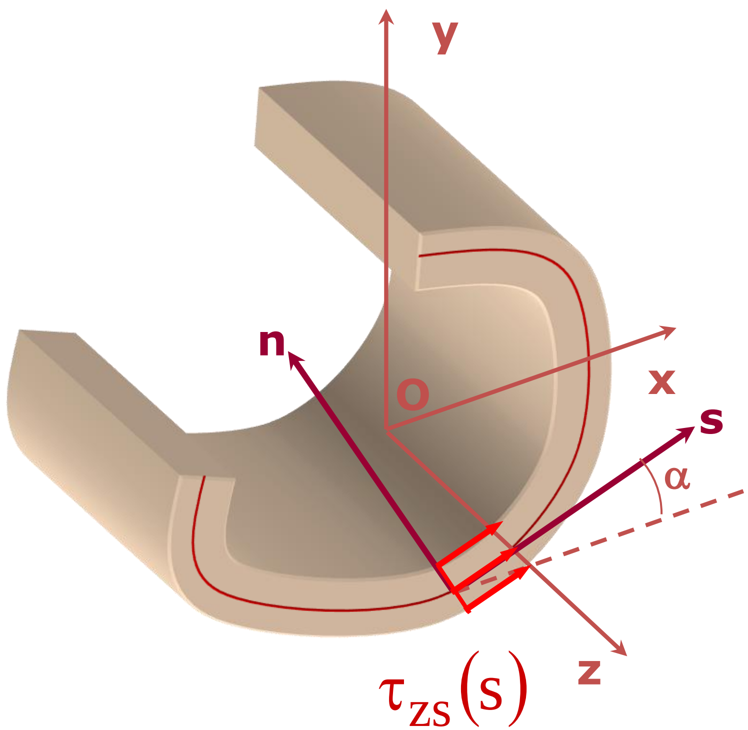
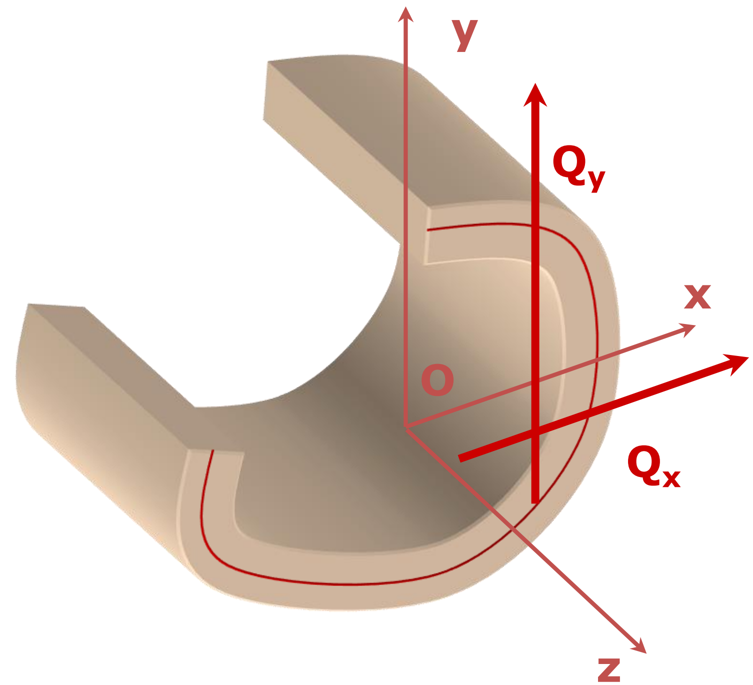
CHAPTER 2. Bending, shear and torsion of thin-walled beams

Bending and shear of open and closed, thin-walled beams (II)

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Concept of Shear centre



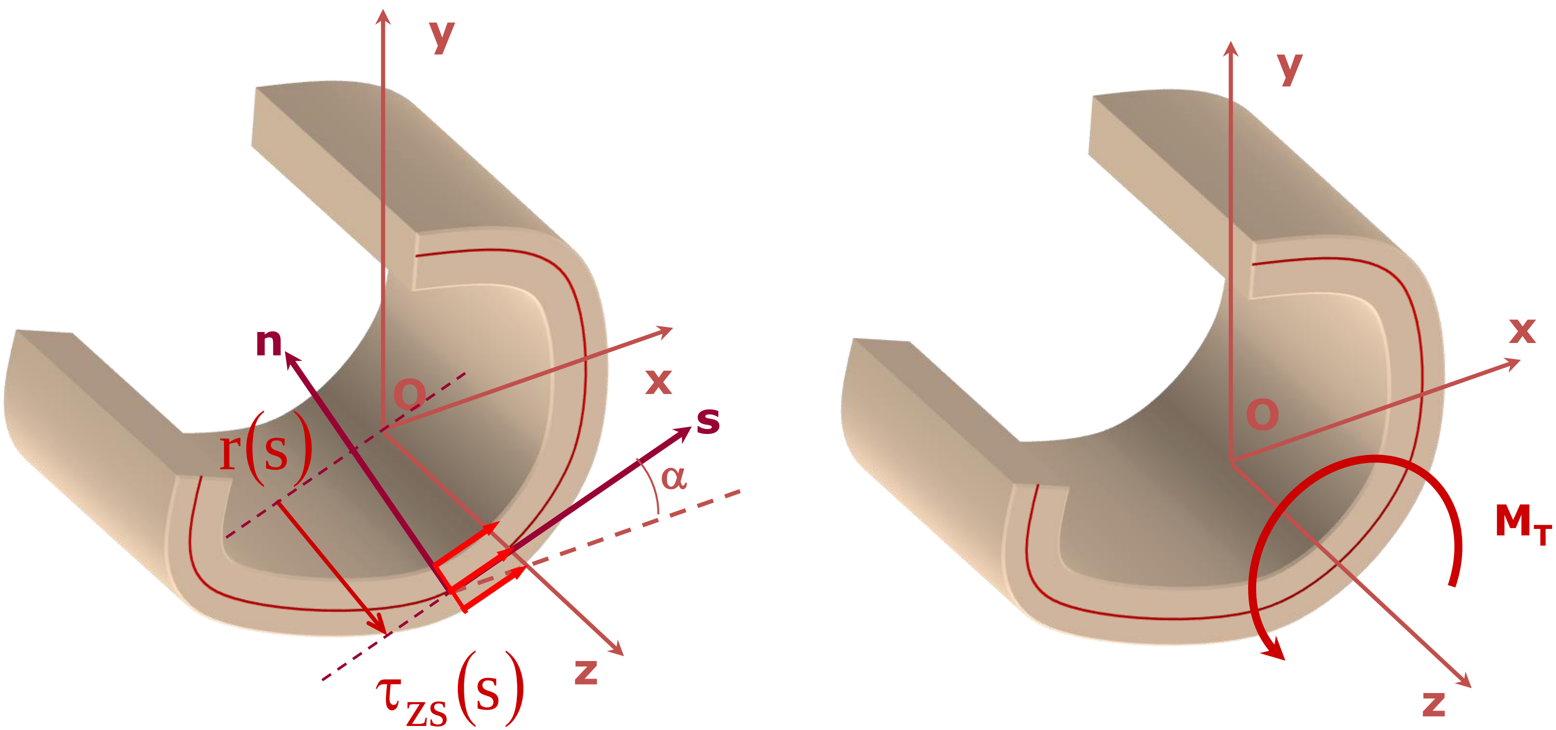


$$\int_0^{S_p} \tau_{zs}(s)\cdot\cos\alpha\cdot e(s)\cdot ds = Q_x$$

$$\int_0^{S_p} \tau_{zs}(s)\cdot \mathrm{sen}\alpha\cdot e(s)\cdot ds = Q_y$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Concept of Shear centre



$$\int_0^{S_p} \tau_{zs}(s)\cdot e(s)\cdot r(s)\cdot ds = M_T$$

A torsion moment is induced by transverse loads

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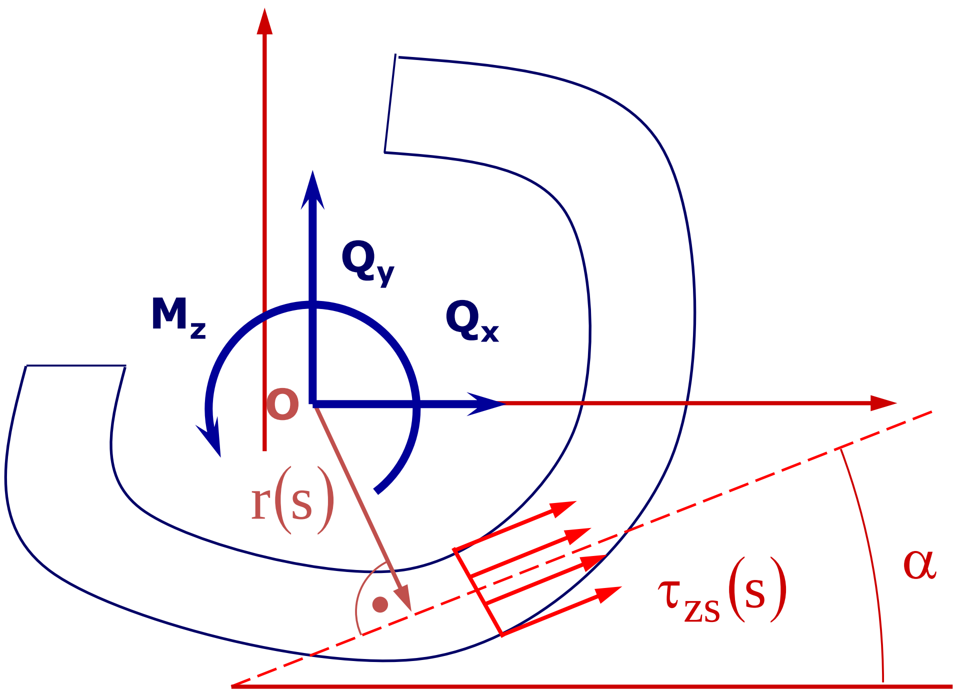
Concept of Shear centre



