

Applied Differential Calculus

Lecture 2: Second-order ordinary differential equations

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Outline

- Linear second order ODEs.
- Variation of parameters.
- Method of undetermined coefficients.
- Reduction of order.
- Supplementary material: Resonance.

General properties

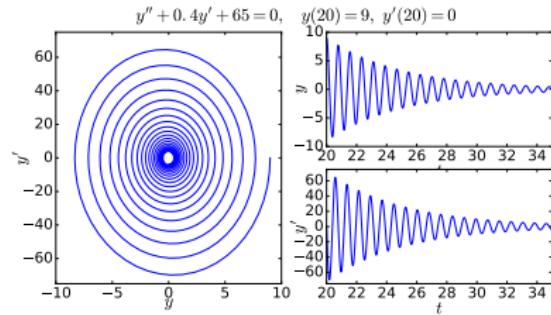
- Existence and uniqueness: $a(t)$, $b(t)$, $f(t)$ continuous on interval I .
Then for each t_0 in I and each set of values of y_0 , v_0 , the IVP:

$$y'' + a(t)y' + b(t)y = f(t), \quad y(t_0) = y_0, \quad y'(t_0) = v_0,$$

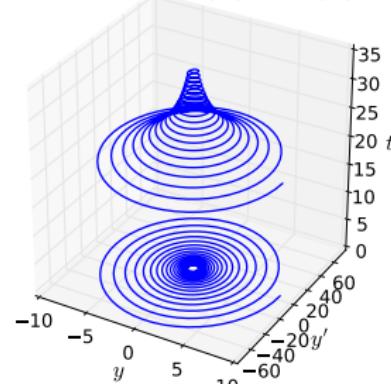
has a unique solution $y(t)$ for $t \in I$.

- Superposition principle for undriven ODE: $y_1(t)$, $y_2(t)$ are solutions of the homogeneous ODE with $f(t) = 0$. Then $y(t) = c_1y_1(t) + c_2y_2(t)$ is also a solution (c_i are arbitrary constants).
- For independent solutions $y_i(t)$, with Wronskian determinant $W(y_1, y_2) = y_1y'_2 - y_2y'_1 \neq 0$, $y(t) = c_1y_1(t) + c_2y_2(t)$ is the general solution. Abel formula: $\frac{d}{dt}W(y_1, y_2) = -a(t)W(y_1, y_2)$, $f(t) = 0$.
- Particular solution $y_p(t)$ of the driven ODE gives general solution: $y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t)$ if $y_1(t)$, $y_2(t)$ are independent solutions of the homogeneous ODE.

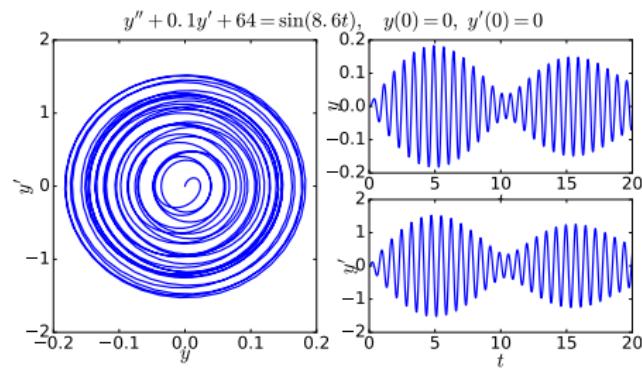
Visualizing solutions: damped spring



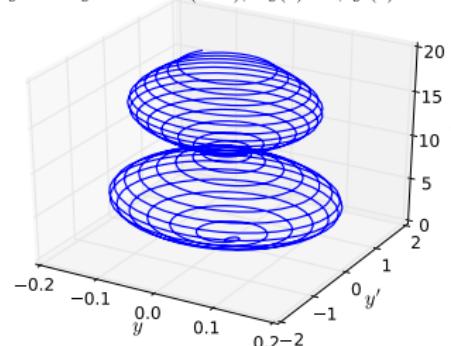
$$y'' + 0.4y' + 65 = 0, \quad y(20) = 9, \quad y'(20) = 0$$



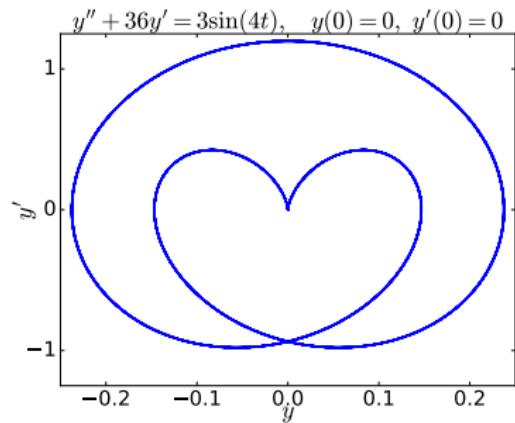
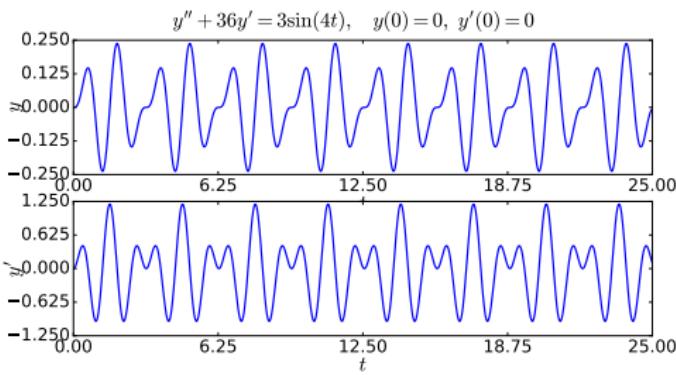
Visualizing solutions: driven damped spring



