uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

SOLUCIONES DE LOS EJERCICIOS DE CÁLCULO I

Para Grados en Ingeniería

Capítulo 4: Integración en una variable

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Figuras realizadas con Arturo de Pablo Martínez

4. Integración en una variable

4.1. Cálculo de primitivas.

Problema 4.1.1

1. $IPP \Rightarrow I = x\Big(\frac{1}{2}\operatorname{tg}(2x) - x\Big) - \int \Big(\frac{1}{2}\operatorname{tg}(2x) - x\Big)\,dx = \frac{1}{4}\Big(2x\operatorname{tg}(2x) + \log|\cos(2x)| - 2x^2\Big)$.

2. $\operatorname{tg} x = t \Rightarrow I = \int t^3(t^2+1)\,dt = \frac{1}{6}\operatorname{tg}^6 x + \frac{1}{4}\operatorname{tg}^4 x$.

3. $\sqrt{x} = t \Rightarrow I = \int \frac{2t(t+1)}{t^2+3}\,dt = 2\int \Big(1 + \frac{t}{t^2+3} - \frac{3}{t^2+3}\Big)\,dt = 2\sqrt{x} + \log|x+3| - 2\sqrt{3}\operatorname{arc\,tg}(\sqrt{x}/3)$.

4. $\sqrt{1-(x+1)^2} = t$ y utilizando el desarrollo $(x+3)^3 = (x+1+2)^3 = (x+1)^3 + 6(x+1)^2 + 12(x+1) + 8 \Rightarrow I = \int \Big(t^2 - 13 - 6(1-t^2)^{1/2} - 8(1-t^2)^{-1/2}\Big)\,dt = \frac{1}{3}t^3 - 13t - 11\operatorname{arc\,sen} t - 3t(1-t^2)^{1/2} = \frac{1}{3}(1-(x+1)^2)^{3/2} - 13(1-(x+1)^2)^{1/2} - 11\operatorname{arc\,sen}((1-(x+1)^2)^{1/2}) - 3(1-(x+1)^2)^{1/2}(x+1)^2$, pues $\int \sqrt{1-t^2}\,dt = \frac{1}{2}\operatorname{arc\,sen} t + \frac{1}{2}t\sqrt{1-t^2}$ después del cambio $t = \operatorname{sen} u$. También puede ser apropiado el cambio desde el inicio $x+1 = \operatorname{sen} u$.

5. Descomponer en suma de fracciones simples, o más fácil CV $x - 1 = t \Rightarrow$
$I = \int \Big(\frac{1}{t} + \frac{2}{t^2} + \frac{1}{t^3}\Big)\,dt = \log|x+1| - \frac{2}{x-1} - \frac{1}{2(x-1)^2}$.

6. $x = \sec t \Rightarrow I = \int(\sec t + \sec^3 t)\,dt = \frac{1}{2}(\sec t\operatorname{tg} t + 3\log|\sec t + \operatorname{tg} t|) = \frac{1}{2}(x\sqrt{x^2-1} + 3\log|x + \sqrt{x^2-1}|)$, véase 56. Como esta última integral se puede hacer mediante CV $\operatorname{sen} t = u$, un único cambio desde el principio (nada intuitivo) sería $x = \frac{1}{\sqrt{1-u^2}} \Rightarrow I = \int \frac{2-u^2}{(1-u^2)^2}\,du = \int \Big(\frac{3/4}{1+u} + \frac{1/4}{(1+u)^2} + \frac{3/4}{1-u} + \frac{1/4}{(1-u)^2}\Big)\,du = \frac{1}{4}\Big(3\log\Big|\frac{1+u}{1-u}\Big| + \frac{2u}{1-u^2}\Big) = \frac{1}{4}\Big(3\log\Big|\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}\Big| + 2x\sqrt{x^2-1}\Big) = \frac{1}{2}(3\log|x+\sqrt{x^2-1}| + x\sqrt{x^2-1})$.

7. $\cos x = t \Rightarrow I = \int \frac{t^8}{t^2-1}\,dt = \frac{1}{7}\cos^7 x + \frac{1}{5}\cos^5 x + \frac{1}{3}\cos^3 x + \cos x - \log|\operatorname{cosec} x - \operatorname{cotg} x|$.

8. $I = -\log|\operatorname{sen} x + \cos x|$.

9. IPP dos veces $\Rightarrow I = \frac{1}{\pi^2}e^x(\operatorname{sen}\pi x - \pi\cos\pi x) - \frac{I}{\pi^2} \Rightarrow I = \frac{1}{\pi^2+1}e^x(\operatorname{sen}\pi x - \pi\cos\pi x)$.

10. $\operatorname{tg} x = t \Rightarrow I = \int(t^2+1)\,dt = \frac{1}{3}\operatorname{tg}^3 x + \operatorname{tg} x$.

11. $I = \int \frac{1-\cos 2x}{2}\,dx = \frac{x}{2} - \frac{1}{4}\operatorname{sen} 2x$.

12. $I = \int \frac{(1-\cos 2x)^2}{4}\,dx = \frac{1}{4}\int\Big(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\Big)\,dx = \frac{3}{8}x - \frac{1}{4}\operatorname{sen} 2x + \frac{1}{32}\operatorname{sen} 4x$.

13. $I = \frac{x}{2} + \frac{1}{4}\operatorname{sen} 2x$.

14. $I = \int \frac{(1+\cos 2x)^3}{8}\,dx = \frac{1}{8}\int\Big(1 + 3\cos 2x + 3\frac{1+\cos 4x}{2} + \cos 2x(1-\operatorname{sen}^2 2x)\Big)\,dx = \frac{5}{16}x + \frac{1}{4}\operatorname{sen} 2x + \frac{3}{64}\operatorname{sen} 4x - \frac{1}{48}\operatorname{sen}^3 2x$.

15. $I = \frac{1}{4}\int \operatorname{sen}^2 2x\,dx = \frac{x}{8} - \frac{1}{32}\operatorname{sen} 4x$.

16. $3+\sqrt{2x+5} = t \Rightarrow I = \int \frac{t-3}{t}\,dt = \sqrt{2x+5} + 3 - 3\log(3+\sqrt{2x+5})$; poniendo $\sqrt{2x+5} = u \Rightarrow I = \int \frac{u}{u+3}\,dt = \sqrt{2x+5} - 3\log(3+\sqrt{2x+5})$; ambas se diferencian en una constante.

17. $\sqrt{\dfrac{x-1}{x+1}} = t \Rightarrow I = \displaystyle\int \frac{4t^2}{(1-t^2)^2}\,dt = \int \Big(\frac{1}{(1+t)^2} + \frac{1}{(1-t)^2} - \frac{1}{1+t} - \frac{1}{1-t} \Big)\,dt =$ $\dfrac{2t}{1-t^2} - \log\Big|\dfrac{1+t}{1-t}\Big| = \sqrt{x^2-1} - \log|x + \sqrt{x^2-1}|$. El cambio desde el inicio (no obvio) $x = \sec u$ lleva a $I = \displaystyle\int (\sec^2 u - \sec u)\,du = \operatorname{tg} u - \log|\sec u + \operatorname{tg} u| = \sqrt{x^2-1} - \log|x + \sqrt{x^2-1}|$.

