uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

SOLUCIONES DE LOS EJERCICIOS DE CÁLCULO II PARA GRADOS DE INGENIERÍA

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3 Integración en $\mathbb{R}^n$

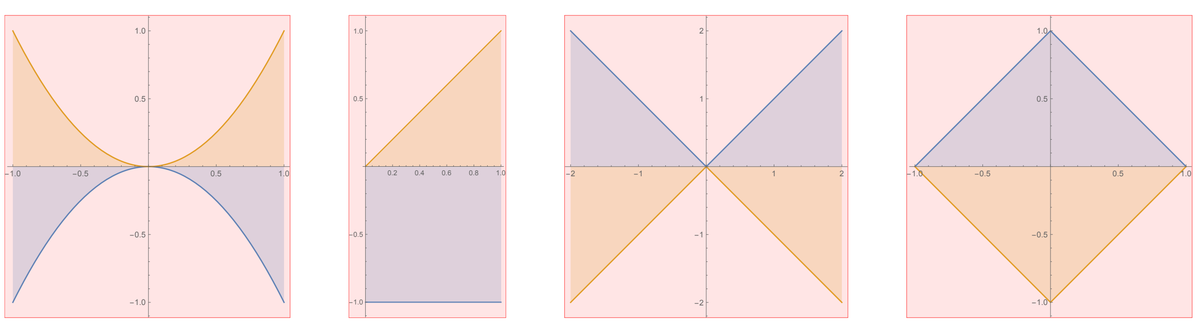
3.1 Integral múltiple.

Problema 3.1

i) $\int_0^1\int_0^1 xy(x+y)\,dydx = \int_0^1 (\frac{x^2}{2}+\frac{x}{3})\,dx = \frac{1}{3}$; *ii)* $\int_0^1\int_0^1 (x^3+3x^2y+y^3)\,dydx = \int_0^1 (x^3+\frac{3x^2}{2}+\frac{1}{4})\,dx = 1$; *iii)* $\int_0^\pi\int_0^\pi (\operatorname{sen}^2 x \operatorname{sen}^2 y)\,dydx = \int_0^\pi (\frac{\pi}{2}\operatorname{sen}^2 x)\,dx = \frac{\pi^2}{4}$;
iv) $\int_0^{\pi/2}\int_0^{\pi/2} (\operatorname{sen}(x+y))\,dydx = \int_0^{\pi/2} (\cos x - \cos(x+\pi/2))\,dx = 2$; *v)* $\int_{-1}^1\int_0^{\pi/2} (x\operatorname{sen} y - ye^x)\,dydx = \int_{-1}^1 (x-\frac{\pi^2}{8}\mathrm{e}^x)\,dx = -\frac{\pi^2(e-e^{-1})}{8}$.

Problema 3.2

i) $\int_{-1}^1\int_{-x^2}^{x^2} (x^2-y)\,dydx = \int_{-1}^1 2x^4\,dx = \frac{4}{5}$; *ii)* $\int_0^1\int_{-1}^x (xy-x^3)\,dydx = \int_0^1 (-x^4-\frac{x^3}{2}-\frac{x}{2})\,dx = -\frac{23}{40}$; *iii)* $\int_{-2}^0\int_x^{-x} (2x-\operatorname{sen}(x^2y))\,dydx + \int_0^2\int_{-x}^x (2x-\operatorname{sen}(x^2y))\,dydx = -\int_{-2}^0 4x^2\,dx + \int_0^2 4x^2\,dx = 0$;
iv) $\int_{-1}^0\int_{-x-1}^{x+1} y\operatorname{sen} x\,dydx + \int_0^1\int_{x-1}^{1-x} y\operatorname{sen} x\,dydx = 0$.



Problema 3.3

i) $m|D| \le \int_D f \le M|D|$, donde $M = \max_D f = 5$, $m = \min_D f = 1$, $|D| = 4\pi$; *ii)* en este caso, $M = 12$, $m = 0$, $|A| = 4$; *iii)* dividimos $A = \bigcup_{k=1}^4 A_k$, donde $A_1 = [0,1]\times[1,2]$, $A_2 = [1,2]\times[1,2]$, $A_3 = [0,1]\times[2,3]$, $A_4 = [1,2]\times[2,3]$; se tiene $m_1 = 0$, $M_1 = 2$, $m_2 = 1$, $M_2 = 8$, $m_3 = 0$, $M_3 = 3$, $m_4 = 2$, $M_4 = 12$, con $|A_k| = 1$ para todo k; así $0+1+0+2 \le \int_A f \le 2+8+3+12$.