$$y'' + 0.1y' + 64 = \sin(8.6t), \quad y(0) = 0, \quad y'(0) = 0$$

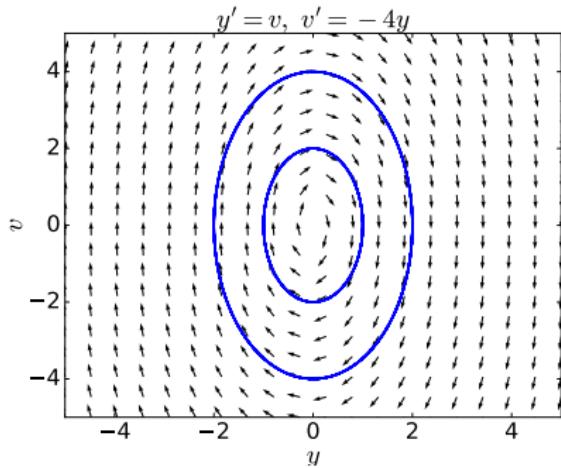
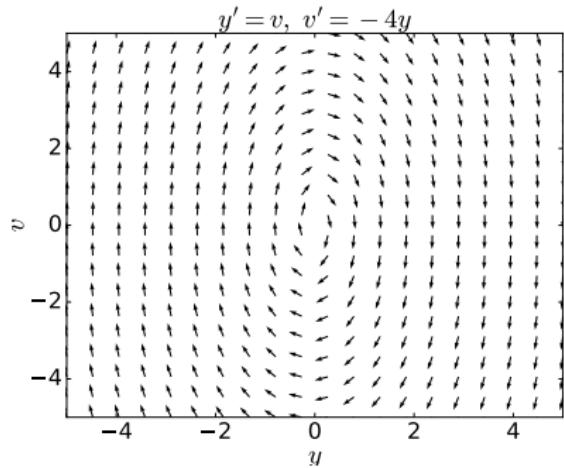


Visualizing solutions: driven undamped spring



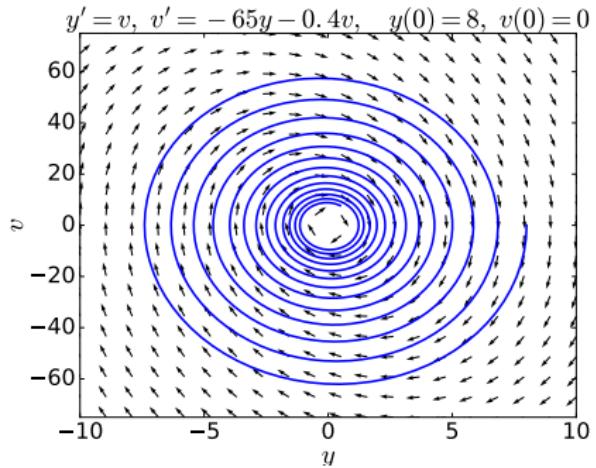
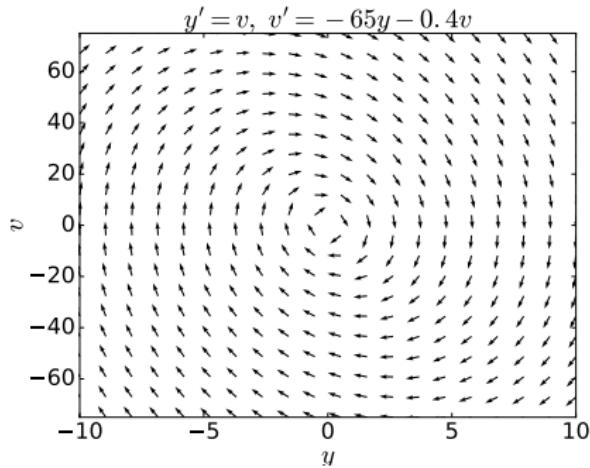
Visualizing solutions: vector field undamped spring

$y'' + 4y = 0$, equivalent to $y' = v$, $v' = -4y$.



