Applied Differential Calculus
Lecture 1: First-order ordinary differential equations

Authors:
Manuel Carretero, Luis L. Bonilla, Filippo Terragni, Sergei Iakunin,
Rocío Vega

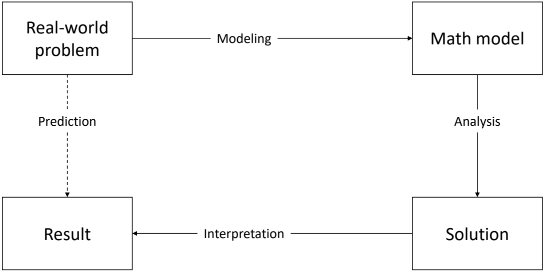
Bachelor's Degree in Computer Science and Engineering and
Dual Bachelor in Computer Science and Engineering and Business Administration.

uc3m Universidad Carlos III de Madrid

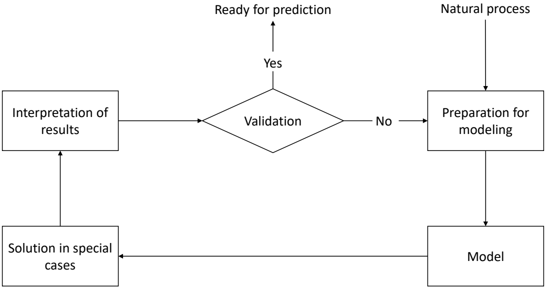
Outline

- Modeling.
- General concepts. Existence and uniqueness. Direction field for a first-order ODE.
- Exact solutions for special ODEs.
- Numerical solutions: Euler, Heun and Runge-Kutta methods.
- Long term behavior of solutions. Qualitative analysis of solutions.

Problem solving via mathematical modeling

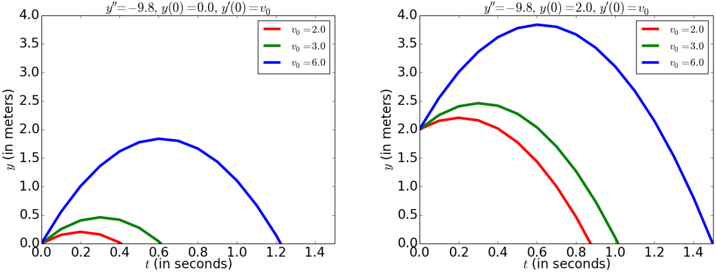


Problem solving via mathematical modeling



Galileo's model

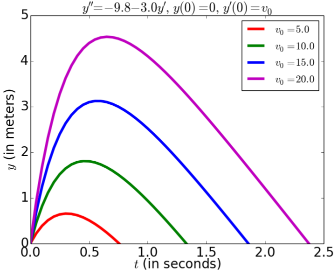
$$y'' = -g, \quad 0 \le t \le T,$$
$$y(0) = 0, \quad y'(0) = v_0.$$



(a) $y_0 = 0,\ v_0 = 2,\ 4,\ 6.$ (b) $y_0 = 2,\ v_0 = 2,\ 4,\ 6.$

Newton's model

$$m\,y'' = -mg - cy', \quad 0 \le t \le T,$$
$$y(0) = 0, \quad y'(0) = v_0.$$



(a) $c/m = 3$; (b) $c/m = 10$.

Modeling a fish population, 1

$$y' = (\text{Birth rate}) - [(\text{Death rate}) + (\text{Harvesting rate})],$$

measured in tons per year. Simplest model: birth and death rates proportional to $y(t)$. $a = b - d > 0$ is growth rate:

$$y' = ay - H, \quad y(t_0) = y_0.$$

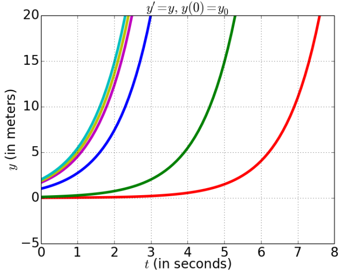
Solution by integrating factor:

$$-e^{\mu(t)}H = e^{\mu(t)}(y' - ay) = \frac{d}{dt}\left(e^{\mu(t)}y\right) = e^{\mu(t)}(y' + \mu' y) \Longrightarrow \mu' = -a.$$