18. $\sqrt[3]{x} = t \Rightarrow I = \displaystyle\int 3t^2 \operatorname{arc\,tg} t\,dt$; ahora $IPP \Rightarrow I = t^3 \operatorname{arc\,tg} t - \displaystyle\int \frac{t^3}{1+t^2}\,dt = x \operatorname{arc\,tg}(x^{1/3}) - \dfrac{1}{2}x^{2/3} + \dfrac{1}{2}\log(1 + x^{2/3})$.

19. $\sqrt{\sqrt{x}+1} = t \Rightarrow I = \displaystyle\int 4t^2(t^2-1)\,dt = \frac{4}{5}(\sqrt{x}+1)^{5/2} - \frac{4}{3}(\sqrt{x}+1)^{3/2}$.

20. $\sqrt{x+2} = t \Rightarrow I = \displaystyle\int \frac{2t^2}{1+t}\,dt = x + 2 - 2\sqrt{x+2} + 2\log(1 + \sqrt{x+2})$.

21. $\sqrt{2+\mathrm{e}^x} = t \Rightarrow I = \displaystyle\int \frac{2t^2}{t^2-2}\,dt = 2\sqrt{2+\mathrm{e}^x} + 2\sqrt{2}\log(\sqrt{2+\mathrm{e}^x} + \sqrt{2}) - \sqrt{2}\,x$.

22. $\operatorname{sen} x = t \Rightarrow I = \displaystyle\int \mathrm{e}^t(1-t^2)\,dt$, ahora IPP dos veces $\Rightarrow I = -(1 - \operatorname{sen} x)^2 \mathrm{e}^{\operatorname{sen} x}$.

23. $I = \displaystyle\int \operatorname{sen} x(1 - \cos^2 x)^2\,dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x$.

24. $I = \displaystyle\int \cos x(1 - \operatorname{sen}^2 x)\operatorname{sen}^2 x\,dx = \frac{1}{3}\operatorname{sen}^3 x - \frac{1}{5}\operatorname{sen}^5 x$.

25. $I = \displaystyle\int (\sec^2 x - 1)\,dx = \operatorname{tg} x - x$.

26. $I = \displaystyle\int \operatorname{tg} x(\sec^2 x - 1)\,dx = \frac{1}{2}\operatorname{tg}^2 x + \log|\cos x|$.

27. $x = \operatorname{sen} t \Rightarrow I = \displaystyle\int \operatorname{sen}^3 t \cos^2 t\,dt = -\frac{1}{3}\cos^3 t + \frac{1}{5}\cos^5 t = -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2}$. También se puede hacer CV $\sqrt{1-x^2} = u \Rightarrow I = \displaystyle\int u^2(u^2-1)\,du = \frac{u^5}{5} - \frac{u^3}{3} = \frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2}$.

28. $t = \operatorname{tg}(x/2) \Rightarrow \operatorname{sen} x = \dfrac{2t}{1+t^2},\ \cos x = \dfrac{1-t^2}{1+t^2},\ dx = \dfrac{2dt}{1+t^2} \Rightarrow I = \displaystyle\int \frac{3+2t-3t^2}{t(t^2+3)}\,dt = \int \Big(\frac{1}{t} + \frac{-4t+2}{t^2+3}\Big)\,dt = \log|t| - 2\log(t^2+3) + \frac{2}{\sqrt{3}}\operatorname{arc\,tg}(t/\sqrt{3}) =$
$\log\Big|\dfrac{\operatorname{tg}(x/2)}{(\operatorname{tg}^2(x/2)+3)^2}\Big| + \dfrac{2}{\sqrt{3}}\operatorname{arc\,tg}\Big(\dfrac{1}{\sqrt{3}}\operatorname{tg}(x/2)\Big)$.

29. $t = \operatorname{tg}(x/2) \Rightarrow I = \displaystyle\int \frac{2(3+2t-3t^2)}{(1+t^2)(3+t^2)}\,dt = \log\Big|\frac{\operatorname{tg}^2(x/2)+1}{\operatorname{tg}^2(x/2)+3}\Big| + 3t - \frac{12}{\sqrt{3}}\operatorname{arc\,tg}\Big(\frac{1}{\sqrt{3}}\operatorname{tg}(x/2)\Big)$.

30. $\operatorname{sen} 3x = t \Rightarrow I = \dfrac{1}{3}\displaystyle\int \frac{t^2}{(1-t^2)^3}\,dt = \frac{1}{48}\Big(\frac{4t}{(1-t^2)^2} - \frac{2t}{1-t^2} + \log\Big|\frac{1-t}{1+t}\Big|\Big) =$
$\dfrac{1}{24}\Big(2\sec 3x \operatorname{tg} 3x - \operatorname{tg} 3x - \log|\sec 3x + \operatorname{tg} 3x|\Big)$.

31. $I = \displaystyle\int \Big(4x + 7 + \frac{4x^2 - 29x + 27}{(x+3)(x-3)(x-2)}\Big)\,dx = 2x^2 + 7x + \log\Big|\frac{(x+3)^5(x-2)^3}{(x-3)^4}\Big|$.

32. $\log x = t \Rightarrow I = \displaystyle\int \mathrm{e}^t \cos t\,dt$, que se resuelve mediante IPP dos veces; también se puede hacer desde el principio IPP dos veces $\Rightarrow I = x\cos(\log x) + x\operatorname{sen}(\log x) - I \Rightarrow I = \dfrac{x}{2}\Big(\cos(\log x) + \operatorname{sen}(\log x)\Big)$.

33. $\mathrm{e}^x = t \Rightarrow I = \displaystyle\int \frac{t^3}{t^2+t+2}\,dt = \int \Big(t - 1 - \frac{t - 1/2}{t^2+t+2} + \frac{5/2}{(t+1/2)^2 + 7/4}\Big)\,dt =$

$\frac{1}{2}e^{2x} - e^x - \frac{1}{2}\log(e^{2x}+e^x+2) + \frac{5}{\sqrt{7}}\operatorname{arc\,tg}\Big(\frac{2e^x+1}{\sqrt{7}}\Big)$.

34. $\sqrt{1+\sqrt[3]{x}} = t \Rightarrow I = \int 6t^2(t^2-1)\,dt = \frac{6}{5}(1-x^{1/3})^{5/2} - 2(1-x^{1/3})^{3/2}$.

35. $x = \operatorname{tg} t \Rightarrow I = \int \operatorname{sen}^3 t \cos t\,dt = \frac{x^3}{3(1+x^2)^{3/2}}$.

36. $I = 2\operatorname{arc\,tg}(x-1)$.

37. $I = \operatorname{tg} x$.

38. $\sqrt[3]{x+2} = t \Rightarrow I = \int \frac{3t}{t^3-1}\,dt = \log\Big|(x+2)^{1/3}-1\Big| - \frac{1}{2}\log\Big|(x+2)^{2/3}+(x+2)^{1/3}+1\Big| +$ $\sqrt{3}\operatorname{arc\,tg}\Big(\frac{2(x+2)^{1/3}+1}{\sqrt{3}}\Big)$.