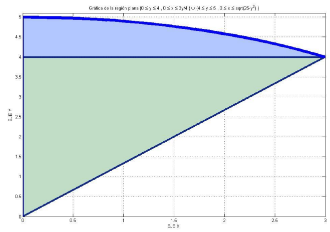
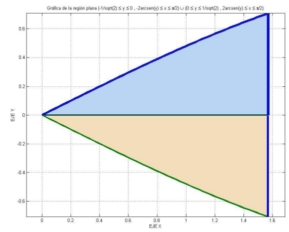
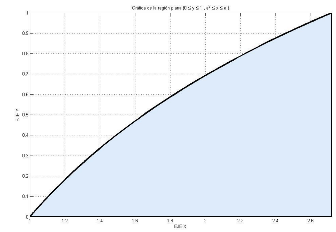
Problema 3.4

$$\int_0^1\int_0^x x\,dydx + \int_0^1\int_x^1 y\,dydx = \frac{2}{3}.$$

Problema 3.5

$$i)\ \int_0^4\int_0^{3y/4} f(x,y)\,dxdy + \int_4^5\int_0^{\sqrt{25-y^2}} f(x,y)\,dxdy;\ \ ii)\ \int_0^1\int_x^1 f(x,y)\,dydx;$$

$$iii)\ \int_{-1/\sqrt{2}}^0\int_{-2\,\mathrm{arcsen}(y)}^{\pi/2} f(x,y)\,dxdy + \int_0^{1/\sqrt{2}}\int_{2\,\mathrm{arcsen}(y)}^{\pi/2} f(x,y)\,dxdy;\ iv)\ \int_0^1\int_{\mathrm{e}^y}^{\mathrm{e}} f(x,y)\,dxdy.$$

Problema 3.6

$$\int_R f = \int_0^1\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2}}\,dydx = \int_0^1 2\,dx = 2 = \int_0^2\int_0^{\sqrt{2y-y^2}} \frac{1}{\sqrt{1-x^2}}\,dxdy = \cdots \text{ más difícil;}$$

$$\int_R g = \int_0^1\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} \mathrm{sen}(y-1)\,dydx = 0 = \int_0^2\int_0^{\sqrt{2y-y^2}} \mathrm{sen}(y-1)\,dxdy = \cdots \text{ más difícil.}$$

Problema 3.7

$$\int_0^\pi\int_0^y \frac{\mathrm{sen}\,y}{y}\,dxdy = \int_0^\pi \mathrm{sen}\,y\,dy = 2.$$

Problema 3.8

$$i)\ \text{por simetría } 3\int_0^1\int_0^1\int_0^1 z^2\,dxdydz = 3\int_0^1 z^2\,dz = 1;\ ii)\ \int_0^1\int_0^1 (y^2+2yz+z^2+y+z+\frac{1}{3})\,dydz = \int_0^1 (z^2+2z+\frac{7}{6})\,dz = \frac{5}{2}.$$

Problema 3.9

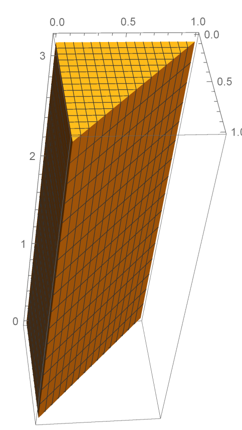
$$i)\ \int_0^1\int_0^1\int_0^1 x^3\,dxdydz = \int_0^1 x^3\,dx = \frac{1}{4};\ ii)\ \int_0^1\int_0^1\int_0^1 \mathrm{e}^{-xy}y\,dxdydz = \int_0^1 (1-\mathrm{e}^{-y})\,dy = \frac{1}{\mathrm{e}};$$

$$iii)\ \int_0^1\int_{-1}^1\int_1^2 (2x+3y+z)\,dxdydz = \int_0^1\int_{-1}^1 (3+3y+z)\,dydz = \int_0^1 2(3+z)\,dz = 7;$$

$$iv)\ \int_0^1\int_0^1\int_0^1 ze^{x+y}\,dxdydz = (e-1)\int_0^1\int_0^1 ze^y\,dydz = (e-1)^2\int_0^1 z\,dz = \frac{(e-1)^2}{2}.$$

Problema 3.10

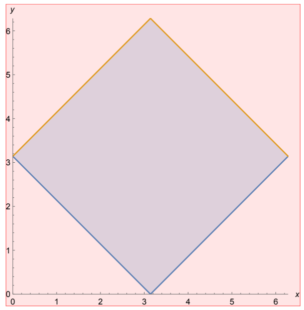
$$\int_0^1\int_0^{1-x}\int_0^\pi x^2\cos x\,dzdydx = \pi(4\,\mathrm{sen}\,1 + 5\cos 1 - 6).$$