$$\tau_{zs}(s)$$

Shear force, Q_x

Shear force, Q_y

Torsion moment, M_z

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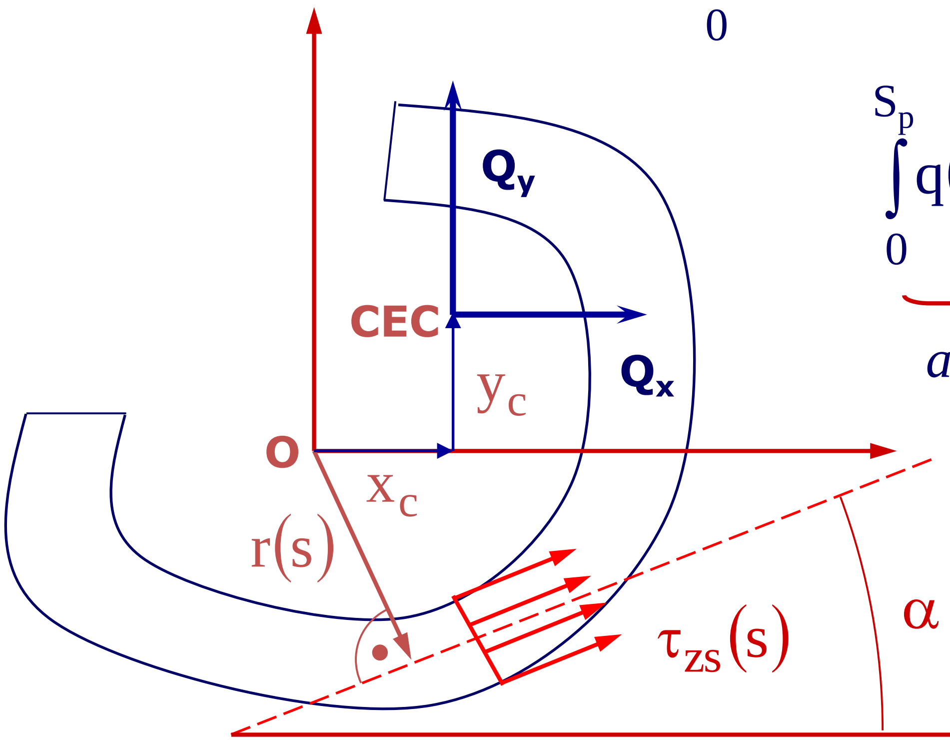
Concept of Shear centre

$$\int_0^{S_p} \tau_{zs}(s)\cdot e(s)\cdot r(s)\cdot ds = Q_y\cdot x_c - Q_x\cdot y_c$$

$$\underbrace{\int_0^{S_p} q(s)\cdot r(s)\cdot ds}_{a\cdot Q_y + b\cdot Q_x} = Q_y\cdot x_c - Q_x\cdot y_c$$



$$\boxed{\begin{aligned} x_c &= a \\ y_c &= -b \end{aligned}}$$

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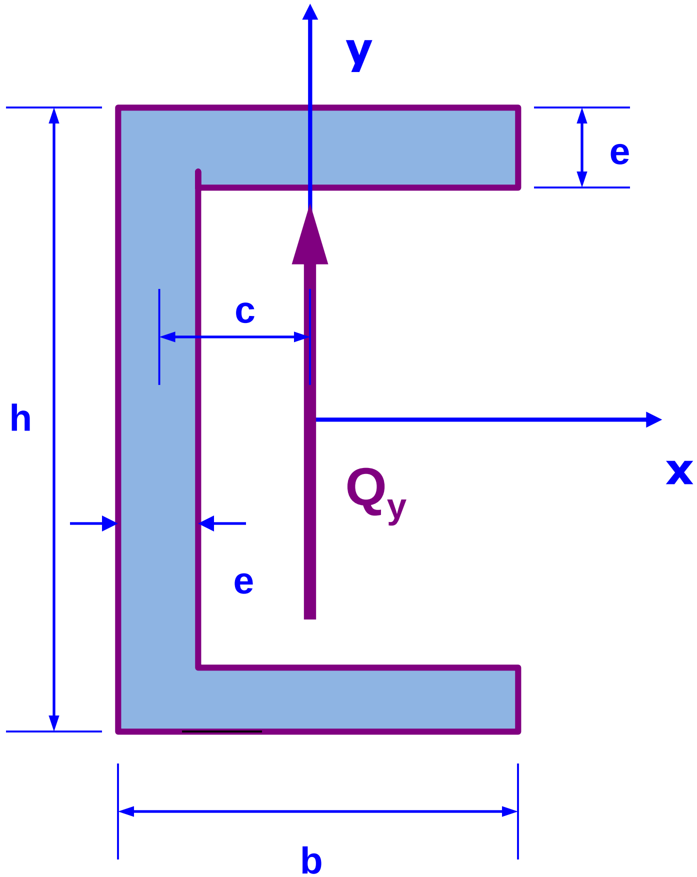
Concept of Shear centre

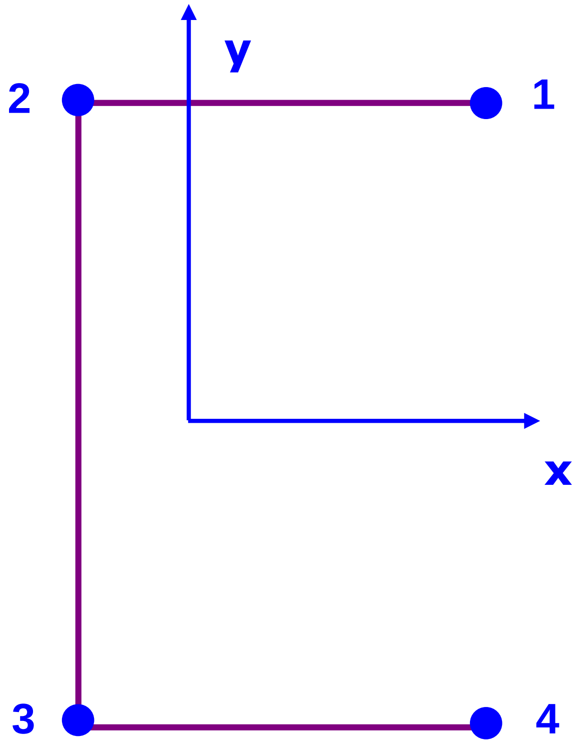
Example



$$e = 1 \ \ \text{cm}$$

$$b = 10 \ \ \text{cm}$$

$$h = 20 \ \ \text{cm}$$



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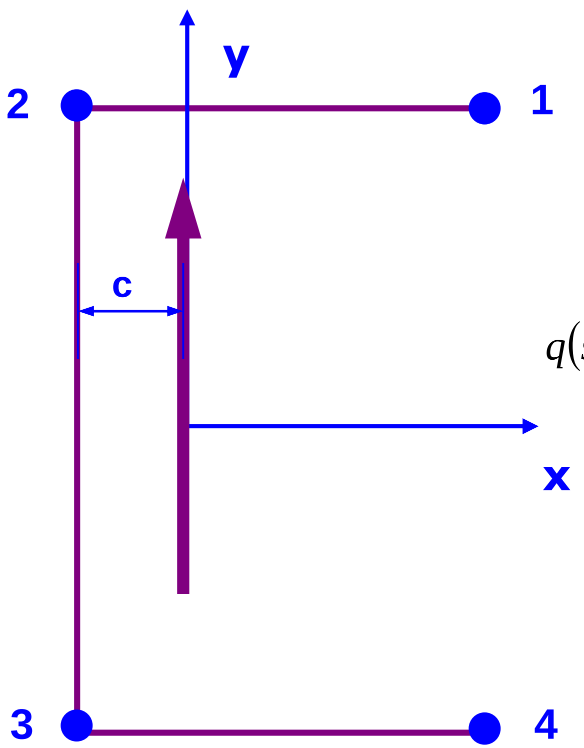
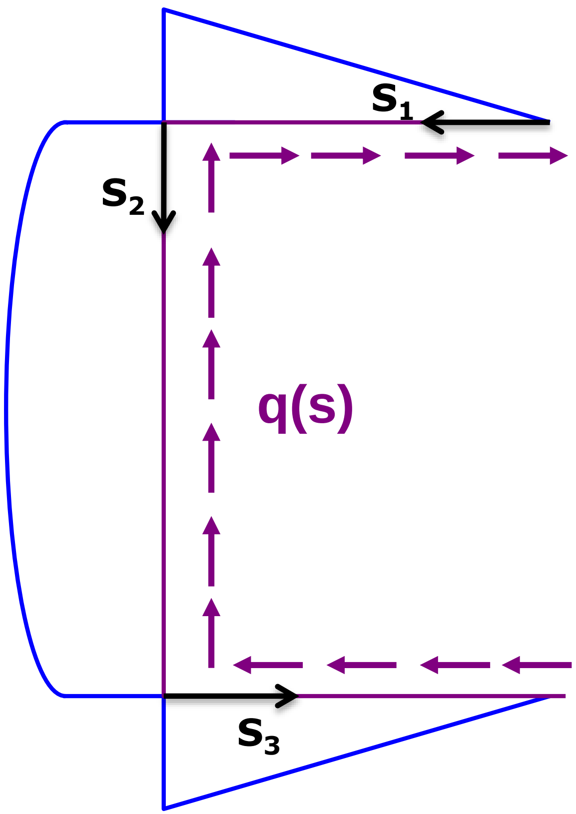
Example

$$q(s) = -\frac{Q_y \cdot e \cdot h}{2 \cdot I_x} s_1$$



$$q(s) = -\frac{Q_y \cdot e}{2 \cdot I_x}\left[h \cdot b + hs - s_2^2\right]$$



$$q(s) = -\frac{Q_y \cdot e \cdot h^2}{2 \cdot I_x}\left[b - s_3\right]$$

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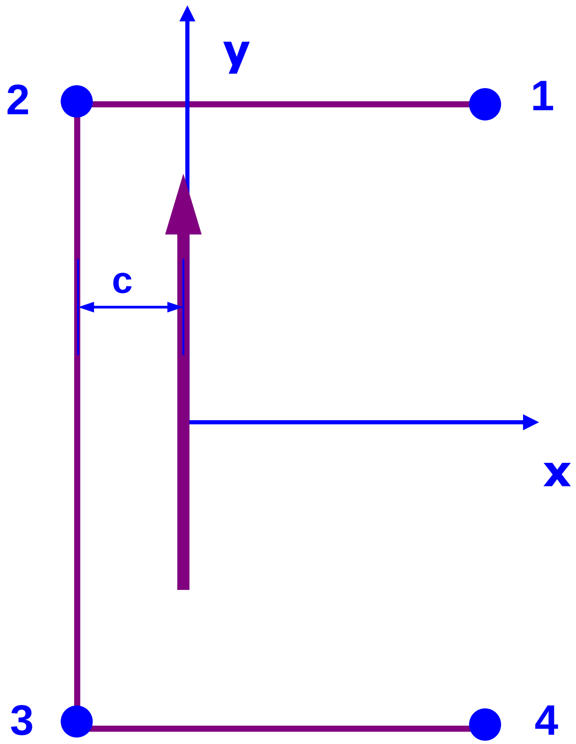
Concept of Shear centre

Example



$$\int_0^{S_p} q(s)\cdot r(s)\cdot ds = -Q_y \cdot c$$

$$\int_0^{S_p} q(s)\cdot r(s)\cdot ds = 2\cdot\int_0^{b} q(s_1)\cdot\frac{h}{2}\cdot ds_1 = \frac{Q_y\cdot e\cdot h^2\cdot b^2}{4\cdot I_x}$$

$$\frac{Q_y\cdot e\cdot h^2\cdot b^2}{4\cdot I_x} = -Q_y\cdot c$$

$$c = \frac{e\cdot h^2\cdot b^2}{4\cdot I_x} = -3.75cm$$

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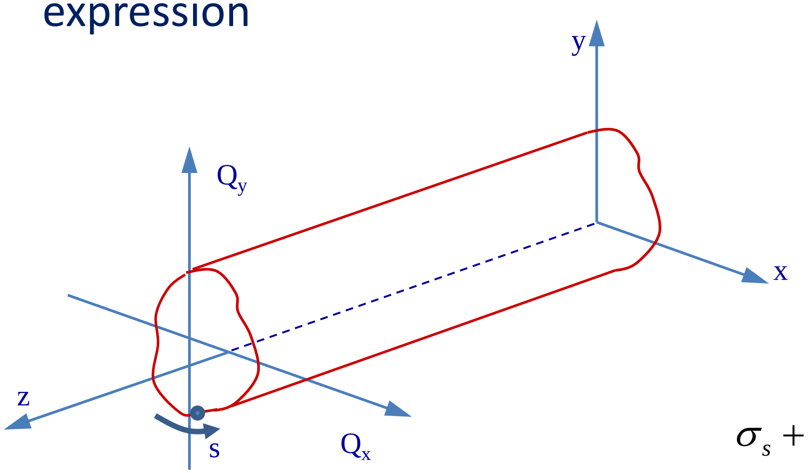
Alternative method to calculate Shear stresses in closed sections

The solution for a shear loaded closed section beam follows a similar pattern to that described in previous slides but with two important differences.