39. $I = -\frac{1}{3}(x^2+1)^{-3/2}$.

40. $x = \operatorname{sen} t \Rightarrow I = \int \operatorname{tg}^2 t\,dt = \frac{x}{\sqrt{1-x^2}} - \operatorname{arc\,sen} x$.

41. $\sqrt{e^x-1} = t \Rightarrow I = \int \frac{2t^2}{t^2+1}\,dt = 2(\sqrt{e^x-1} - \operatorname{arc\,tg}(\sqrt{e^x-1}))$;
otro cambio posible, $e^x = \sec^2 t \Rightarrow I = \int 2\operatorname{tg}^2 t\,dt = 2(\operatorname{tg} t - t) = 2\Big(\sqrt{e^x-1} - \operatorname{arc\,tg}(\sqrt{e^x-1})\Big)$.

42. $I = \frac{3}{x} + 5\log|x-1| - 3\log|x|$.

43. $\sqrt{1-\sqrt{x}} = t \Rightarrow I = -4\int t(1+t)\,dt = -2(1-\sqrt{x}) - \frac{4}{3}(1-\sqrt{x})^{3/2}$;
otro cambio posible, $\sqrt{x} = t \Rightarrow I = 2\int \Big(1+\sqrt{1-t}\Big)\,dt = 2t - \frac{4}{3}(1-t)^{4/3}$.

44. $I = \int \frac{(1+\operatorname{sen} x)(1-\cos x)}{\operatorname{sen}^2 x}\,dx = \int \Big(\frac{1}{\operatorname{sen}^2 x} + \frac{1}{\operatorname{sen} x} - \frac{\cos x}{\operatorname{sen}^2 x} - \frac{\cos x}{\operatorname{sen} x}\Big)\,dx =$
$-\operatorname{cotg} x - \log|\operatorname{cosec} x + \operatorname{cotg} x| + \operatorname{cosec} x + \log|\operatorname{cosec} x| = \frac{1-\cos x}{\operatorname{sen} x} - \log(1+\cos x)$.

45. $\sqrt{x-1} = t \Rightarrow I = \int 2t^2(t^2+1)^2\,dt = \frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2}$.

46. $\operatorname{tg} x = t \Rightarrow I = \int (1+t^2)^2\,dt = \operatorname{tg} x + \frac{2}{3}\operatorname{tg}^3 x + \frac{1}{5}\operatorname{tg}^5 x$.

47. $1+x^2 = t \Rightarrow I = \int \frac{t-1}{2t^3}\,dt = -\frac{1}{2(1+x^2)} + \frac{1}{4(1+x^2)^2} = -\frac{1+2x^2}{4(1+x^2)^2}$. También CV $x = \operatorname{tg} u \Rightarrow I = \int \frac{\operatorname{tg}^3 u}{\sec^4 u}\,dt = \int \operatorname{sen}^3 u \cos u\,du = \frac{\operatorname{sen}^4 u}{4} = \frac{x^4}{4(1+x^2)^2}$; basta comprobar que ambos resultados difieren en una constante.

48. $e^x = t \Rightarrow I = \int \frac{t}{t^2-4}\,dt = \frac{1}{2}\log\big|e^{2x}-4\big|$.

49. $\sqrt{1+x} = t \Rightarrow I = \int \frac{2}{t^2+1}\,dt = 2\operatorname{arc\,tg}\big(\sqrt{1+x}\,\big)$.

50. $1+\sqrt[3]{1-x} = t \Rightarrow I = \int \Big(-3t+6-\frac{3}{t}\Big)\,dt = -\frac{3}{2}\Big(1+\sqrt[3]{1-x}\Big)^2 + 6\Big(1+\sqrt[3]{1-x}\Big) -$ $3\log\Big(1+\sqrt[3]{1-x}\Big) = 9(1-x)^{1/3} + \frac{9}{2}(1-x)^{2/3} - 3\log(1+(1-x)^{1/3}) + C$.

51. *IPP* dos veces $\Rightarrow I = \frac{1}{2}e^x\operatorname{sen} 2x + \frac{1}{4}e^x\cos 2x - \frac{1}{4}I \Rightarrow I = \frac{1}{5}e^x(2\operatorname{sen} 2x + \cos 2x)$.

52. *IPP* $\Rightarrow I = \frac{1}{9}x^3(3\log x - 1)$.

53. $I = \int \operatorname{sen} x(1-\cos^2 x)\cos^2 x\,dx = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x$.

54. $I = \int \Big(\frac{1+\cos^2 x}{2}\Big)^2\,dx = \frac{1}{4}\int \Big(1+2\cos 2x + \frac{1+\cos 4x}{2}\Big)\,dx = \frac{3}{8}x + \frac{1}{4}\operatorname{sen} 2x + \frac{1}{32}\operatorname{sen} 4x$.

55. $\operatorname{tg} x = t \Rightarrow I = \displaystyle\int \frac{t^4}{1+t^2}\,dt = \int \Big(t^2 - 1 + \frac{1}{1+t^2}\Big)\,dt = \frac{1}{3}\operatorname{tg}^3 x - \operatorname{tg} x + x.$

56. $\operatorname{sen} x = t \Rightarrow I = \displaystyle\int \frac{1}{(1-t^2)^2}\,dt = \frac{1}{4}\log\Big|\frac{1-t}{1+t}\Big| - \frac{1}{4}\Big(\frac{1}{1+t} - \frac{1}{1+t}\Big) = \frac{1}{2}\log|\sec x + \operatorname{tg} x| + \frac{1}{2}\sec x \operatorname{tg} x$; otra forma, $IPP \Rightarrow I = \sec x \operatorname{tg} x - \displaystyle\int \sec x \operatorname{tg}^2 x\,dx = \sec x \operatorname{tg} x + \int \sec x\,dx - I \Rightarrow I = \frac{1}{2}\Big(\sec x \operatorname{tg} x + \int \sec x\,dx\Big).$

57. $I = \displaystyle\int \frac{1+\operatorname{sen} x}{\cos^2 x}\,dx = \operatorname{tg} x + \sec x.$

58. $\log x = t \Rightarrow I = \displaystyle\int \mathrm{e}^t \operatorname{sen} t\,dt = \frac{1}{2}\mathrm{e}^t(\operatorname{sen} t - \cos t) = \frac{1}{2}x(\operatorname{sen}(\log x) - \cos(\log x)).$

59. $x = \operatorname{sen} t \Rightarrow I = \displaystyle\int \frac{1}{\operatorname{sen}^2 t}\,dt = -\operatorname{cotg} t = -\frac{\sqrt{1-x^2}}{x}.$