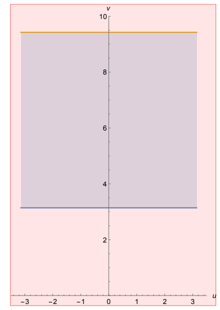


3.2 Cambios de variables en la integral múltiple.

Problema 3.11

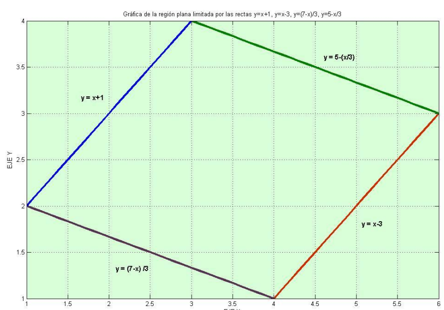
Poniendo $\left\{ \begin{array}{l} u = x - y \\ v = x + y \end{array} \right\}$ el jacobiano es $J = 1/2$, la integral es $\dfrac{1}{2}\displaystyle\int_{-\pi}^{\pi}\int_{\pi}^{3\pi} u^2 \operatorname{sen}^2 v \, dv du = \dfrac{\pi^4}{3}$.

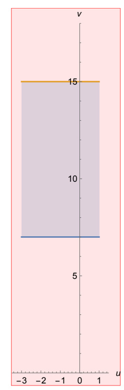




Problema 3.12

Poniendo $\left\{ \begin{array}{l} u = y - x \\ v = 3y + x \end{array} \right\}$ el jacobiano es $J = 1/4$, la integral es $\dfrac{1}{4}\displaystyle\int_{-3}^{1}\int_{7}^{15} u \, dv du = -8$.

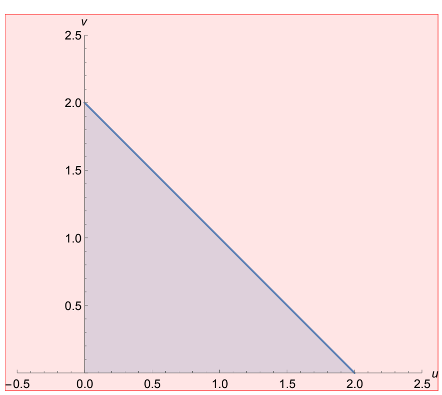


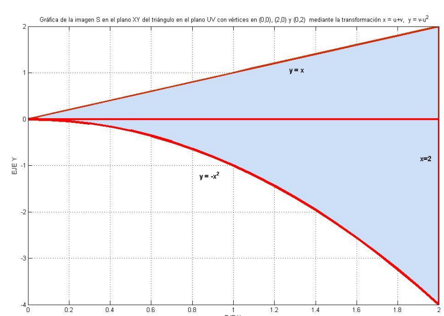


Problema 3.13

i) $J(u,v) = \begin{vmatrix} 1 & 1 \\ -2u & 1 \end{vmatrix} = 1 + 2u$; *iii)* $A = \displaystyle\int_S 1 \, dx dy = \int_0^2 \int_0^{2-v} (1+2u) \, du dv = \frac{14}{3}$;

iv) $\displaystyle\int_0^2 \int_0^{2-v} \frac{1+2u}{(1+u+u^2)^2} \, du dv = 2 + \frac{\sqrt{3}}{9}(\pi - 6 \operatorname{arctg}(5/\sqrt{3}))$.





Problema 3.14

$$\int_0^{\pi/2} \int_1^2 r \log(r^2) \, dr d\theta = 2\pi(\log 2 - 3/8).$$

Problema 3.15

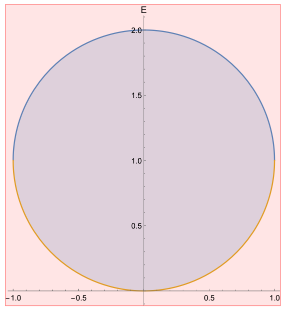
Poniendo $\left\{ \begin{array}{l} x = ar\cos t \\ y = br\,\mathrm{sen}\, t \end{array} \right\}$ el jacobiano es $J = abr$, y la integral es

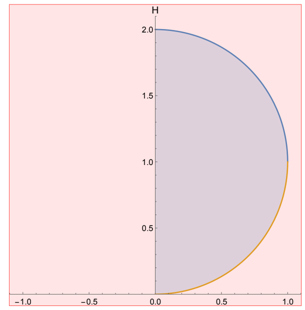
$$\int_0^{2\pi}\int_0^1 \left(\frac{\mathrm{sen}^4 t r^2}{1+r^2} + ab^2r^3\cos t\,\mathrm{sen}^2 t\right) abr\,drdt = \frac{3}{8}\pi ab(1-\log 2).$$

Problema 3.16

f es impar en x y E es simétrico en esa variable, luego $\int_E f = 0$;

$$\int_H f = \int_0^{\pi/2}\int_0^{2\,\mathrm{sen}\,\theta} \cos\theta e^r r\,drd\theta = 2.$$