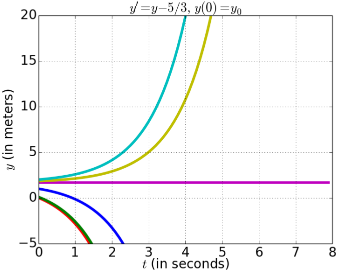
Then $\mu = -at$ and we get $e^{-at}y = c + \frac{H}{a}e^{-at}$ or $y(t) = \frac{H}{a} + ce^{at}$.

$$y(t) = \frac{H}{a} + \left(y_0 - \frac{H}{a}\right)e^{a(t-t_0)}.$$

Modeling a fish population, 2



(a)



(b)

Exponential population growth with no harvesting: IVP with $a = 1$, $H = 0$ in (a) and $H = 5/3$ in (b), and various values for y_0.

$$y(t) = \frac{H}{a} + \left(y_0 - \frac{H}{a}\right) e^{at} \quad (t_0 = 0).$$

Fish population with logistic growth

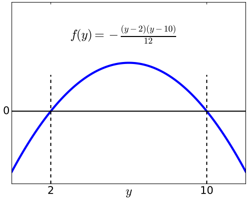
Limitation to growth due to overcrowding

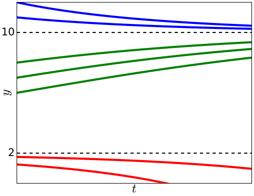
$$y' = ay - cy^2 - H, \quad y(0) = y_0, \quad t \geq 0.$$

Solution formula is complicated. We find it later by separation of variables.
Qualitative behavior of solution curves. We do one example: $a = 1$, $c = \frac{1}{12}$, $H = \frac{5}{3}$, so that

$$y' = -\frac{1}{12}(y-10)(y-2) := f(y), \quad y(0) = y_0, \quad t \geq 0.$$

$f(y) = 0$ for $y = 10$, $y = 2$. These are equilibrium solutions.





Separation of variables

By analysis of phase line, solution curves go to equilibrium solutions with $f' < 0$ as $t \to +\infty$ and to equilibrium solutions with $f' > 0$ as $t \to -\infty$.

Separation of variables:

$$y' = -\frac{1}{12}(y-10)(y-2) \Longrightarrow -\frac{dt}{12} = \frac{dy}{(y-10)(y-2)}.$$

$$-\frac{t}{12} + C = \frac{1}{8}\int \left(\frac{1}{y-10} - \frac{1}{y-2}\right) dy = \frac{1}{8}\ln\left|\frac{y-10}{y-2}\right|.$$

$$\frac{10-y}{y-2} = Ke^{-2t/3} \Longrightarrow y(t) = \frac{10+2Ke^{-2t/3}}{1+Ke^{-2t/3}},$$

with $K = \frac{10-y_0}{y_0-2}e^{2t_0/3}$ provided $2 < y_0 < 10$.
As t increases from $-\infty$ to $+\infty$, solution curves go from 2 to 10.

Theorem

Theorem of existence and uniqueness. Provided f and $\partial f/\partial y$ are continuous in a rectangle R and (t_0, y_0) is inside R, the IVP:

$$y' = f(t, y),$$
$$y(t_0) = y_0,$$

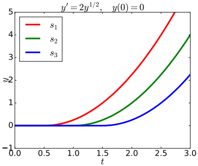
has a unique solution for $|t - t_0| < \delta$ (for some $\delta > 0$ that leaves t in the rectangle R). The IVP has a solution (existence) but no more than one solution in R on any t-interval containing t_0 (uniqueness).