60. $I = \sqrt{1+x^2}.$

61. $\sqrt{\mathrm{e}^{2x}-1} = t \Rightarrow I = \displaystyle\int \frac{1}{t^2+1}\,dt = \operatorname{arc\,tg} t = \operatorname{arc\,tg}\Big(\sqrt{\mathrm{e}^{2x}-1}\Big)$. También CV $\mathrm{e}^x = \sec t \Rightarrow I = \displaystyle\int dt = \operatorname{arcsec}(\mathrm{e}^x)$. Obsérvese que $\operatorname{arcsec}(\mathrm{e}^x) = \operatorname{arc\,cos}(\mathrm{e}^{-x}) = \operatorname{arc\,tg}\Big(\sqrt{\mathrm{e}^{2x}-1}\Big).$

62. $\mathrm{e}^x = t \Rightarrow I = \displaystyle\int \frac{t^3}{t^2+2t+2}\,dt = \int \Big(t - 2 + \frac{2t+2}{t^2+2t+2} + \frac{2}{t^2+2t+2}\Big)\,dt = \frac{1}{2}\mathrm{e}^{2x} - 2\mathrm{e}^x + \log(\mathrm{e}^{2x} + 2\mathrm{e}^x + 2) + 2\operatorname{arc\,tg}(\mathrm{e}^x + 1).$

63. $I = \displaystyle\int \Big(x - \frac{x}{(x^2-1)^2}\Big)\,dx = \frac{x^2}{2} + \frac{1}{2(x^2-1)}.$

64. $(1-2x)^{1/6} = t \Rightarrow I = -3\displaystyle\int \frac{t^2}{t-1}\,dt = -\frac{3}{2}(1-2x)^2 - 3(1-2x)^{1/6} - 3\log\Big|(1-2x)^{1/6} - 1\Big|.$

65. $x = 3\operatorname{sen} t \Rightarrow I = \displaystyle\int \frac{1}{9\operatorname{sen}^2 t}\,dt = -\frac{1}{9}\operatorname{cotg} t = -\frac{\sqrt{9-x^2}}{9x}.$

66. $I = \displaystyle\frac{1}{3}\int \Big(-\frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{x+1}{x^2+x+1}\Big)\,dx = -\frac{1}{3}\log|x-1| - \frac{1}{3(x-1)} + \frac{1}{6}\log|x^2 + x + 1| + \frac{\sqrt{3}}{9}\operatorname{arc\,tg}\Big(\frac{2x+1}{\sqrt{3}}\Big).$

67. Si $m \neq -1$, $IPP \Rightarrow I = \dfrac{x^{m+1}}{(m+1)^2}((m+1)\log|x| + 1)$; si $m = -1$, $I = \dfrac{1}{2}\log^2|x|.$

68. $\operatorname{sen} x = t \Rightarrow I = -\dfrac{1}{3\operatorname{sen}^3 x} + \dfrac{1}{\operatorname{sen} x}.$

69. $x^{3/2} = t \Rightarrow I = \displaystyle\frac{2}{3}\int t \operatorname{sen} t\,dt = \frac{2}{3}\Big(x^{3/2}\cos(x^{3/2}) - \operatorname{sen}(x^{3/2})\Big).$

70. $\log x = t \Rightarrow I = \displaystyle\int \mathrm{e}^t \cos^2 t\,dt = \frac{1}{2}\int \mathrm{e}^t(1+\cos 2t)\,dt = \frac{x}{2} + \frac{x}{10}\Big(2\operatorname{sen}(2\log x) + \cos(2\log x)\Big).$

71. IPP tres veces $\Rightarrow I = x\Big(\log^3 x - 3\log^2 x + 6\log x - 6\Big)$. Es más intuitivo hacer primero CV $\log x = t$ y continuar con IPP; lo mismo se puede decir de la siguiente primitiva.

72. IPP dos veces $\Rightarrow I = \dfrac{x^2}{2}\Big(\log^2 x - \log x + 1\Big).$

Problema 4.1.2 Hallando la primitiva en cada intervalo, se tiene

$$f(x) = \begin{cases} \dfrac{x}{x^2+4} + C_1 & \text{si} \quad x < 0 \\ 2\mathrm{e}^{\sqrt{x}}(\sqrt{x} - 1) + C_2 & \text{si} \quad x > 0 \end{cases}$$

El valor $f(0) = 0$ implica $C_1 = 0$, $C_2 = 2$.
El Teorema fundamental del cálculo permitirá, más adelante, escribir directamente

$f(x) = f(0) + \int_0^x f'(t)\,dt$, por lo que, si $x < 0$ se tiene

$$f(x) = \int_0^x \frac{4-t^2}{(4+t^2)^2}\,dt = \frac{x}{x^2+4}\,,$$

mientras que para $x > 0$ es

$$f(x) = \int_0^x \mathrm{e}^{\sqrt{t}}\,dt = 2\mathrm{e}^{\sqrt{x}}(\sqrt{x}-1)+2\,,$$

y se obtiene el mismo resultado. Las primitivas son fáciles con los cambios obvios ($t = 2\operatorname{tg} u$ y $\sqrt{t} = u$, respectivamente).

Problema 4.1.3 Tomando la partición $P_n = \{x_i = a + \frac{i}{n}(b-a),\ i = 0, 1, \cdots, n\}$, tenemos

$$\begin{aligned}
U_n &= \sum_{i=1}^{n}(a + \frac{i}{n}(b-a))\frac{b-a}{n} = \frac{a(b-a)}{n}\sum_{i=1}^{n} 1 + \frac{(b-a)^2}{n^2}\sum_{i=1}^{n} i = a(b-a) + \frac{(b-a)^2(n+1)}{2n};\\
\inf_{P_n} U_n &= \lim_{n\to\infty} U_n = a(b-a) + \frac{(b-a)^2}{2} = \frac{1}{2}(b^2-a^2);\\
L_n &= \sum_{i=1}^{n}(a + \frac{i-1}{n}(b-a))\frac{b-a}{n} = a(b-a) + \frac{(b-a)^2 n}{2(n-1)};\\
\sup_{P_n} L_n &= \lim_{n\to\infty} L_n = \frac{1}{2}(b^2-a^2).
\end{aligned}$$

Finalmente $\int_a^b x\,dx = \frac{1}{2}(b^2-a^2)$.

Problema 4.1.4 *i)* $\int_{-a}^{a} g(x)\,dx = \int_{-a}^{0} g(x)\,dx + \int_0^a g(x)\,dx = -\int_0^a g(y)\,dy + \int_0^a g(x)\,dx = 0.$ Así

$$\int_6^{10} \operatorname{sen}[\operatorname{sen}\{(x-8)^3\}]\,dx = \int_{-2}^{2} \operatorname{sen}(\operatorname{sen} y^3)\,dy = 0,$$

pues el integrando es impar.
ii) $\int_{-a}^{a} h(x)\,dx = \int_{-a}^{0} h(x)\,dx + \int_0^a h(x)\,dx = \int_0^a h(y)\,dy + \int_0^a h(x)\,dx.$

Problema 4.1.5 *i)* Poner $x - c = y$; el área es invariante por traslaciones.
ii) Poner $a + b - x = y$; el área es invariante por simetrías.
iii) Aplicar *ii)* con $a + b = 0$. La función $g(x) = f(x) - f(-x)$ es impar, por lo que se aplica el problema anterior.
iv) $-|f(x)| \le f(x) \le |f(x)| \Rightarrow -\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$. Versión integral de la desigualdad triangular, el valor absoluto de la suma es menor o igual que la suma de los valores absolutos.
v) Cambiando $x \to \frac{x}{a}$ en la segunda integral,

$$\int_1^a \frac{dx}{x} + \int_1^b \frac{dx}{x} = \int_1^a \frac{dx}{x} + \int_a^{ab} \frac{dx}{x} = \int_1^{ab} \frac{dx}{x},$$

es decir, $\log a + \log b = \log(ab)$.