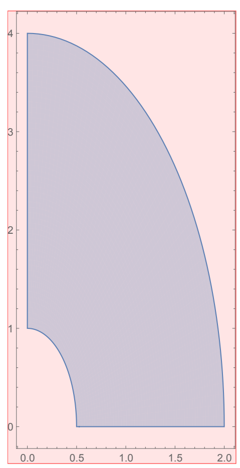


Problema 3.17

$$\int_0^{\pi/2}\int_0^{2\,\mathrm{sen}\,\theta} \frac{r\sqrt{1+\mathrm{sen}^2\theta}}{\mathrm{sen}\,\theta}\,drd\theta = 1 + \pi/2.$$

Problema 3.18

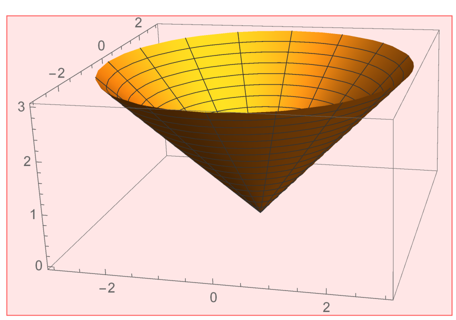
Poniendo $\left\{ \begin{array}{l} x = \frac{r}{2}\cos t \\ y = r\,\mathrm{sen}\, t \end{array} \right\}$ el jacobiano es $J = \frac{r}{2}$, y la integral es $\int_0^{\pi/2}\int_1^4 \frac{1}{4}\cos t\,drdt = \frac{3}{4}$.



Problema 3.19

i) $\int_0^{2\pi}\int_0^3\int_r^3 r\sqrt{r^2+z^2}\,dzdrd\theta = \frac{27}{2}\pi(2\sqrt{2}-1)$; *ii)* $\int_0^{2\pi}\int_0^3\int_0^z r\sqrt{9-r^2}\,drdzd\theta = 54\pi - \frac{81}{8}\pi^2$;

iii) $\int_0^{2\pi}\int_0^3\int_0^z rze^{r^2+z^2}\,drdzd\theta = \frac{\pi}{4}(e^9-1)^2$.



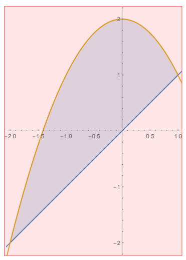
Problema 3.20

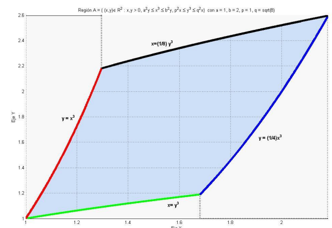
$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \mathrm{e}^{-\rho^2}\rho^2 \operatorname{sen}\varphi\, d\rho d\varphi d\theta = \frac{\pi}{3}(2-\sqrt{2})(1-\mathrm{e}^{-27}).$$ Ver figura del Problema 3.28.

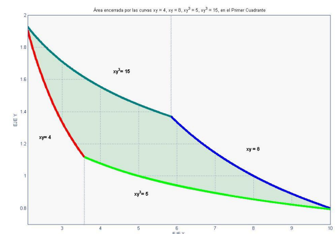
3.3 Aplicaciones.

Problema 3.21

i) $\displaystyle\int_{-2}^{1}\int_{x}^{2-x^2} dydx = \frac{9}{2}$; *ii)* poniendo $\left\{\begin{array}{l} u = x^3/y \\ v = y^3/x \end{array}\right\}$, el jacobiano es $J = \dfrac{1}{8\sqrt{uv}}$, y la integral queda $\displaystyle\int_{a^2}^{b^2}\int_{p^2}^{q^2} \frac{1}{8\sqrt{uv}}\, dvdu = \frac{1}{2}(b-a)(q-p)$; *iii)* poniendo $\left\{\begin{array}{l} u = xy \\ v = xy^3 \end{array}\right\}$, el jacobiano es $J = \dfrac{1}{2v}$, y la integral queda $\displaystyle\int_4^8\int_5^{15} \frac{1}{2v}\, dvdu = 2\log 3$.