What if assumptions of theorem do not hold?

$$y' = 2y^{1/2}, \quad y(t_0) = y_0.$$

$f(y) = 2y^{1/2}$ continuous for all $y \geq 0$ but $f'(y) = y^{-1/2}$ not continuous at $y = 0$. One of the theorem conditions fails for rectangles about any point $(t_0, 0)$. Separation of variables give $t - t_0 = y^{1/2} - y_0^{1/2}$, i.e., $y = (y_0^{1/2} + t - t_0)^2$ for any $y_0 > 0$. This solution holds for $y_0 = 0$ but there are infinitely many solutions of the same IVP: $y = 0$ and:

$$y(t) = \begin{cases} 0, & t < s \\ (t-s)^2, & t \geq s \geq t_0. \end{cases}$$



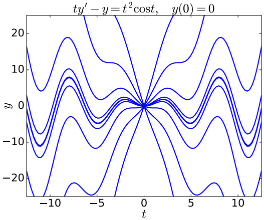
What if assumptions of theorem do not hold?

$ty' - y = t^2 \cos t, \quad t > 0$ $\quad \left(f(t, y) = \dfrac{y}{t} + t \cos t \text{ not continuous at } t = 0\right)$.

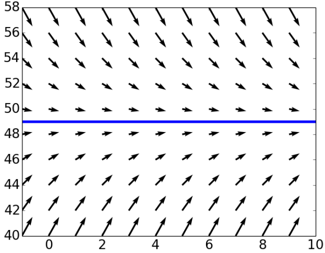
Multiplying by $1/t^2$, we get $(y/t)' = \cos t$, and therefore

$$y = t \sin t + Ct, \quad t > 0.$$

This formula gives solutions also for $t \leq 0$ and all these solutions satisfy $y(0) = 0$. The IVP with: $y(0) = 0$ has infinitely many solutions whereas any IVP with initial condition $y(0) = y_0 \neq 0$ has no solution at all!



Direction field



Direction field in the (x, y) plane for the ODE $y' = 9.8 - 0.2y$ showing the equilibrium solution $y^* = 9.8/0.2 = 49$. This ODE describes how a body weighting 1 kg falls, if its velocity is $y(x)$ and its friction force is ky, with $k = 0.2$ kg/s. $y^* = 49$ m/s is the limiting velocity.

Direction field

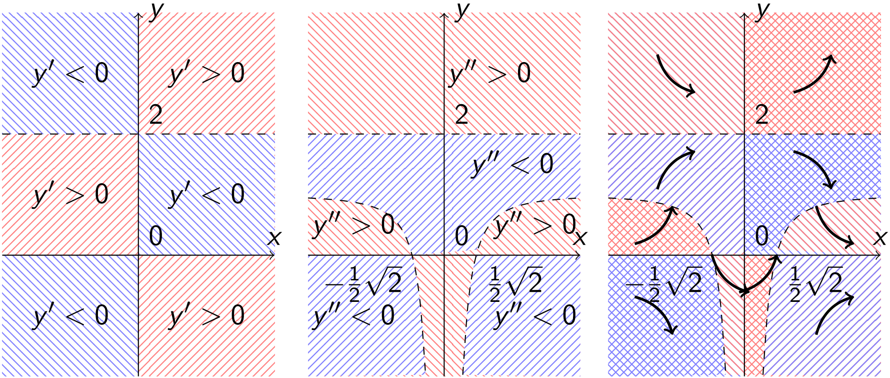


Figure: Sectors: (left) decreasing and increasing $y(x)$, (center) concave and convex $y(x)$, (right) combined information about signs of y' and y'' for the ODE $y' = xy(y-2)$. We have used $y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f = y(y-2)(1+2x^2y-2x^2)$.

Direction field and solution curves

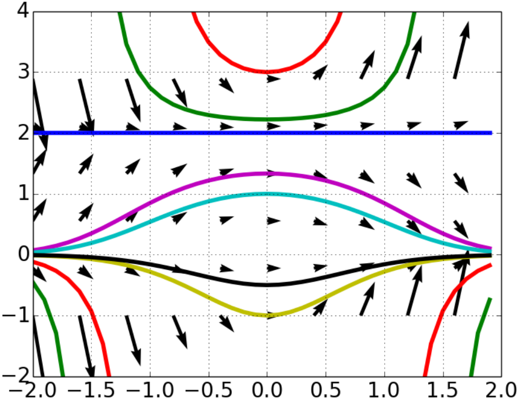


Figure: Slope field and trajectories in the (x, y) plane for the ODE $y' = xy(y-2)$.