Problema 4.1.6 Hay que utilizar la fórmula $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n f(i/n)$.

i) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \frac{1}{1+(i/n)^2} = \int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$. *ii)* $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \frac{1}{1+i/n} = \int_0^1 \frac{1}{1+x}\,dx = \log 2$.

iii) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathrm{e}^{2i/n} = \int_0^1 \mathrm{e}^{2x}\,dx = \frac{\mathrm{e}^2-1}{2}$; también se puede calcular la suma geométrica:
$\lim_{n\to\infty} \frac{1}{n}\frac{\mathrm{e}^{2(n+1)/n}-\mathrm{e}^{2/n}}{\mathrm{e}^{2/n}-1} = \lim_{x\to 0} \frac{x(\mathrm{e}^{2(x+1)}-\mathrm{e}^{2x})}{\mathrm{e}^{2x}-1} = \frac{\mathrm{e}^2-1}{2}$.

iv) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{\sqrt{1-(i/n)^2}} = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}$.

Problema 4.1.7

$$\begin{aligned} L &= \exp\Big(\lim_{n\to\infty}\frac{1}{n}\log\big(\prod_{k=1}^n(1+k/n)\big)\Big) = \exp\Big(\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\log(1+k/n)\Big) \\ &= \exp\Big(\int_0^1 \log(1+x)\,dx\Big) = \mathrm{e}^{2\log 2-1} = \frac{4}{\mathrm{e}}. \end{aligned}$$

Problema 4.1.8 *i)* $x \le 0 \Rightarrow F(x) = \int_{-1}^x (-1)\,dt = -x-1$;
$x > 0 \Rightarrow F(x) = F(0) + \int_0^x 1\,dt = -1+x$.
$\Rightarrow F(x) = \begin{cases} -x-1 & \text{si} \quad -1 \le x \le 0 \\ x-1 & \text{si} \quad 0 \le x \le 1. \end{cases}$

ii) $x \le 0 \Rightarrow F(x) = \int_{-1}^x (-t\mathrm{e}^t)\,dt = \mathrm{e}^x(1-x) - \frac{2}{\mathrm{e}}$;
$x > 0 \Rightarrow F(x) = F(0) + \int_0^x t\mathrm{e}^{-t}\,dt = 2 - \frac{2}{\mathrm{e}} - \mathrm{e}^{-x}(x+1)$.
$\Rightarrow F(x) = \begin{cases} \mathrm{e}^x(1-x) - \frac{2}{\mathrm{e}} & \text{si} \quad -1 \le x \le 0 \\ 2 - \frac{2}{\mathrm{e}} - \mathrm{e}^{-x}(x+1) & \text{si} \quad 0 \le x \le 1. \end{cases}$

iii) $x \le \frac{1}{2} \Rightarrow F(x) = \int_{-1}^x (\frac{1}{2}-t)\,dt = \frac{1}{2}(2+x-x^2)$;
$x > \frac{1}{2} \Rightarrow F(x) = F(1/2) + \int_{1/2}^x (t-\frac{1}{2})\,dt = \frac{1}{4}(2x^2-2x+5)$.
$\Rightarrow F(x) = \begin{cases} \frac{1}{2}(2+x-x^2) & \text{si} \quad -1 \le x \le \frac{1}{2} \\ \frac{1}{4}(2x^2-2x+5) & \text{si} \quad \frac{1}{2} \le x \le 1. \end{cases}$

iv) $x \le 0 \Rightarrow F(x) = \int_{-1}^x t^2\,dt = \frac{1}{3}(x^3+1)$;
$x > 0 \Rightarrow F(x) = F(0) + \int_0^x (t^2-1)\,dt = \frac{1}{3}(x^3-3x+1)$.
$\Rightarrow F(x) = \begin{cases} \frac{1}{3}(x^3+1) & \text{si} \quad -1 \le x \le 0 \\ \frac{1}{3}(x^3-3x+1) & \text{si} \quad 0 \le x \le 1. \end{cases}$

v) $x \le 0 \Rightarrow F(x) = \int_{-1}^x 1\,dt = x+1$;
$x > 0 \Rightarrow F(x) = F(0) + \int_0^x (t+1)\,dt = \frac{1}{2}(x^2+2x+2)$.

$$\Rightarrow F(x) = \begin{cases} x+1 & \text{si} \quad -1 \le x \le 0 \\ \frac{1}{2}(x^2+2x+2) & \text{si} \quad 0 \le x \le 1. \end{cases}$$

vi) $F(x) = \int_{-1}^{x} 1\,dt = x+1.$

vii) $x \le \frac{1}{2} \Rightarrow F(x) = \int_{-1}^{x} \cos(\pi t/2)\,dt = \frac{2}{\pi}(1+\operatorname{sen}(\pi x/2));$

$x > \frac{1}{2} \Rightarrow F(x) = F(1/2) + \int_{1/2}^{x} \cos(\pi t/2)\,dt = \frac{2}{\pi}(1+\sqrt{2}-\cos(\pi x/2)).$

$$\Rightarrow F(x) = \begin{cases} \frac{2}{\pi}(1+\operatorname{sen}(\pi x/2)) & \text{si} \quad -1 \le x \le \frac{1}{2} \\ \frac{2}{\pi}(1+\sqrt{2}-\cos(\pi x/2)) & \text{si} \quad \frac{1}{2} \le x \le 1. \end{cases}$$

Problema 4.1.9

i) $\sqrt{e^x-1} = t \quad \Rightarrow \quad I = \int_0^1 \frac{2t^2}{1+t^2}\,dt = 2(t-\operatorname{arc\,tg} t)\Big|_0^1 = 2-\frac{\pi}{2};$

también

$e^x = \sec^2 t \quad \Rightarrow \quad I = 2\int_0^{\pi/4} \operatorname{tg}^2 t\,dt = 2(\operatorname{tg} t - t)\Big|_0^{\pi/4} = 2-\frac{\pi}{2}.$

ii) $x = \sec t \quad \Rightarrow \quad I = \int_0^{\pi/3} \operatorname{tg}^2 t\,dt = (\operatorname{tg} t - t)\Big|_0^{\pi/3} = \sqrt{3}-\frac{\pi}{3};$

también

$\sqrt{x^2-1} = t \quad \Rightarrow \quad I = \int_0^{\sqrt{3}} \frac{t^2}{1+t^2}\,dt = (t-\operatorname{arc\,tg} t)\Big|_0^{\sqrt{3}} = \sqrt{3}-\frac{\pi}{3}.$

4.2. Teorema fundamental del cálculo.

Problema 4.2.1 *i)* $|F(x)-F(y)| \le \int_y^x |f(t)|\,dt \le M|x-y|.$

$0 \le \lim_{x\to y} |F(x)-F(y)| \le \lim_{x\to y} M|x-y| = 0.$

ii) F es derivable si y sólo si f es continua, y en ese caso $F'(x) = f(x)$. Un ejemplo del caso contrario es $f(x) = \begin{cases} 1 & x > a \\ -1 & x < a \end{cases} \Rightarrow F(x) = |x-a|$; f no es continua en $x = a$, F es no derivable en el mismo punto.