- First, the shear flow includes the effect of shear force and the induced torsional moment (if shear force is not applied at shear centre)
- Secondly, the shear loads may be applied through points in the cross-section other than the shear centre so that torsional as well as shear effects are included. This is possible since shear stresses produced by torsion in closed section beams have exactly the same form as shear stresses produced by shear, unlike shear stresses due to shear and torsion in open section beams.

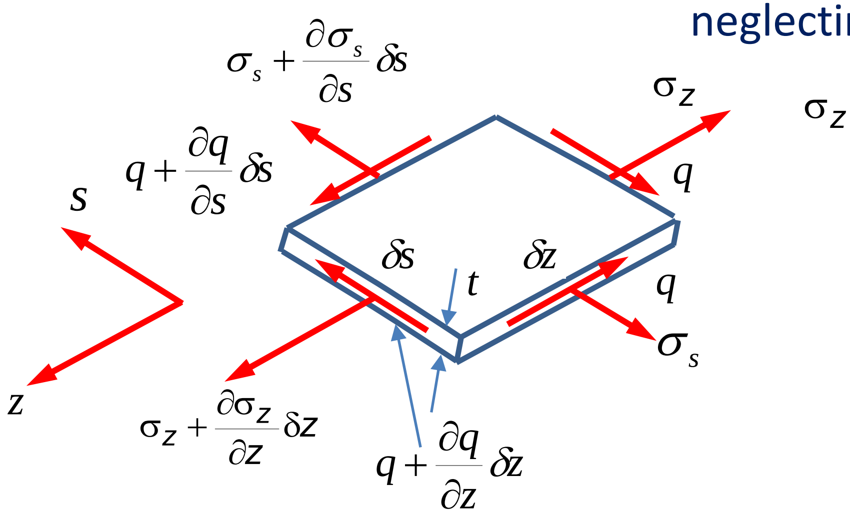
Alternative method to calculate Shear stresses in closed sections

The shear flow can be calculated including the torsional as well as shear effects in one expression



In a beam subjected to transversal loads, both shear flow and normal stress appear

The equilibrium of a differential element in the z direction, neglecting body forces, yields to:



$$\frac{\partial q}{\partial s} + t \cdot \frac{\partial \sigma_z}{\partial s} = 0$$

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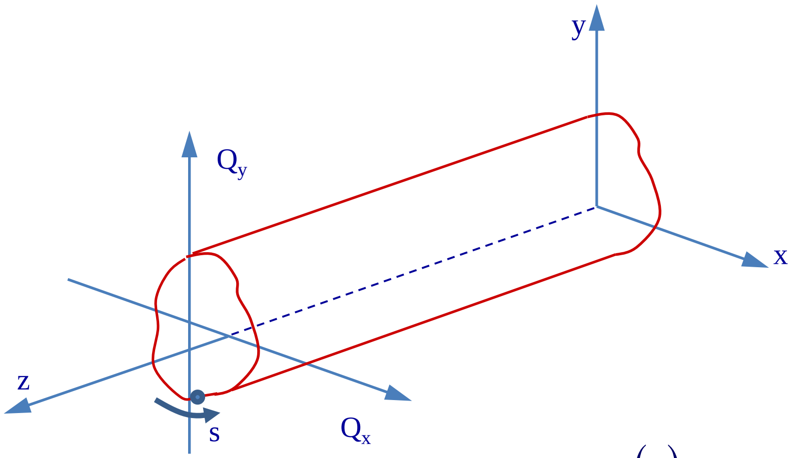
Alternative method to calculate Shear stresses in closed sections



$$\frac{\partial q}{\partial s}+t\cdot\frac{\partial \sigma_z}{\partial s}=0$$

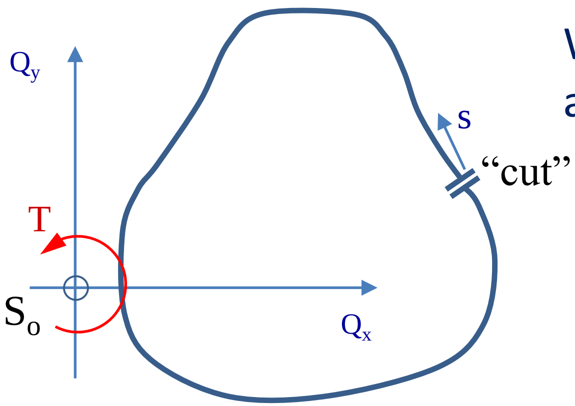
Integrating as before

$$q(s)-q_{s_0}=-\left(\frac{Q_x\cdot I_x-Q_y\cdot P_{xy}}{I_x\cdot I_y-P_{xy}^2}\right)\cdot\int_0^S x\cdot t\cdot ds-\left(\frac{Q_y\cdot I_y-Q_x\cdot P_{xy}}{I_x\cdot I_y-P_{xy}^2}\right)\cdot\int_0^S y\cdot t\cdot ds$$

$$q(s)=-\left(\frac{Q_x\cdot I_x-Q_y\cdot P_{xy}}{I_x\cdot I_y-P_{xy}^2}\right)\cdot\int_0^S x\cdot t(s)\cdot ds-\left(\frac{Q_y\cdot I_y-Q_x\cdot P_{xy}}{I_x\cdot I_y-P_{xy}^2}\right)\cdot\int_0^S y\cdot t(s)\cdot ds+q_{s_0}$$

$$q(s)=q_b(s)+q_{s_0}$$

Alternative method to calculate Shear stresses in closed sections



We obtain q_b by supposing that the closed beam section is cut at some convenient point thereby producing an open section

$$q_b = -\left(\frac{Q_x \cdot I_x - Q_y \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2}\right) \cdot \int_0^S x \cdot t(s) \cdot ds - \left(\frac{Q_y \cdot I_y - Q_x \cdot P_{xy}}{I_x \cdot I_y - P_{xy}^2}\right) \cdot \int_0^S y \cdot t(s) \cdot ds$$

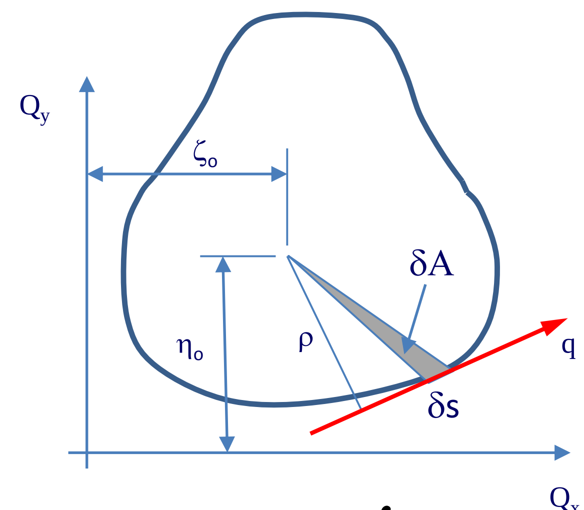
The value of shear flow at the cut (s = 0) is then found by equating applied and internal moments taken about some convenient moment centre

$$Q_x \cdot \eta_o - Q_y \cdot \xi_o = \oint q(s) \cdot p(s) \cdot ds$$

$$Q_x \cdot \eta_o - Q_y \cdot \xi_o = \oint q_b(s) \cdot p(s) \cdot ds + q_{s_o} \cdot \oint p(s) \cdot ds$$

$$\boxed{q_{s_o} = \frac{Q_x \cdot \eta_o - Q_y \cdot \xi_o - \oint q_b(s) \cdot p(s) \cdot ds}{2 \cdot \Omega}}$$

δ



$$2 \cdot \Omega = \oint p(s) \cdot ds$$

Ω is the area enclosed by the mid-line of the beam section wall

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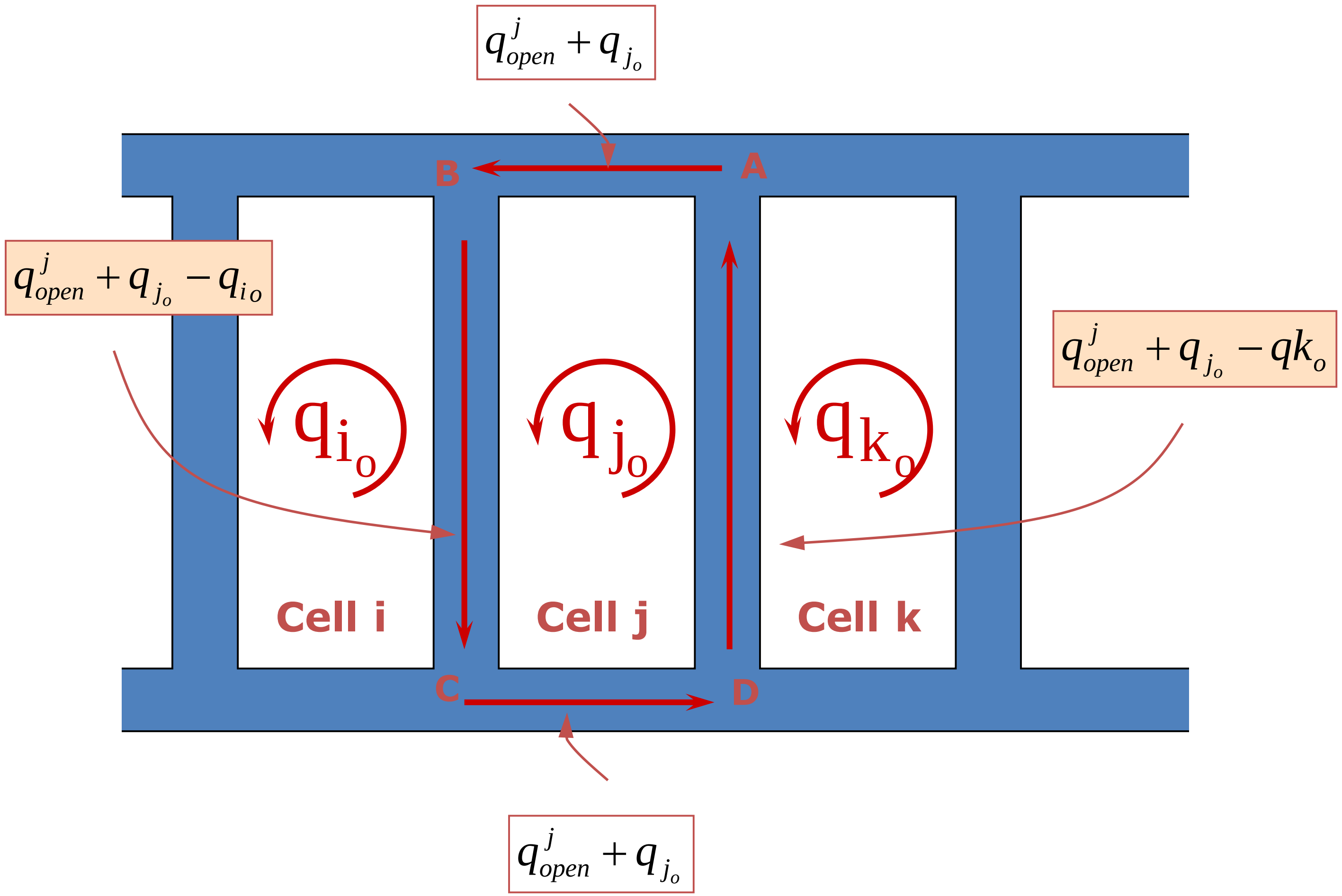
Multi-cell closed sections