Direction field and solution curves

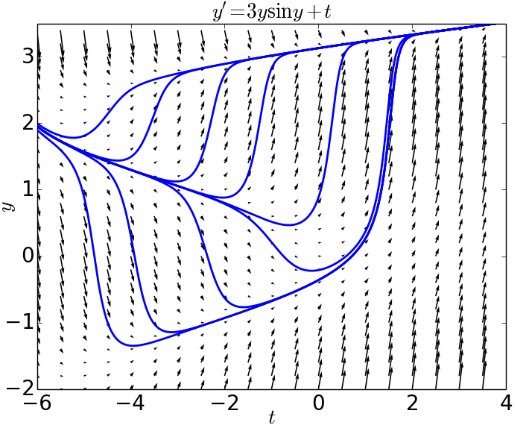


Figure: Solution curves come arbitrarily close for large positive and negative values of $t - t_0$.

Direction field for periodic harvesting

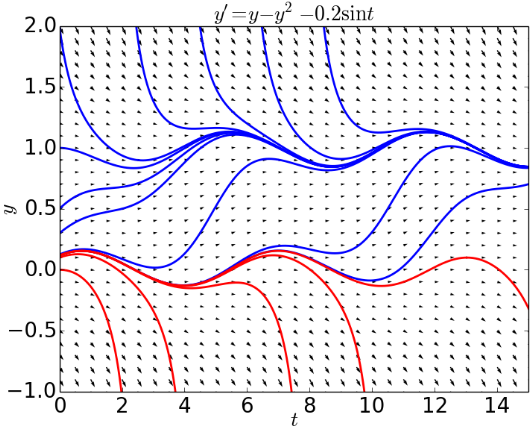


Figure: $y \sim 1 + 0.2\cos t$ as $t \to +\infty$ for $y(0) > 0.1$, $y \to -\infty$ otherwise.

Exact solutions

- Separable ODE:
  $M(x) + N(y)y' = 0 \Longrightarrow \int M(x)dx = -\int N(y)dy + c$.
- Exact ODE: $M(x,y) + N(x,y)y' = 0$, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
- Linear ODE: $y' + a(x)y = F(x)$. Use integrating factor $I = e^{\int a(x)dx}$ to get trivial ODE $(Iy)' = F(x)I(x)$. Then
  $y = e^{-\int a(x)dx}[\int F(x')e^{\int a(x')dx'}dx' + c]$.
- Bernoulli ODE: $y' = a(x)y + b(x)y^P$. Use $u = y^{1-P}$ to get linear ODE.
- Riccati ODE: $y' = a(x)y + b(x)y^2 + c(x)$. Find particular solution, transform in Bernoulli by $y = y_p(x) + u(x)$.
- Substitutions: For $y' = F(y/x)$, $u = y/x$ gives $u' = [F(u) - u]/x$.
  $x = av + bw + c$, $y = dv + ew + f$, with appropriate $a, \ldots, f$, converts $y' = (Ax + By + C)/(Dx + Ey + F)$ in a separable ODE for $w(v)$.

Euler method for $\frac{dy}{dt} = f(t, y)$

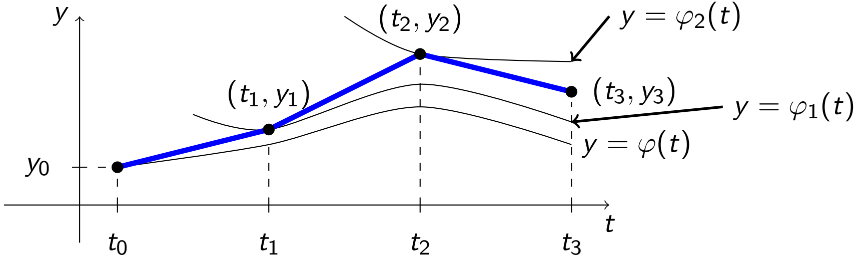


Figure: The explicit (forward) Euler method $y_{j+1} = y_j + f(t_j, y_j)h$, $y_0 = y(t_0)$, approximates a given trajectory $y = \varphi(t)$ by segments that move it to nearby trajectories $\varphi_1(t)$, $\varphi_2(t)$, ... in the (t, y) plane.