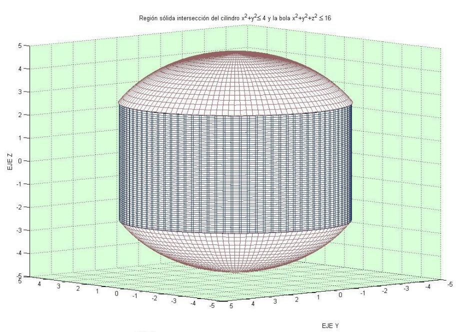


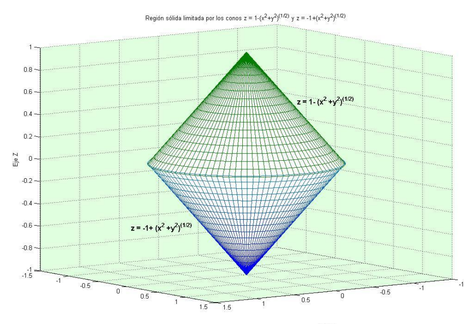


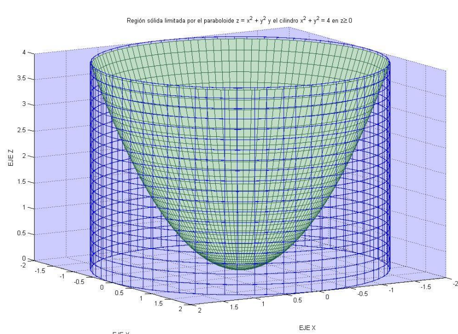
Problema 3.22

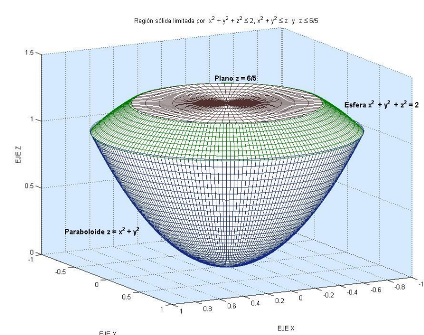
i) $\displaystyle 2\int_0^{2\pi}\int_0^2\int_0^{\sqrt{16-r^2}} r\, dzdrd\theta = \frac{32}{3}\pi(8-3\sqrt{3})$; *ii)* $\displaystyle 2\int_0^{2\pi}\int_0^1\int_0^{1-r} r\, dzdrd\theta = \frac{2\pi}{3}$;

iii) $\displaystyle\int_0^{2\pi}\int_0^2\int_0^{r^2} r\, dzdrd\theta = 8\pi$; *iv)* $\displaystyle\int_0^{2\pi}\int_0^1\int_0^{\sqrt{z}} r\, drdzd\theta + \int_0^{2\pi}\int_1^{6/5}\int_0^{\sqrt{2-z^2}} r\, drdzd\theta = \frac{493\pi}{750}$.





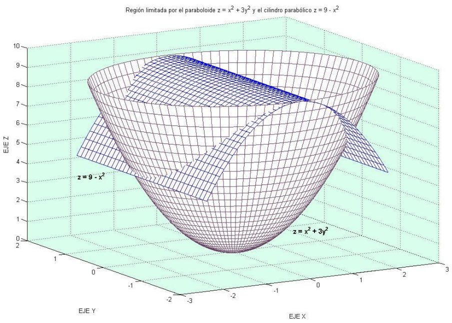


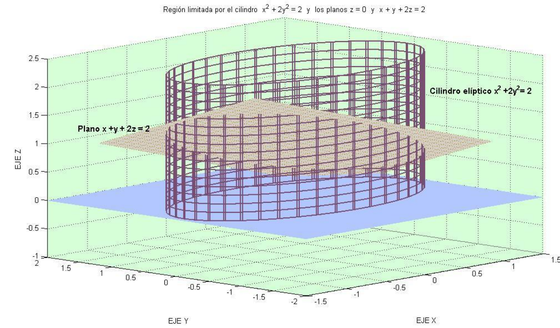


Problema 3.23

i) primero observamos que la intersección de las superficies $9-x^2 = x^2+3y^2$ es la elipse $2x^2+3y^2 = 9$; el volumen es entones, por simetría, $\displaystyle 4\int_0^{3/\sqrt{2}}\int_0^{\sqrt{(9-2x^2)/3}}\int_{x^2+3y^2}^{9-x^2} dzdydx = \frac{27}{4}\pi\sqrt{6}$;

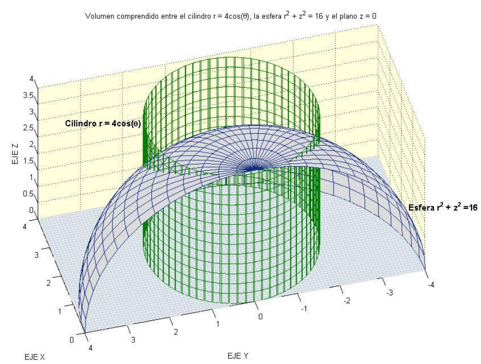
ii) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{1-x^2/2}}^{\sqrt{1-x^2/2}}\int_0^{(2-x-y)/2} dzdydx = \pi\sqrt{2}$.