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Multi-cell closed sections

Solution process for a section with n cells:

- The section in divided by n-cuts

$$q_{open}^{j}$$

- Each cell is solved as open

- Imposing in each cell:

$$\int_0^{S_p} \gamma_{zs} \cdot ds = 0$$

System of n-equations with n-unknowns

$$\int_A^B \frac{\left(q_{open}^{j} + q_{j_o}\right)}{G} \cdot ds + \int_B^C \frac{\left(q_{open}^{j} + q_{j_o} - q_{i_o}\right)}{G} \cdot ds +$$

$$\int_C^D \frac{\left(q_{open}^{j} + q_{j_o}\right)}{G} \cdot ds + \int_D^A \frac{\left(q_{open}^{j} + q_{j_o} - q_{k_o}\right)}{G} \cdot ds = 0$$

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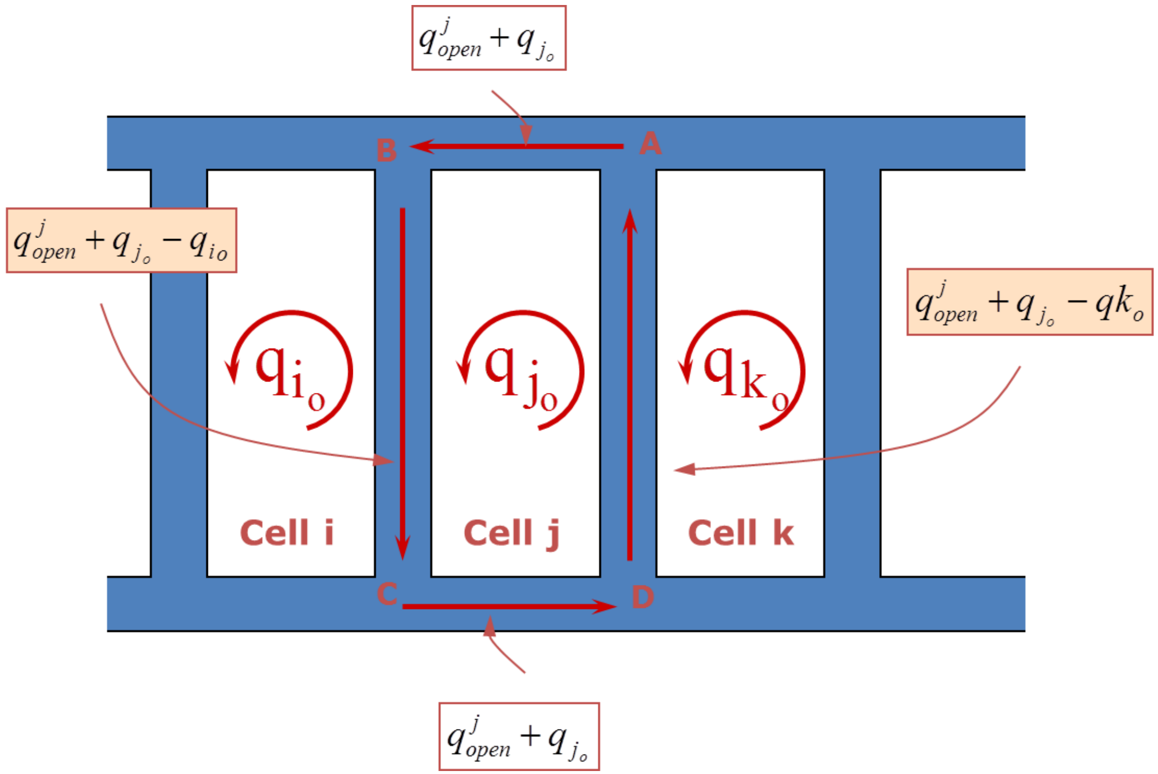
Multi-cell closed sections



$$\int_A^B \frac{\left(q_{open}^{j}+q_{j_o}\right)}{G}\cdot ds+\int_B^C \frac{\left(q_{open}^{j}+q_{j_o}-q_{i_o}\right)}{G}\cdot ds+$$

$$\int_C^D \frac{\left(q_{open}^{j}+q_{j_o}\right)}{G}\cdot ds+\int_D^A \frac{\left(q_{open}^{j}+q_{j_o}-q_{k_o}\right)}{G}\cdot ds=0$$

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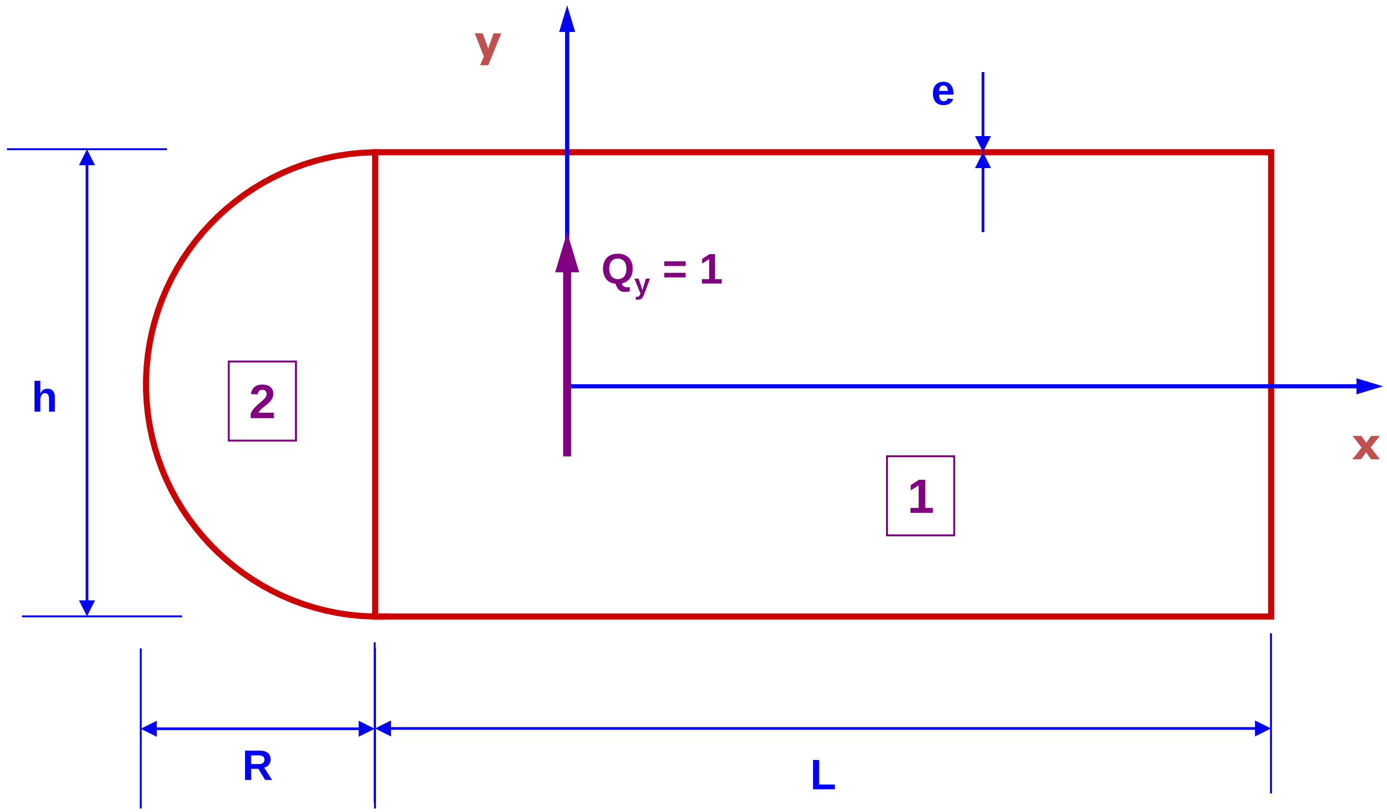
Multi-cell closed sections

Example



$$e = 0,5 \quad \text{cm}$$
$$L = 50 \quad \text{cm}$$
$$h = 12 \quad \text{cm}$$
$$R = 6 \quad \text{cm}$$

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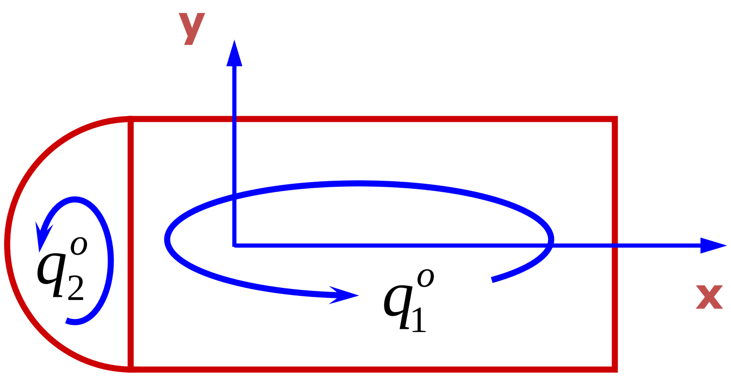
Multi-cell closed sections