Truncation error. Suppose IVP $y' = f(t, y)$, $y(t_0) = y_0$, is approximated by one-step numerical scheme $y_{j+1} = y_j + \Phi(t_j, y_j, y_{j+1}, h)\, h$. Define the local *truncation error* or discretization error of the scheme as:

$$\tau_{j+1} = \frac{y(t_{j+1}) - y(t_j)}{h} - \Phi(t_j, y(t_j), y(t_{j+1}), h), \quad j = 0, 1, \ldots, N-1,$$

where we substitute the exact solution $y(t_j)$ instead of y_j. If the τ_j vanish as $h \to 0$, we say that the difference equations are *consistent* with the differential equation. A consistent one-step scheme is also *convergent*, i.e., the *global truncation error* $e_j = y(t_j) - y_j$ tends to 0 with h.

Note: $e_{j+1} - e_j = h\Phi(t_j, y(t_j), y(t_{j+1}), h) - h\Phi(t_j, y_j, y_{j+1}, h) + h\tau_{j+1}$. For the Euler method $\Phi(t_j, y_j, y_{j+1}, h) = f(t_j, y_j)$, the Taylor theorem gives

$$\frac{y(t_{j+1}) - y(t_j)}{h} = \frac{y(t_j + h) - y(t_j)}{h} = \frac{dy(t_j)}{dt} + \frac{h}{2}\frac{d^2y(\xi)}{dt^2}$$
$$= f(y(t_j), t_j) + \frac{h}{2}\frac{d^2y(\xi)}{dt^2} \Longrightarrow \tau_{j+1} = \frac{h}{2}\frac{d^2y(\xi)}{dt^2} = O(h).$$

Heun and RK2 methods

Heun:

$$y_{j+1} = y_j + \frac{h}{2}[f(t_j, y_j) + f(t_j + h, p_{j+1})], \quad p_{j+1} = y_j + hf(t_j, y_j).$$

RK2:

$$y_{j+1} = y_j + Ahf(t_j, y_j) + Bh\, f(t_j + Ph, y_j + Qhf(t_j, y_j)),$$

with

$$A + B = 1, \quad BP = \frac{1}{2}, \quad BQ = \frac{1}{2}.$$

Heun is RK2 with $A = \frac{1}{2}$. Modified Euler (midpoint) is RK2 with $A = 0$.

Long term behavior and control of linear IVPs

First order linear ODE:

$$y' + p(t)y = q(t), \quad y(0) = y_0, \quad t \geq 0.$$

Solution:

$$y(t) = y_0 e^{-P(t)} + e^{-P(t)} \int_0^t e^{P(s)} q(s)\, ds, \quad P(t) = \int_0^t p(s)\, ds.$$

Bounded Input-Bounded Output (BIBO): $p(t)$ continuous, $q(t)$ piecewise continuous on $t \geq 0$, p_0, M positive numbers,

$$p(t) \geq p_0, \quad |q(t)| \leq M, \quad t \geq 0.$$

Then the solution of the IVP:

$$|y(t)| \leq |y_0| + \frac{M}{p_0}, \quad t \geq 0.$$

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Control of a chemical reactor