Problema 3.24
$8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dzdydx = \frac{4}{3}\pi abc$; también se podrían haber utilizado coordenadas esféricas adaptadas, $x = a\rho\cos\theta\operatorname{sen}\varphi$, $y = b\rho\operatorname{sen}\theta\operatorname{sen}\varphi$, $z = c\rho\cos\varphi$, con jacobiano $J = abc\rho^2\operatorname{sen}\varphi$, y volumen $\int_0^{2\pi}\int_0^{\pi}\int_0^{R} abc\rho^2\operatorname{sen}\varphi\, d\rho d\varphi d\theta = \frac{4}{3}\pi abc$; en el caso de la bola de radio R quedaría $\frac{4}{3}\pi R^3$.

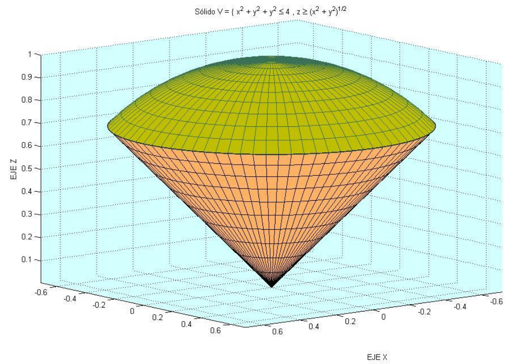
Problema 3.25
$\int_{-\pi/2}^{\pi/2}\int_0^{4\cos\theta}\int_0^{\sqrt{16-r^2}} r\, dzdrd\theta = \frac{64}{9}(3\pi - 4)$.



Problema 3.26
$\int_0^{\pi/2}\int_0^2 kr^2\, drd\theta = \frac{4k\pi}{3}$.

Problema 3.27
i) $M = \int_{-2}^{1}\int_{x^2}^{2-x} \rho\, dydx = \frac{9\rho}{2}$; $x_{CM} = \frac{2}{9\rho}\int_{-2}^{1}\int_{x^2}^{2-x} \rho x\, dydx = -\frac{1}{2}$, $y_{CM} = \frac{2}{9}\int_{-2}^{1}\int_{x^2}^{2-x} y\, dydx = \frac{8}{5}$; *ii)* $M = 2\int_0^2\int_{x^2-3}^{5-x^2} \rho\, dydx = \frac{64\rho}{3}$; $x_{CM} = 0$ por simetría, $y_{CM} = \frac{3}{32}\int_0^2\int_{x^2-3}^{5-x^2} y\, dydx = 1$; *iii)* $M = \int_0^{\pi}\int_0^{\operatorname{sen}^2 x} \rho\, dydx = \frac{\pi\rho}{2}$; $x_{CM} = \frac{2}{\pi}\int_0^{\pi}\int_0^{\operatorname{sen}^2 x} x\, dydx = \frac{\pi}{2}$ (o por simetría); $y_{CM} = \frac{2}{\pi}\int_0^{\pi}\int_0^{\operatorname{sen}^2 x} y\, dydx = \frac{3}{8}$; *iv)* $M = \int_0^{\pi/4}\int_{\operatorname{sen} x}^{\cos x} \rho\, dydx = (\sqrt{2}-1)\rho$; $x_{CM} = \frac{1}{\sqrt{2}-1}\int_0^{\pi/4}\int_{\operatorname{sen} x}^{\cos x} x\, dydx = \frac{\pi}{4}(2+\sqrt{2}) - \sqrt{2} - 1$; $y_{CM} = \frac{1}{\sqrt{2}-1}\int_0^{\pi/4}\int_{\operatorname{sen} x}^{\cos x} y\, dydx = \frac{\sqrt{2}+1}{4}$.

Problema 3.28
En coordenadas esféricas, como $x^2 + y^2 = \rho^2\operatorname{sen}^2\varphi$, el momento de inercia es
$\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \alpha\rho^2\operatorname{sen}^2\varphi\rho^2\operatorname{sen}\varphi\, d\rho d\varphi d\theta = \frac{16}{15}\pi\alpha(8 - 5\sqrt{2})$.

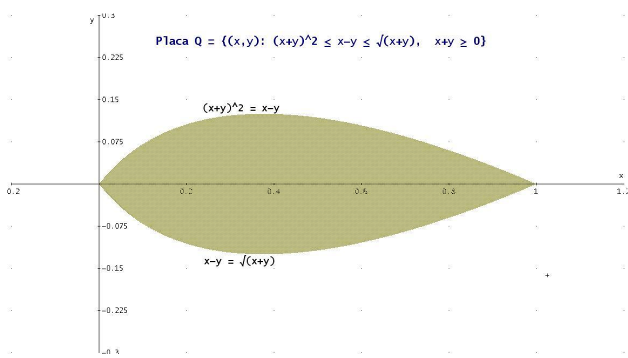


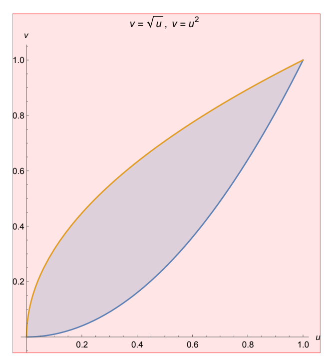
Problema 3.29
Por simetría, la masa es $2\int_{-1}^{1}\int_{x}^{1}(y-x)\,dydx=\dfrac{8}{3}$.