Example



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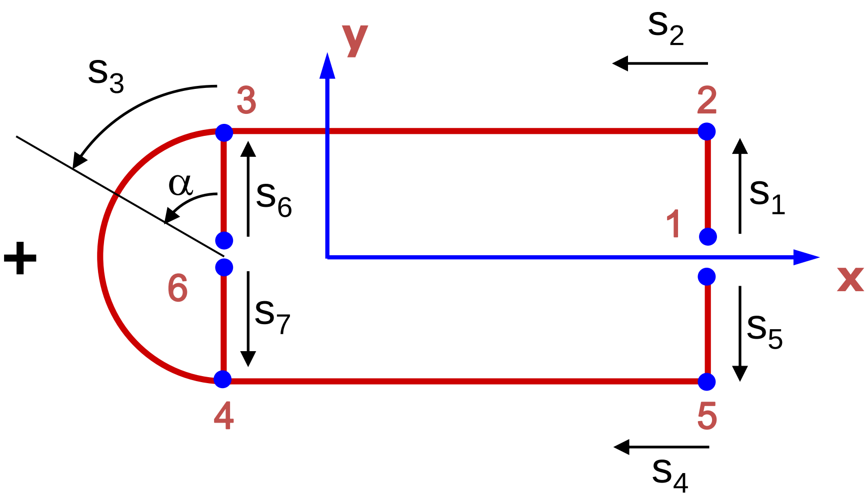
Cuts at points 1 and 6



$$m_{ex}^{*} = \int_{0}^{s} y(s) \cdot e(s) \cdot ds$$

$$q_a(s) = -\frac{m_{ex}^{*}(s)}{I_x} \cdot Q_y$$

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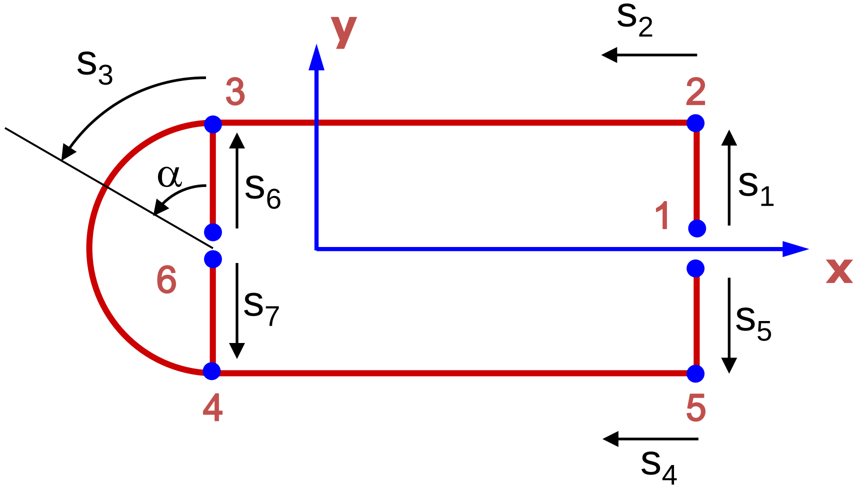
Example



Skin 12

$$m_{ex}^{*} = \frac{e \cdot s_1^2}{2}$$

$$q_a(s) = -\frac{e \cdot s_1^2}{2 \cdot I_x}$$

Skin 23

$$m_{ex}^{*} = \frac{e \cdot h^2}{8} + \frac{h \cdot e \cdot s_2}{2}$$

$$q_a(s) = -\frac{e \cdot h}{2 \cdot I_x} \cdot \left(s_2 + \frac{h}{4} \right)$$

Skin 63

$$m_{ex}^{*} = \frac{e \cdot s_6^2}{2}$$

$$q_a(s) = -\frac{e \cdot s_6^2}{2 \cdot I_x}$$

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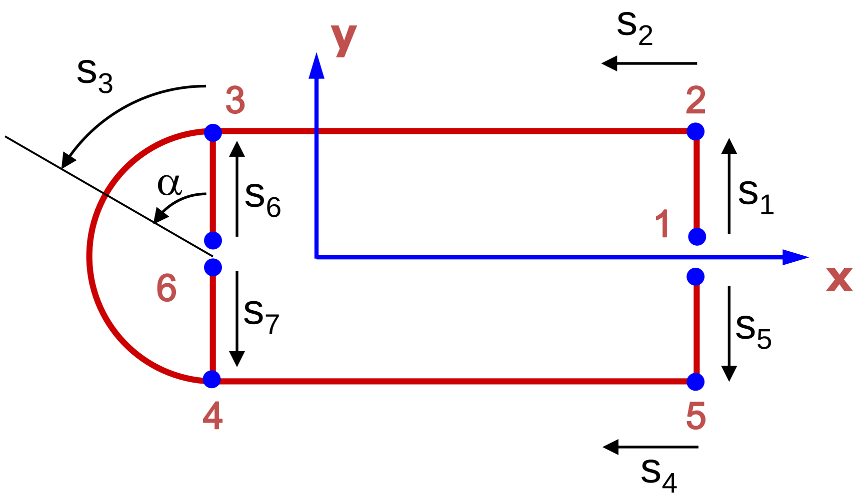
Example



Skin 15

$$m_{ex}^{*} = -\frac{e \cdot s_5^2}{2}$$

$$q_a(s) = \frac{e \cdot s_5^2}{2 \cdot I_x}$$

Skin 54

$$m_{ex}^{*} = -\left(\frac{e \cdot h^2}{8} + \frac{h \cdot e \cdot s_4}{2}\right)$$

$$q_a(s) = \frac{e \cdot h}{2 \cdot I_x} \cdot \left(s_4 + \frac{h}{4}\right)$$

Skin 64

$$m_{ex}^{*} = -\frac{e \cdot s_7^2}{2}$$

$$q_a(s) = \frac{e \cdot s_7^2}{2 \cdot I_x}$$

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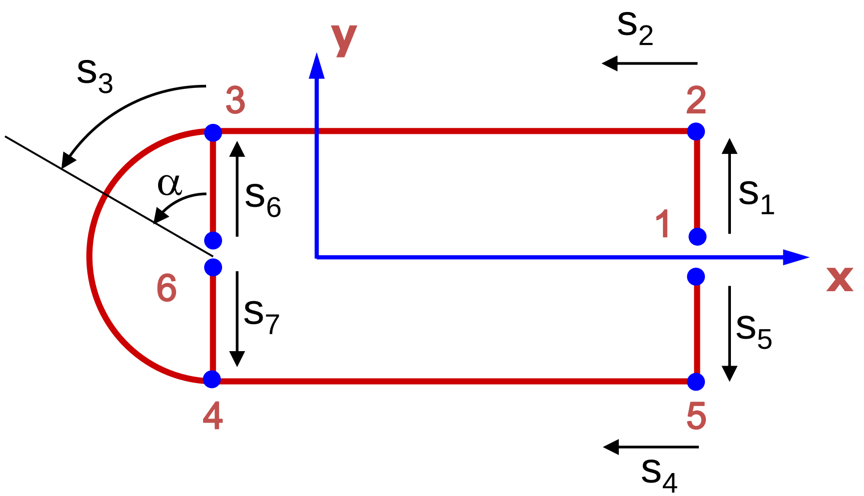
Multi-cell closed sections

Example



Skin 34

$$m_{ex}^{*} = \frac{e \cdot h^2}{4} + \frac{h \cdot e \cdot L}{2} + \int_0^{\alpha} R \cdot \cos\alpha \cdot e \cdot R \cdot d\alpha = \frac{e \cdot h^2}{4} + \frac{h \cdot e \cdot L}{2} + R^2 \cdot e \cdot sen\alpha$$

$$q_a(\alpha) = -\frac{1}{I_x} \cdot \left(\frac{e \cdot h^2}{4} + \frac{h \cdot e \cdot L}{2} + R^2 \cdot e \cdot sen\alpha \right)$$

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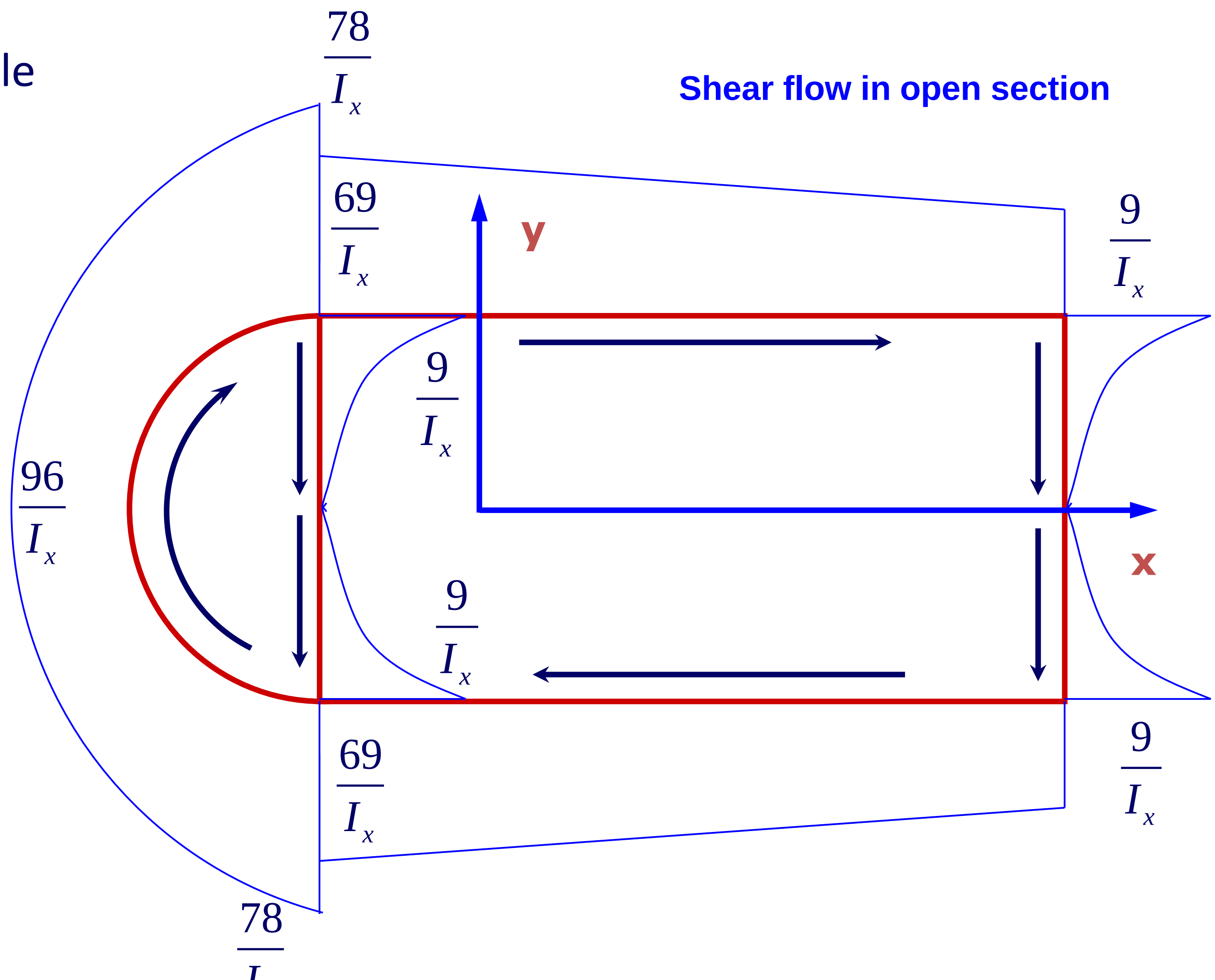
Multi-cell closed sections