Problema 4.2.2 *i)* $F'(x) = \frac{e^{x^3}}{x^3}3x^2 - \frac{e^{x^2}}{x^2}2x.$

ii) $F'(x) = \frac{3x^2}{1+\operatorname{sen}^2(x^3)} - \frac{-3x^2}{1+\operatorname{sen}^2(x^3)} = \frac{6x^2}{1+\operatorname{sen}^2(x^3)}.$

iii) $F'(x) = \frac{\operatorname{sen}^3 x}{1+\operatorname{sen}^6(\int_1^x \operatorname{sen}^3 t\,dt) + (\int_1^x \operatorname{sen}^3 t\,dt)^2}.$

iv) $F'(x) = \frac{e^{\int_1^{x^2} \operatorname{tg}\sqrt{t}\,dt}\operatorname{tg} x\, 2x}{\int_1^{x^2} \operatorname{tg}\sqrt{t}\,dt}.$

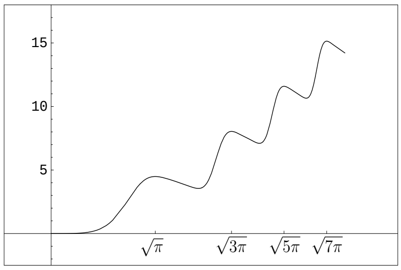
v) $F'(x) = 2x\int_0^x f(t)\,dt + x^2 f(x).$

vi) $F'(x) = \cos\Big(\int_0^x \operatorname{sen}\big(\int_0^y \operatorname{sen}^3 t\,dt\big)\,dy\Big)\,\operatorname{sen}\big(\int_0^x \operatorname{sen}^3 t\,dt\big).$

Problema 4.2.3 $f'(x) = e^{-(x-1)^2} - e^{-2(x-1)} = 0 \Rightarrow (x-1)^2 = 2(x-1) \Rightarrow x = 1$ o $x = 3$. Se obtiene que f crece en $(1,3)$ y decrece en $(3,\infty)$. El máximo se alcanza en $x = 3$. Para estudiar el mínimo necesitamos utilizar los valores $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$, $\displaystyle\int_0^\infty e^{-2x}\,dx = \frac{1}{2}$. Comparamos entonces $\displaystyle\lim_{x\to\infty} f(x) = \frac{\sqrt{\pi}-1}{2}$ y $f(1)=0$, por lo que el mínimo se alcanza en $x=1$.

Problema 4.2.4 *i)* La función $F(x) = \displaystyle\int_0^x e^{t^2}dt - 1$ es monótona creciente ($F'(x) = e^{x^2} > 0$) y verifica $F(0) = -1$, $F(1) > 0$.

ii) $G'(x) = 2x \operatorname{sen} x^2 e^{\operatorname{sen} x^2} = 0 \Rightarrow x_n = \sqrt{n\pi}$, $n \in \mathbb{N}$ ($x > 0$). El signo de la derivada entre cada par de puntos críticos viene dado por el signo de $\operatorname{sen} x^2$, por lo que $G'(x) > 0$ en $x \in \displaystyle\bigcup_{k=0}^{\infty} (\sqrt{2k\pi}, \sqrt{(2k+1)\pi})$ y $G'(x) < 0$ en el complementario. Así los puntos críticos x_n son mínimos si $n = 2k$ y máximos si $n = 2k+1$. Evaluando la segunda derivada se obtiene el mismo resultado pues $G''(x_n) = 4(-1)^n n\pi$.



Problema 4.2.5 $y' = -2x\,\operatorname{tg}(x^4) \Rightarrow r \equiv y = -2\sqrt[4]{\pi/4}(x - \sqrt[4]{\pi/4})$.

Problema 4.2.6 Utilizando L'Hôpital,

$$i)\quad L = \lim_{x\to 0} \frac{e^{x^2}-1}{3x^2} = \frac{1}{3}, \qquad ii)\quad L = \lim_{x\to 0}\frac{\operatorname{sen} x^3}{4x^3} = \frac{1}{4}.$$

Problema 4.2.7

$$L^+ = \lim_{x\to 0^+} \frac{\displaystyle\int_0^{x^2} \operatorname{tg}(\sqrt{t})\,dt}{2x^3} = \lim_{x\to 0^+}\frac{2x\,\operatorname{tg}|x|}{6x^2} = \lim_{x\to 0}\frac{2x\,\operatorname{tg} x}{6x^2} = \frac{1}{3},$$
$$L^- = \lim_{x\to 0^-}\frac{2x\,\operatorname{tg}|x|}{6x^2} = \lim_{x\to 0}\frac{-2x\,\operatorname{tg} x}{6x^2} = -\frac{1}{3}.$$

Problema 4.2.8 *i)* Utilizando el desarrollo del seno, e intercambiando suma e integral,

$$f(x) = \int_0^{x^2}\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n+1)!}\,dt = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2(2n+1)}}{(2n+1)(2n+1)!}.$$

ii) Utilizando el desarrollo anterior y el desarrollo del coseno, $L = \displaystyle\lim_{x\to 0}\frac{x^2 + o(x^2)}{x^2/2 + o(x^2)} = 2$.

iii) La serie converge pues $\displaystyle\lim_{n\to\infty}\frac{f(1/n)}{1/n^2} = \lim_{x\to 0}\frac{f(x)}{x^2} = 1$.

Problema 4.2.9 Si $F(x) = \displaystyle\int_{-1/x}^{x}\frac{dt}{a^2+t^2}$ no depende de x,

$$0 = F'(x) = \frac{1}{a^2+x^2} - \frac{1}{x^2(a^2+1/x^2)} \Leftrightarrow a^2x^2 + 1 = a^2 + x^2\ \forall x \ \Leftrightarrow a = \pm 1.$$

Problema 4.2.10 *i)* $g(x) = 3 + \displaystyle\int_0^x \sum_{n=2}^{\infty}\frac{t^{2n}}{n!}\,dt = 3 + \sum_{n=2}^{\infty}\frac{x^{2n+1}}{(2n+1)\,n!}$.

ii) El polinomio anterior indica

$$g(x) = 3 + \frac{x^5}{5!\,5} + o(x^5),$$

es decir, $g^{k)}(0) = 0$ para $k = 1, \cdots, 4$, $g^{5)}(0) = 1/5$. Así $x = 0$ es un punto de inflexión de g.