Problema 3.30
$M=\displaystyle\int_1^2\int_1^3 xy\,dydx=6$; $x_{CM}=\dfrac{1}{6}\displaystyle\int_1^2\int_1^3 x^2y\,dydx=\dfrac{14}{9}$; $y_{CM}=\dfrac{1}{6}\displaystyle\int_1^2\int_1^3 xy^2\,dydx=\dfrac{13}{6}$.

Problema 3.31
$M=\displaystyle\int_0^1\int_{-x}^{x}y^2\,dydx=\dfrac{1}{6}$; $x_{CM}=6\displaystyle\int_0^1\int_{-x}^{x}xy^2\,dydx=\dfrac{4}{5}$; $y_{CM}=6\displaystyle\int_0^1\int_{-x}^{x}y^3\,dydx=0$ (o por simetría); $I_x=\displaystyle\int_0^1\int_{-x}^{x}y^4\,dydx=\dfrac{1}{15}$; $I_y=\displaystyle\int_0^1\int_{-x}^{x}x^2y^2\,dydx=\dfrac{1}{9}$.

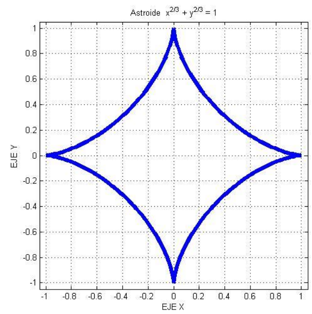
Problema 3.32
Despejando la variable y el conjunto se puede escribir también como $Q=\{0\le x\le 1,\ \frac{1}{2}(2x+1-\sqrt{8x+1})\le y\le\frac{1}{2}(\sqrt{8x+1}-(2x+1))\}$; por tanto la masa sería $M=\displaystyle\int_0^1\int_{\frac{1}{2}(2x+1-\sqrt{8x+1})}^{\frac{1}{2}(\sqrt{8x+1}-(2x+1))}(x^2-y^2)\,dydx$. Sin embargo esta integral no es fácil de calcular. Por otro lado, la propia definición original sugiere el cambio de variables $\left\{\begin{array}{l}u=x+y\\ v=x-y\end{array}\right\}$, mediante el cual el conjunto queda $Q^*=\{0\le u\le 1,\ u^2\le v\le\sqrt{u}\}$ y la densidad $\rho=uv$; ahora el jacobiano es $J=\dfrac{1}{2}$ y la masa es $M=\dfrac{1}{2}\displaystyle\int_0^1\int_{u^2}^{\sqrt{u}}uv\,dvdu=\dfrac{1}{24}$.

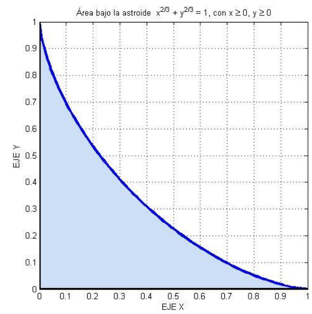




Problema 3.33
i) $\displaystyle\int_0^1\int_0^{(1-x^{2/3})^{3/2}}dydx=\int_0^1(1-x^{2/3})^{3/2}\,dx=\int_0^{\pi/2}3\operatorname{sen}^2u\cos^4u\,du=\dfrac{3\pi}{32}$, después del cambio $x^{1/3}=\operatorname{sen}u$; también se puede usar la transformación en la integral original de dos variables $\left\{\begin{array}{l}x=r\cos^3t\\ y=r\operatorname{sen}^3t\end{array}\right\}$ sugerida por la descripción del conjunto; el jacobiano es $J=3r\operatorname{sen}^2t\cos^2t$ y el área

$\int_0^1 \int_0^{\pi/2} 3r \operatorname{sen}^2 t \cos^2 t \, dt dr = \dfrac{3\pi}{32}$ *ii)* con la transformación anterior, y por simetría, $x_{CM} = y_{CM} = \dfrac{32}{3\pi} \int_0^1 \int_0^{\pi/2} 3r^2 \operatorname{sen}^2 t \cos^5 t \, dt dr = \dfrac{256}{315\pi}$.