Example



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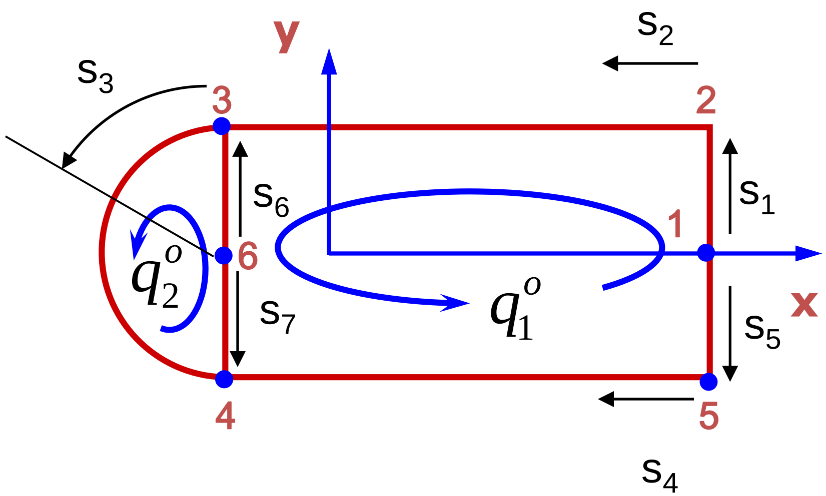
Multi-cell closed sections

Example



Skin 12 $q(s) = q_a^{12}(s_1) + q_1^o$

Skin 23 $q(s) = q_a^{23}(s_2) + q_1^o$

Skin 63 $q(s) = q_a^{63}(s_6) + q_2^o - q_1^o$

Skin 64 $q(s) = q_a^{64}(s_7) + q_1^o - q_2^o$

Skin 15 $q(s) = q_a^{15}(s_5) - q_1^o$

Skin 34 $q(s) = q_a^{34}(\alpha) + q_2^o$

Skin 54 $q(s) = q_a^{54}(s_4) - q_1^o$

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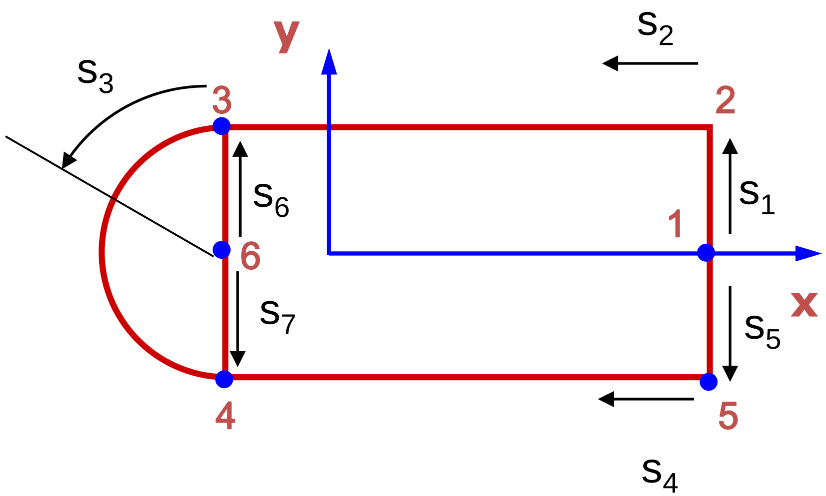
Multi-cell closed sections

Example



In cell 1

$$\int_0^{h/2}\left(q_a^{12}(s_1)+q_1^o\right)\cdot ds_1+\int_0^{L}\left(q_a^{23}(s_2)+q_1^o\right)\cdot ds_2+$$

$$-\int_0^{h/2}\left(q_a^{63}(s_6)+q_2^o-q_1^o\right)\cdot ds_6+\int_0^{h/2}\left(q_a^{64}(s_7)-q_2^o+q_1^o\right)\cdot ds_7+$$

$$-\int_0^{L}\left(q_a^{45}(s_4)-q_1^o\right)\cdot ds_4-\int_0^{L}\left(q_a^{51}(s_5)-q_1^o\right)\cdot ds_5$$

Imposing in each cell:

$$\int_0^{S_p}\frac{q(s)}{e(s)}\cdot ds=0$$

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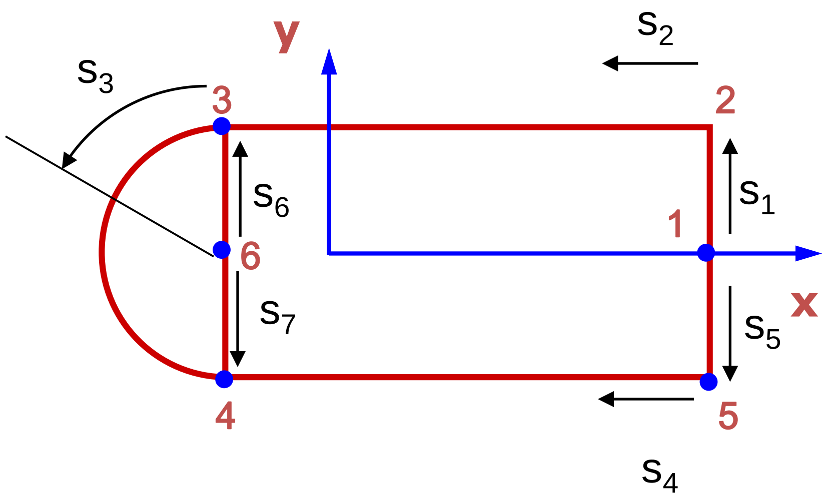
Multi-cell closed sections

Example



In cell 2

$$\int_0^{\pi}\left(q_a^{34}(\alpha)+q_2^o\right)\cdot R\cdot d\alpha-\int_0^{h/2}\left(q_a^{64}(s_7)+q_1^o-q_2^o\right)\cdot ds_2+$$

$$+\int_0^{h/2}\left(q_a^{63}(s_6)-q_1^o+q_2^o\right)\cdot ds_6=0$$

Imposing in each cell:

$$\int_0^{S_p}\frac{q(s)}{e(s)}\cdot ds=0$$

$$q_1^o=\frac{37,58}{I_x}$$

$$q_2^o=\frac{77,44}{I_x}$$

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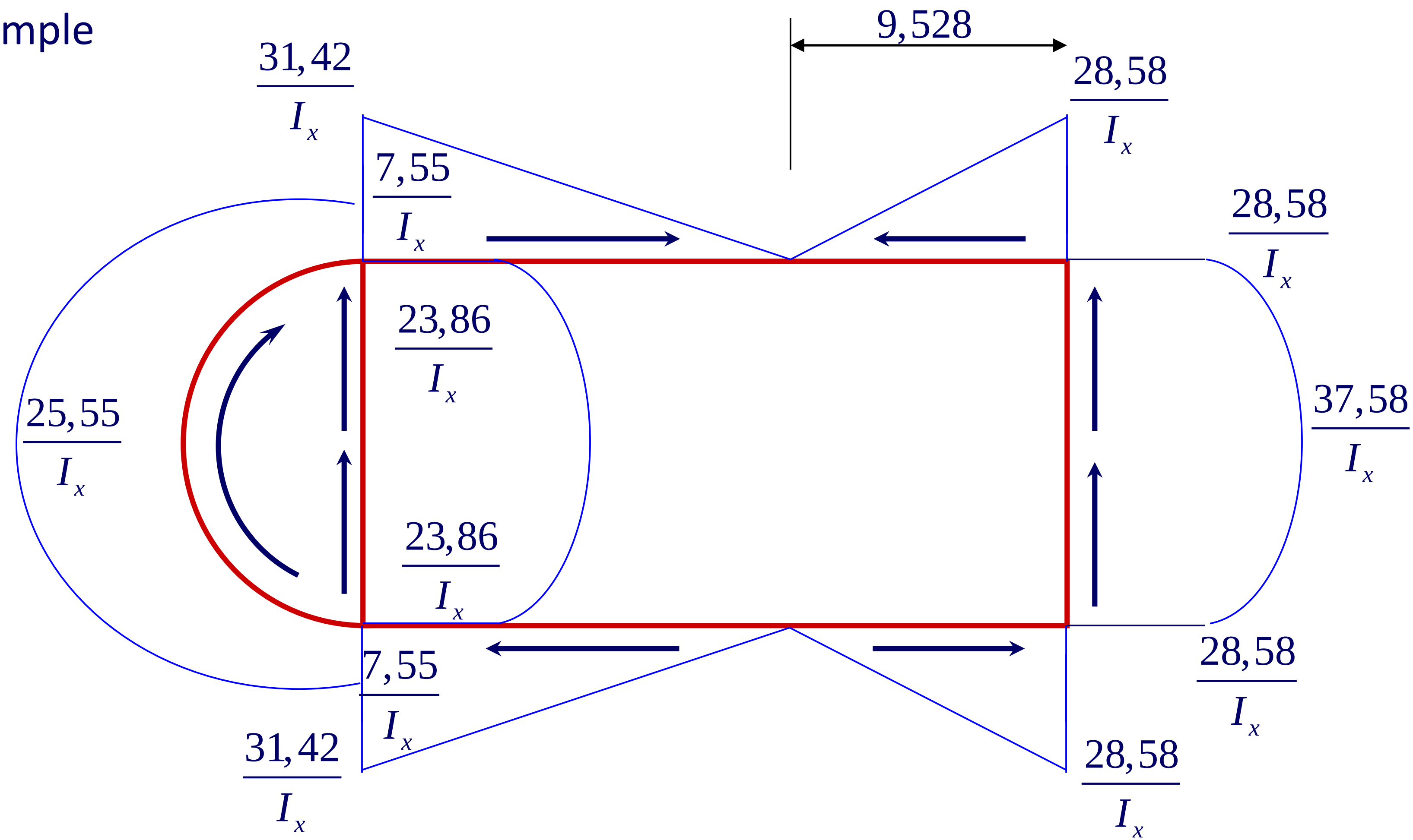
Multi-cell closed sections