Problema 4.2.11 *i)* Sustituyendo en $x = 0$, $\int_0^{g(0)} (e^{t^2} + e^{-t^2})\,dt = 0 \Rightarrow g(0) = 0$, pues el integrando es positivo. Derivando

$$(e^{g^2(x)} + e^{-g^2(x)})g'(x) - 3x^2 - \frac{3}{1+x^2} = 0,$$

que en $x = 0$ implica $g'(0) = \dfrac{3}{e^{g^2(0)} + e^{-g^2(0)}} = \dfrac{3}{2}$. Por el teorema de la función inversa

$$(g^{-1})'(0) = \frac{1}{g'(0)} = \frac{2}{3}.$$

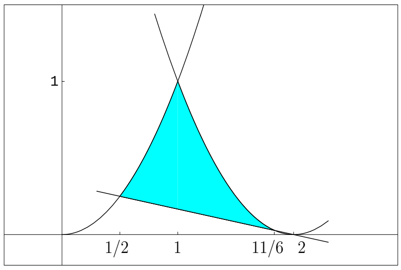
iii) Como g' es continua y no se anula cerca de 0, se tiene que $(g^{-1})'$ es continua. Aplicando L'Hôpital,

$$L = \lim_{x\to 0} \frac{(g^{-1})'(x)}{g'(x)} = \frac{(g^{-1})'(0)}{g'(0)} = \frac{4}{9}.$$

4.3. Aplicaciones.

Problema 4.3.1

i) $A = \displaystyle\int_{1/2}^{1} (x^2 - \frac{2-x}{6})\,dx + \int_1^{11/6} ((x-2)^2 - \frac{2-x}{6})\,dx = \frac{71}{162}$.

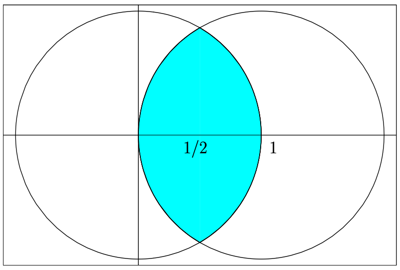


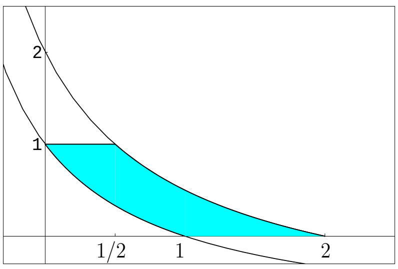
ii) Por simetría, $A = 4\displaystyle\int_{1/2}^{1} \sqrt{1-x^2}\,dx = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$.

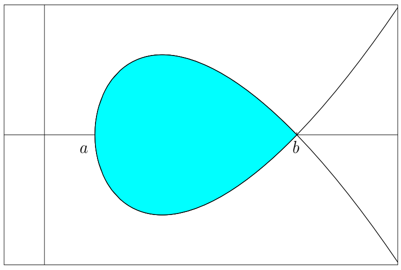
iii) $A = \displaystyle\int_0^{1/2} \frac{2x}{1+x}\,dx + \int_{1/2}^{1} \frac{1}{1+x}\,dx + \int_1^2 (\frac{2-x}{1+x})\,dx = \log 2;$

despejando las curvas en y el área es más fácil de calcular: $A = \displaystyle\int_0^1 \frac{1}{1+y}\,dy = \log 2$.

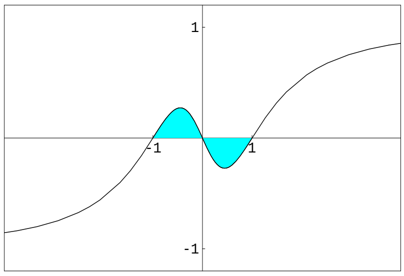
iv) Por simetría, $A = 2\displaystyle\int_a^b (b-x)\sqrt{x-a}\,dx = \frac{8}{15}(b-a)^{5/2}$.







Problema 4.3.2 Por simetría



$$\begin{aligned} A &= 2\int_0^1 \frac{x(1-x^2)}{(x^2+1)^{3/2}}\,dx = 2\int_0^{\pi/4} \frac{\operatorname{tg} t(1-\operatorname{tg}^2 t)}{\sec t}\,dt \\ &= 2\int_0^{\pi/4} (2\operatorname{sen} t - \frac{\operatorname{sen} t}{\cos^2 t})\,dt = 2(3-2\sqrt{2}). \end{aligned}$$

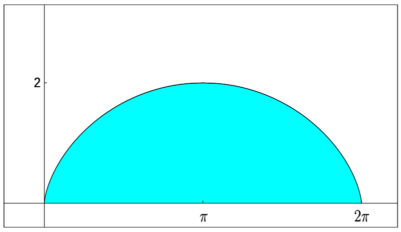
Problema 4.3.3 Utilizamos la fórmula del área en coordenadas polares, $A = \dfrac{1}{2}\displaystyle\int_{\theta_1}^{\theta_2} \rho^2(\theta)\,d\theta$.

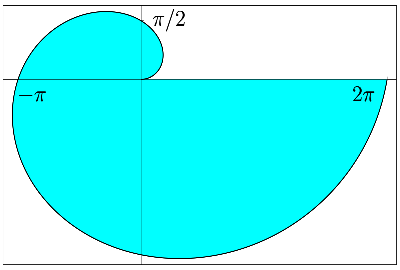
i) $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} \Big(a^2(t-\operatorname{sen} t)\operatorname{sen} t - a^2(1-\cos t)^2\Big)\,dt = 3\pi a^2.$

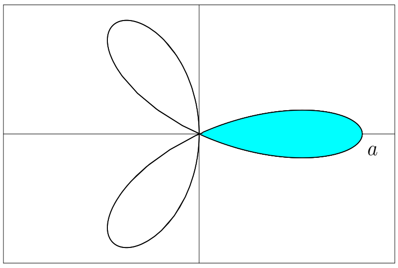
ii) $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} a^2\theta^2\,d\theta = \dfrac{4}{3}\pi^3 a^2.$

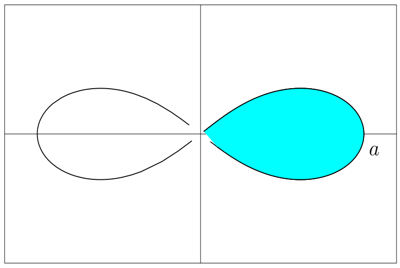
iii) $A = \displaystyle\int_0^{\pi/6} a^2\cos^2 3\theta\,d\theta = \dfrac{1}{12}\pi a^2.$

iv) $A = \displaystyle\int_0^{\pi/4} a^2\cos 2\theta\,d\theta = \dfrac{1}{2}a^2.$