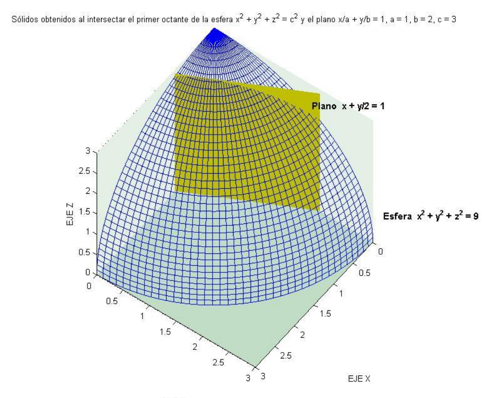


Problema 3.34

La distancia de un punto $P = (x, y)$ a la recta $r \equiv y = x$ es $d(P, r) = \dfrac{|x - y|}{\sqrt{2}}$, por tanto el momento de inercia es $I_r = \int_0^1 \int_0^1 \dfrac{(x-y)^2}{2} \rho \, dx dy = \dfrac{\rho}{12}$.

Problema 3.35

$M_1 = \int_0^a \int_0^{b(1-x/a)} \int_0^{\sqrt{c^2 - x^2 - y^2}} z \, dz dy dx = \dfrac{1}{24} ab(6c^2 - a^2 - b^2)$; la masa del primer octante entero (en esféricas) es $M = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^c \rho \operatorname{sen} \varphi \rho^2 \cos \varphi \, d\rho d\varphi d\theta = \dfrac{\pi c^4}{16}$; finalmente $M_2 = M - M_1$.



Problema 3.36

i) $T(x, y, z) = \alpha(x^2 + y^2 + z^2)$, $T_m = \dfrac{1}{8} \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \alpha(x^2 + y^2 + z^2) \, dx dy dz = \alpha$; *ii)* $T(x, y, z) = \alpha$ en la esfera unidad $x^2 + y^2 + z^2 = 1$.

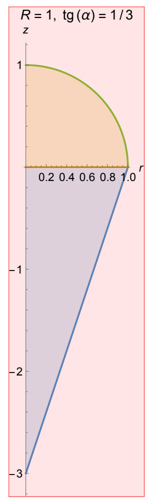
Problema 3.37

$M = \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \rho^4 \operatorname{sen} \varphi \, d\rho d\varphi d\theta = \dfrac{2\pi R^5}{5}$; por simetría $x_{CM} = y_{CM} = 0$;
$z_{CM} = \dfrac{5}{2\pi R^5} \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \rho^5 \operatorname{sen} \varphi \cos \varphi \, d\rho d\varphi d\theta = \dfrac{5R}{12}$.

Problema 3.38
Si consideramos como plano $z = 0$ el plano que separa el helado del barquillo, en coordenadas cilíndricas tenemos que el helado viene descrito por el conjunto $H = \{0 \le r \le R, 0 \le z \le \sqrt{R^2 - r^2}\}$, mientras que el cucurucho es $C = \{0 \le r \le R, \lambda(r - R) \le z \le 0\}$, donde $\lambda = \dfrac{1}{\operatorname{tg}\alpha}$ indica la abertura del mismo; entonces la coordenada z del centro de masas es

$$0 = z_{CM} = \frac{1}{M}\left(\int_0^{2\pi}\int_0^R\int_0^{\sqrt{R^2-r^2}} \rho_h rz\, dzdrd\theta + \int_0^{2\pi}\int_0^R\int_{\lambda(r-R)}^0 \rho_c rz\, dzdrd\theta\right),$$

o lo que es lo mismo, $\dfrac{\rho_c}{\rho_h} = \dfrac{\int_0^R\int_0^{\sqrt{R^2-r^2}} rz\, dzdr}{-\int_0^R\int_{\lambda(r-R)}^0 rz\, dzdr} = \dfrac{\pi R^4/4}{\pi R^4\lambda^2/12} = 3\operatorname{tg}^2\alpha.$



Problema 3.39
i) el conjunto entre $z = \pm r$, $z = 2 \pm 3r$, es decir, $\{0 \le r \le 1/2,\ r \le z \le 2 - 3r\}$; *ii)* Volumen$=\int_0^{2\pi}\int_0^{1/2}\int_r^{2-3r} r\, dzdrd\theta = \frac{\pi}{6}$; (o volumen de dos conos de radio $1/2$ y alturas $1/2$ y $3/2$ respectivamente, $V = \frac{1}{3}\pi\left(\frac{1}{2}\right)^2\left(\frac{1}{2}+\frac{3}{2}\right) = \frac{\pi}{6}$); $x_{CM} = y_{CM} = 0$ por simetría, $z_{CM} = \frac{6}{\pi}\int_0^{2\pi}\int_0^{1/2}\int_r^{2-3r} rz\, dzdrd\theta = \frac{3}{4}$.