Example



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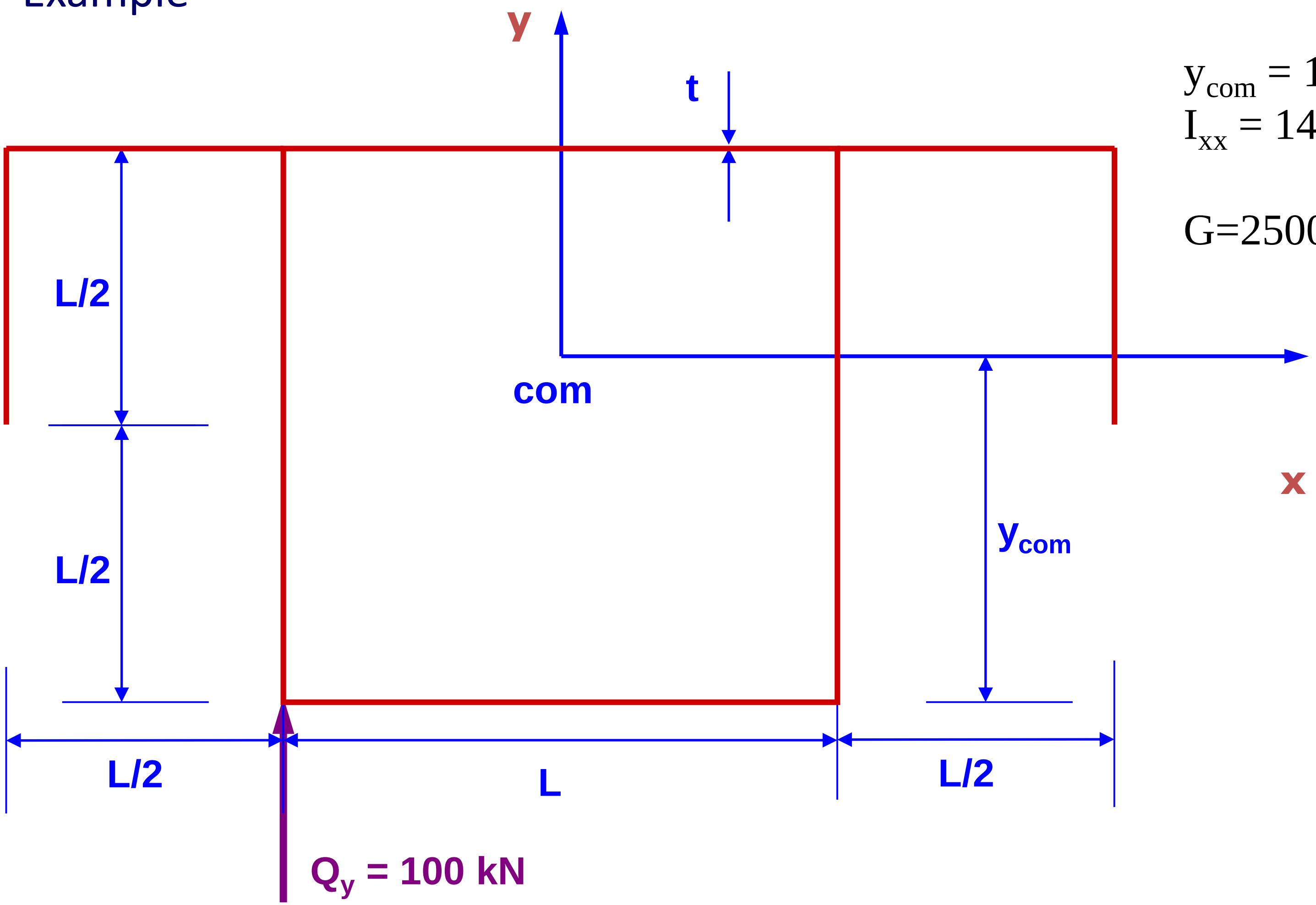
Combined open and closed sections

Example

$t = 2$ mm
$L = 200$ mm

$y_{com} = 125$ mm
$I_{xx} = 14.5 \cdot 10^6$ mm^4

$G = 25000$ N/mm^2



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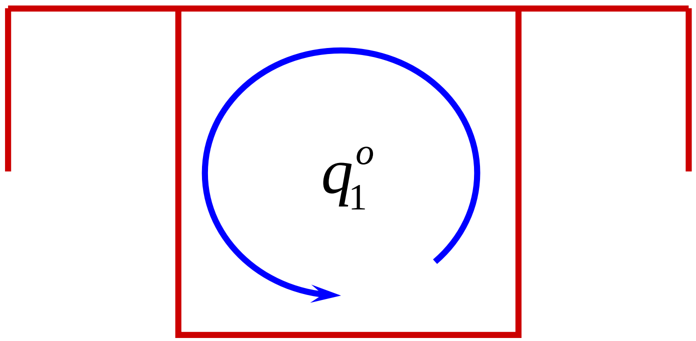
Combined open and closed sections

Shear force Q_y



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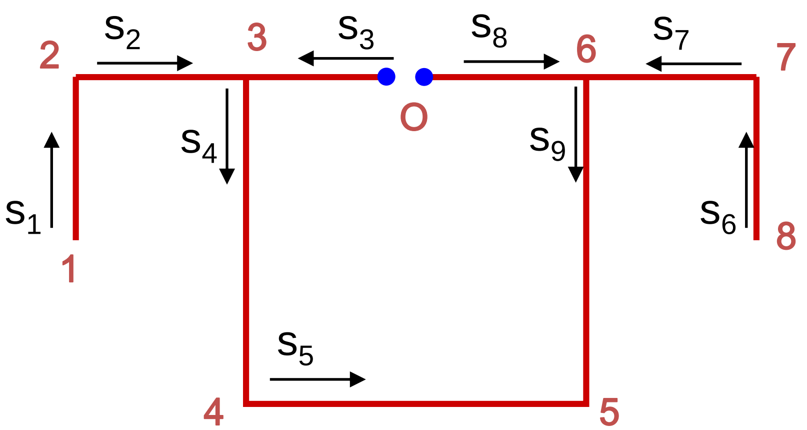
Cuts at point O

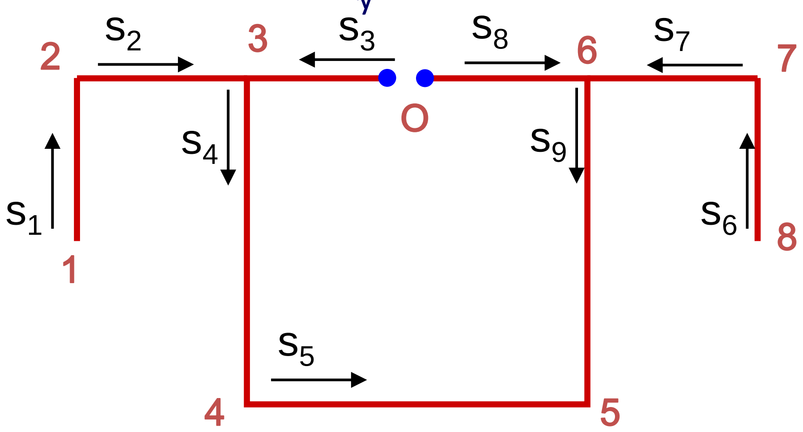


$$m_{ex}^{*} = \int_{0}^{s} y(s) \cdot e(s) \cdot ds$$

$$q_a(s) = -\frac{m_{ex}^{*}(s)}{I_x} \cdot Q_y$$

Combined open and closed sections

Shear force Q_y



Skin 23

$$q_{23}(s_2) = -\frac{Q_y}{I_{xx}}\int_0^{s_2} 2\cdot 75\cdot ds_2 - 34.5 = -34.5 - 1.04\cdot s_2$$

$$q_{23}\left(\frac{L}{2}\right) = q_{3,23} = -138.5N/mm$$

$$q_{67}(s_7) = q_{23}(s_2)$$

$$q_{6,67} = q_{3,23}$$

Skin 12

$$q_{12}(s_1) = -\frac{Q_y}{I_{xx}}\int_0^{s_1} 2(-25+s_1)ds_1 = -69\cdot 10^{-4}\left(s_1^2 - 50s_1\right)$$

$$q_{12}\left(\frac{L}{2}\right) = q_2 = -34.5N/mm$$

$$q_{78}(s_6) = q_{12}(s_1)$$

$$q_7 = q_2$$

Skin O3

$$q_{O3}(s_3) = -\frac{Q_y}{I_{xx}}\int_0^{s_3} 2\cdot 75\cdot ds_3 = -1.04\cdot s_3$$

$$q_{O3}\left(\frac{L}{2}\right) = q_{3,O3} = -104N/mm$$

$$q_{O6}(s_8) = q_{O3}(s_3)$$

$$q_{6,O6} = q_{3,O3}$$

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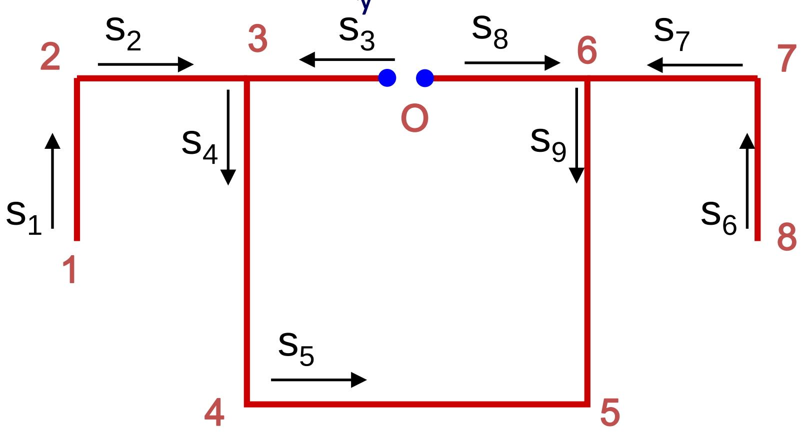
Combined open and closed sections