First order linear ODE (reaction $A \to B$ with leaky entrance of A in reactor):

$$y' = \frac{r(t)}{V(t)} - ky \Longrightarrow y' + ky = \frac{r(t)}{V(t)}.$$

Assume: $V(t) \geq V_0 > 0$ (reactor never runs dry), bounded inflow rate $r(t) \leq r_0$. Then

$$0 \leq \frac{r(t)}{V(t)} \leq \frac{r_0}{V_0},$$

BIBO theorem ($p_0 = k > 0$, $M = r_0/V_0$, $y(0) > 0$):

$$|y(t)| \leq y(0) + \frac{r_0}{kV_0}.$$

Assume $y(t)$ cannot exceed $K > 0$ by design specifications. Criteria are met for:

$$y(0) \leq \frac{K}{2}, \quad r_0 \leq \frac{kV_0K}{2} \Longrightarrow |y(t)| \leq K.$$

Long term behavior for autonomous ODEs

Theorem. Let $f(y)$, $f'(y)$ be continuous for all y. Let $y(t)$ be a solution of the ODE $y' = f(y)$ which is bounded for all $t \geq 0$ (respectively for all $t < 0$). Then as $t \to +\infty$ (resp. $t \to -\infty$),$y(t)$ approaches an equilibrium solution of the ODE (y^* such that $f(y^*) = 0$).

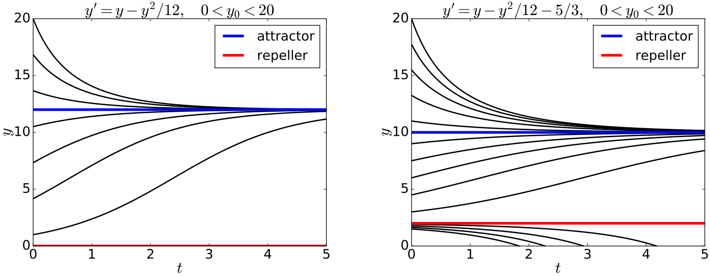


Figure: State (phase) line and solution curves for logistic ODE with harvesting.

Sensitivity to changes in data

Attractor, repeller and semistable equilibrium solutions.

- Attractor: all solution curves in a neighborhood thereof approach it as $t \to +\infty$. If y_0 is an attractor, then small changes in the initial data near y_0 have no effect on the long time behavior of the solution.
- Repeller: all solution curves in a neighborhood thereof depart from it as $t \to +\infty$ (and approach it as $t \to -\infty$). If y_0 is a repeller or a semistable equilibrium solution, then small changes in the initial data near y_0 lead to drastic changes in the solution in the long term.
- On a small enough neighborhood, an attractor/repeller or semistable solution attracts all initial data to one side of it on the phase line and repels all initial data to the other side.

Bifurcations: a logistic harvested model

Bifurcation analysis of the ODE $y' = f(y, c)$. Stages:

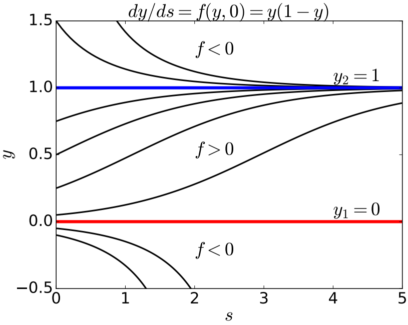
- Track the equilibrium solutions as they move, merge, split up, or disappear with changes in c.
- Describe the qualitative effects of these changes in c on the long term behavior of non-equilibrium solutions.
- Summarize solution behavior as c changes in a bifurcation diagram.

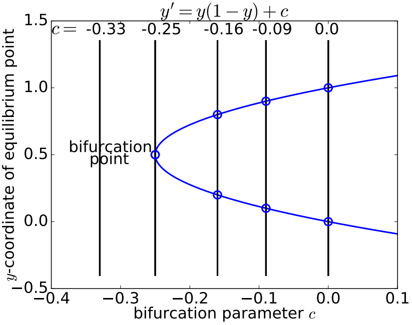
Typical behaviors:

- Saddle-node bifurcation $y' = y(1 - y) + c$.
- Pitchfork bifurcation $y' = (c - y^2)y$.

Bifurcations: a logistic harvested model

- Saddle-node bifurcation $y' = y(1-y) + c$.





Bifurcations

- Pitchfork bifurcation $y' = (c - y^2)y$.