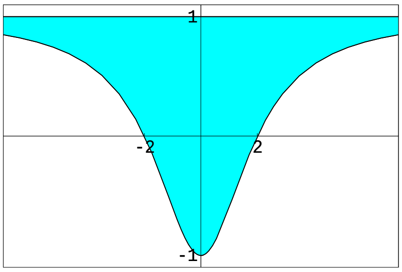








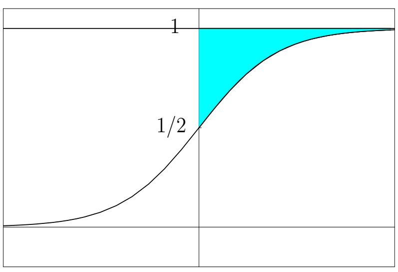
Problema 4.3.4

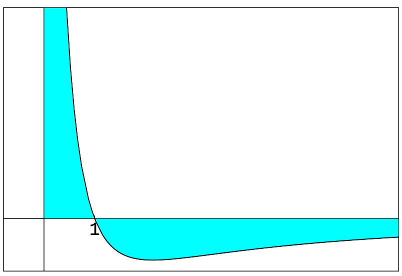


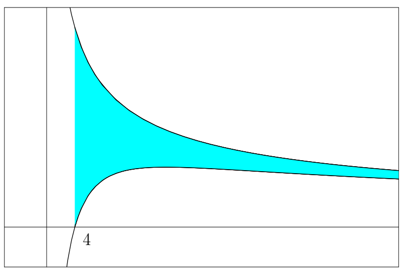
i) $A = 2\displaystyle\int_0^\infty \Big(1-\frac{x^2-4}{x^2+4}\Big)\,dx = 8\operatorname{arc\,tg}(x/2)\Big|_0^\infty = 4\pi.$

ii) $A = \displaystyle\int_0^\infty \Big(1-\frac{1}{1+e^{-x}}\Big)\,dx = -\log(1+e^{-x})\Big|_0^\infty = \log 2.$

iii) $A = \displaystyle\int_0^1 f - \int_1^\infty f$, donde $\displaystyle\int f = \int \frac{1-x}{(x+1)^2\sqrt{x}}\,dx = 2\int \cos 2u\,du = \operatorname{sen} 2u = \frac{2\sqrt{x}}{1+x}$, mediante el cambio $x = \operatorname{tg}^2 u$; así $A = \dfrac{2\sqrt{x}}{1+x}\Big|_0^1 - \dfrac{2\sqrt{x}}{1+x}\Big|_1^\infty = 2$. (Si realizamos también el cambio en los límites de integración, quedaría $A = \operatorname{sen} 2u\Big|_0^{\pi/4} - \operatorname{sen} 2u\Big|_{\pi/4}^{\pi/2} = 2$.) *iv)* $A = 8\displaystyle\int_4^\infty \frac{1}{(x+4)\sqrt{x}}\,dx = 8\int_2^\infty \frac{1}{t^2+4}\,dt = 2\pi.$







Problema 4.3.5 $A = \int_0^1 (\sqrt{x} - x^2)\,dx = \frac{1}{3};\ \ V = \pi \int_0^1 (x - x^4)\,dx = \frac{3\pi}{10}$.

Problema 4.3.6 *i)* $V = \pi \int_0^{2\pi} (1 + \operatorname{sen} x)^2\,dx = 3\pi^2$;

ii) $V = 2\pi \int_0^a \Big((2a + \sqrt{a^2 - x^2}\,)^2 - (2a - \sqrt{a^2 - x^2}\,)^2\Big)\,dx = 16\pi a \int_0^a \sqrt{a^2 - x^2}\,dx = 8\pi^2 a^3$;

iii) $V = 2\pi \int_0^{2R} (4R^2 - x^2)\,dx - 2\pi \int_0^R (4R^2 - x^2)\,dx = \frac{28}{3}\pi R^3$;

iv) $V = \pi \int_0^{\pi} (x^2 - \operatorname{sen}^2 x)\,dx = \frac{1}{6}\pi^2(2\pi^2 - 3)$;

Problema 4.3.7 *i)* $V_x = 2\pi \int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx = \frac{4}{3}\pi a b^2$;
$V_y = 4\pi \int_0^a \frac{b}{a} x \sqrt{a^2 - x^2}\,dx = \frac{4}{3}\pi a^2 b$. *ii)* $A = 2 \int_0^a \frac{2b}{a}\sqrt{a^2 - x^2}\,dx = 2\pi ab$.

Problema 4.3.8 *i)* $A = 4 \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \pi ab$;

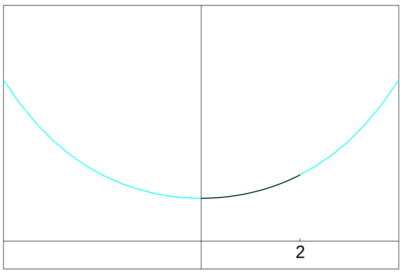
ii) $V = 2 \int_0^b \pi ac(1 - \frac{y^2}{b^2})\,dy = \frac{4}{3}\pi abc$. *iii)* $c = b \Rightarrow V_x = \frac{4}{3}\pi a b^2;\ \ c = a \Rightarrow V_y = \frac{4}{3}\pi a^2 b$.

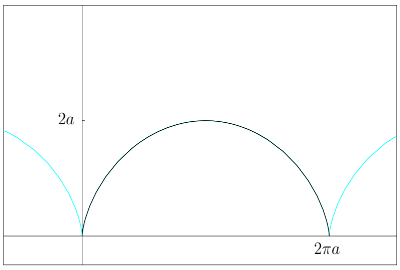
Problema 4.3.9 *i)* $L = \int_0^2 \sqrt{1 + \Big(\frac{e^{x/2} - e^{-x/2}}{2}\Big)^2}\,dx = \int_0^2 \frac{1}{2}(e^{x/2} + e^{-x/2})\,dx = e - \frac{1}{e}$; (escribiendo $y = 2\cosh(x/2)$ la integral es más fácil, $L = 2\operatorname{senh} 1$).

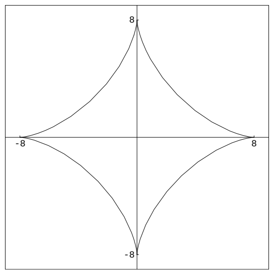
ii) $L = \int_0^{2\pi} \sqrt{a^2(1 - \cos t)^2 + a^2 \operatorname{sen}^2 t}\,dt = a \int_0^{2\pi} \sqrt{2 - 2\cos t}\,dt = 2a \int_0^{2\pi} |\operatorname{sen} t|\,dt$

$= 4a \int_0^{\pi} \operatorname{sen} t\,dt = 8a$.

iii) $L = 4 \int_0^8 \sqrt{1 + x^{-2/3}(4 - x^{2/3})}\,dx = 8 \int_0^8 x^{-1/3}\,dx = 48$.







iv) $L = \int_{a/2}^a \sqrt{1 + \frac{a^2 - x^2}{x^2}}\,dx = a \log 2$.

v) $L = \int_0^{2\pi} \sqrt{(1 + \cos\theta)^2 + \operatorname{sen}^2\theta}\,d\theta = \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\,d\theta = 2 \int_0^{2\pi} |\cos\theta|\,d\theta = 8$.