Shear force Q_y



Skin 45

$$q_{45}(s_5) = -\frac{Q_y}{I_{xx}}\int_0^{s_5} 2\cdot(-125)\cdot ds_5 - 172.5 = -172.5 - 1.725\cdot s_5$$

$$q_{45}\left(\frac{L}{2}\right) = 0$$

$$q_{45}(L) = q_5 = 172.5N/mm$$

Skin 34

$$q_{34}(0) = q_{3,34} = q_{3,23} + q_{3,O3} = -242.5N/mm$$

$$q_{34}(s_4) = -\frac{Q_y}{I_{xx}}\int_0^{s_4} 2(75 - s_4)ds_4 - 242.5 = -242.5 - 1.04s_4 + 69\cdot 10^{-4}s_4^2$$

$$q_{34}(L) = q_4 = -172.5N/mm$$

$$q_{34,\max} = q_{34}(75mm) = -281.7N/mm$$

$$q_{56}(s_9) = q_{34}(s_4)$$

$$q_5 = q_4$$

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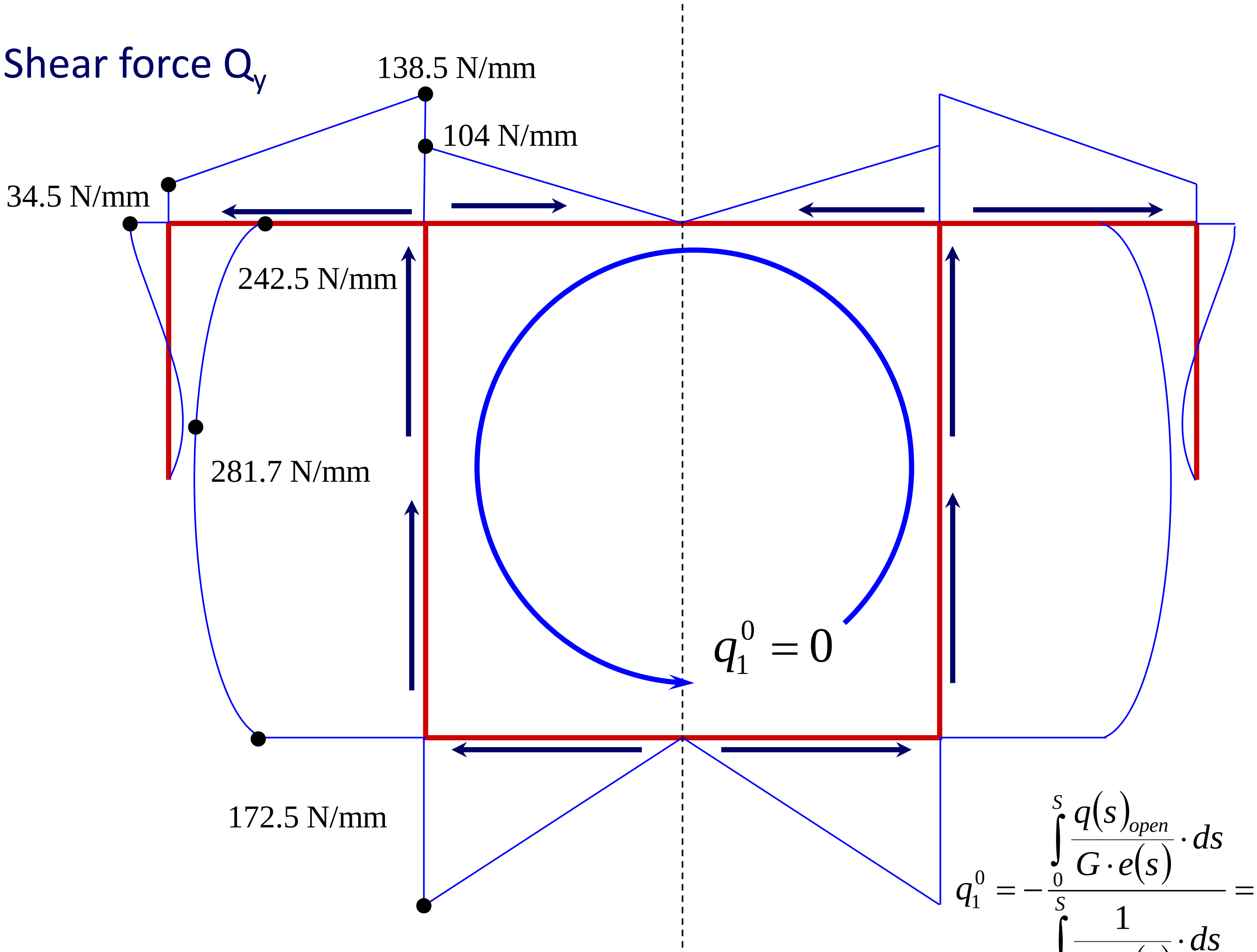
Combined open and closed sections

Shear force Q_y



$$q_1^0 = -\frac{\int_0^S \frac{q(s)_{open}}{G \cdot e(s)} \cdot ds}{\int_0^S \frac{1}{G \cdot e(s)} \cdot ds} = -\frac{\int_0^S q(s)_{open} \cdot ds}{\int_0^S ds} = 0$$

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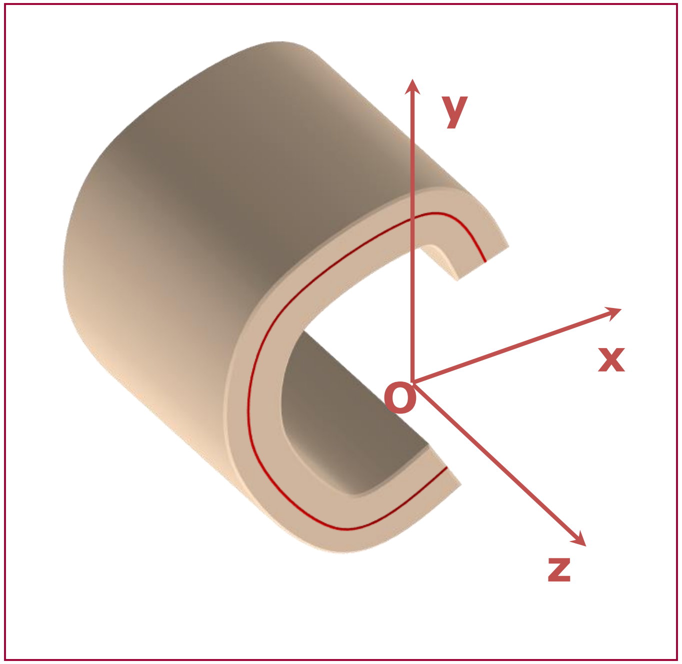
Summary

Shear stresses

$$q(s)=q(0)-K_y\cdot Q_y+-K_x\cdot Q_x$$



$$K_y=\frac{m_{ex}^{*}\cdot I_y-m_{ey}^{*}\cdot P_{xy}}{I_x\cdot I_y-P_{xy}^{2}}$$

$$K_x=\frac{m_{ex}^{*}\cdot P_{xy}-m_{ey}^{*}\cdot I_x}{I_x\cdot I_y-P_{xy}^{2}}$$

$$m_{ex}^{*}=\int_{A^{*}}y(s)\cdot dA=\int_{0}^{s}y(s)\cdot e(s)\cdot ds$$

$$m_{ey}^{*}=\int_{A^{*}}x(s)\cdot dA=\int_{0}^{s}x(s)\cdot e(s)\cdot ds$$

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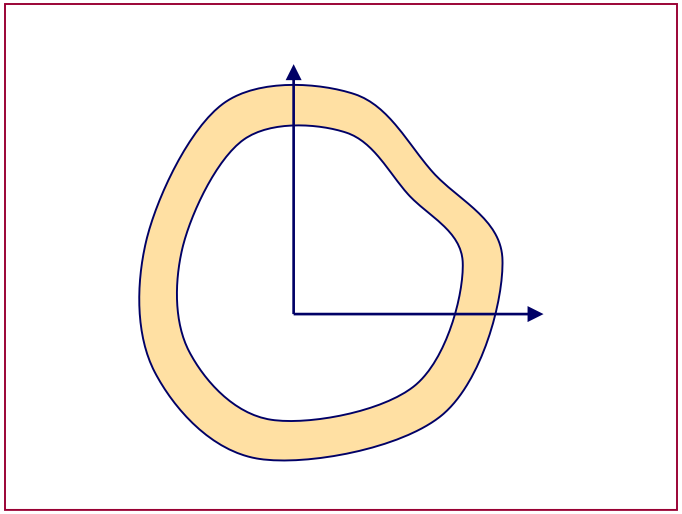
Summary



Single cell closed section

$$q(s) = q(s)_{open} + q(0)$$

$$q(0) = -\frac{\displaystyle\int_0^S \frac{q(s)_{open}}{G \cdot e(s)} \cdot ds}{\displaystyle\int_0^S \frac{1}{G \cdot e(s)} \cdot ds}$$

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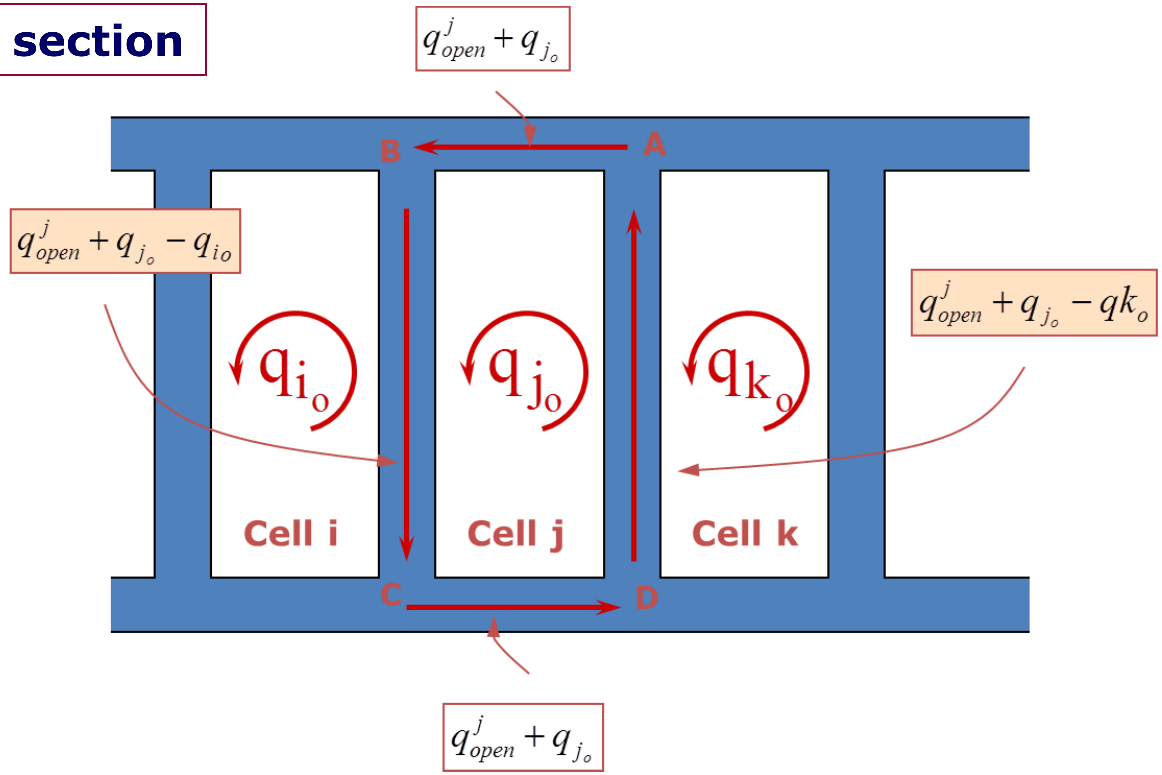
Summary

Multi-cell section



$$\int_A^B \frac{\left(q^j_{open} + q_{j_o}\right)}{G} \cdot ds + \int_B^C \frac{\left(q^j_{open} + q_{j_o} - q_{i_o}\right)}{G} \cdot ds +$$

$$\int_C^D \frac{\left(q^j_{open} + q_{j_o}\right)}{G} \cdot ds + \int_D^A \frac{\left(q^j_{open} + q_{j_o} - q_{k_o}\right)}{G} \cdot ds = 0$$

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