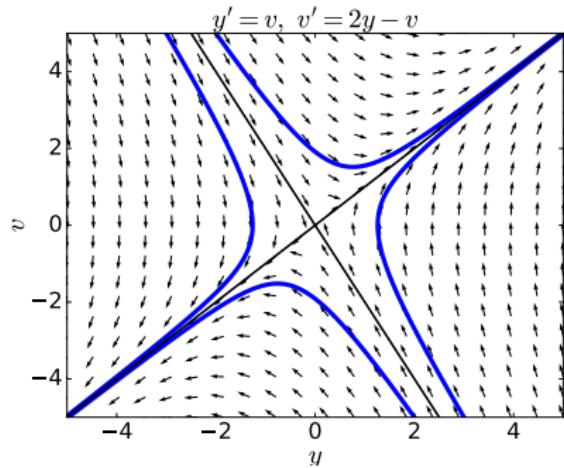
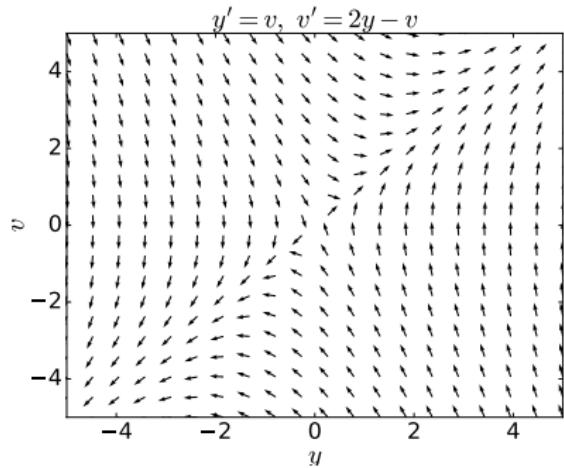
Visualizing solutions: vector field damped spring

$y'' + 0.4y' + 65y = 0$, equivalent to $y' = v$, $v' = -65y - 0.4v$.



Visualizing solutions: saddle point

$y'' + y' - 2y = 0$, equivalent to $y' = v$, $v' = 2y - v$.



Variation of parameters

- Inhomogeneous ODE: $au'' + bu' + cu = F(t)$.
- $u_1(t), u_2(t)$ are independent solutions of the homogeneous ODE.
- Particular solution of the inhomogeneous ODE:

$$u_p(t) = -u_1(t) \int_{t_0}^t \frac{u_2(s)F(s)ds}{a(s)W(u_1, u_2)(s)} + u_2(t) \int_{t_0}^t \frac{u_1(s)F(s)ds}{a(s)W(u_1, u_2)(s)},$$

where $W(u_1, u_2) = u_1u'_2 - u_2u'_1$ is the Wronskian determinant.

- General solution is $u(t) = u_p(t) + c_1u_1(t) + c_2u_2(t)$.

Method of undetermined coefficients

$F(t)$	$u_p(t)$
$P_n(t) = a_0 t^n + a_1 t^{n-1} + \dots + a_n$	$t^s(A_0 t^n + A_1 t^{n-1} + \dots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0 t^n + A_1 t^{n-1} + \dots + A_n)e^{\alpha t}$
$P_n(t) \begin{cases} \cos \beta t \\ \sin \beta t \end{cases}$	$t^s[(A_0 t^n + A_1 t^{n-1} + \dots + A_n) \cos \beta t + (B_0 t^n + B_1 t^{n-1} + \dots + B_n) \sin \beta t]$

Particular solutions of the ODE $au'' + bu' + cu = F(t)$ depending on the form of the source term $F(t)$. Here s is the smallest nonnegative integer ($s = 0, 1$, or 2) that will ensure that no term in $u_p(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $i\beta$ is a root of the characteristic equation, respectively.

Reduction of order

Let $u_1(t)$ be a solution of $a(t)u'' + b(t)u' + c(t)u = 0$. The substitution $u(t) = u_1(t)v(t)$ transform the ODE $a(t)u'' + b(t)u' + c(t)u = F(t)$ in a first-order ODE for v' :

$$a(t)v'' + \left[b(t) + 2\frac{u'_1}{u_1} \right] v' = \frac{F(t)}{u_1(t)}.$$

Example. $u'' - \frac{1+t}{t}u' + \frac{u}{t} = 0$ is solved by $u_1 = 1 + t$. $u = (1 + t)v$ gives:

$$(1 + t)v'' + 2v' - \frac{(1 + t)^2}{t}v' = 0 \implies \frac{v''}{v'} = 1 + \frac{1}{t} - \frac{2}{1 + t}$$

or $\ln v' = t + \ln \frac{t}{(1+t)^2} \implies v' = \frac{te^t}{(1+t)^2} \implies v = \int \frac{te^t dt}{(1+t)^2} = -\frac{te^t}{1+t} + \int e^t dt = \frac{e^t}{1+t}$. Thus $u = e^t$ is the other independent solution.

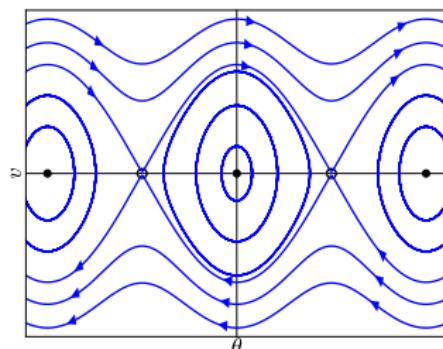
Nonlinear autonomous ODE

$$u'' + V'(u) = 0 \implies 0 = u'[u'' + V'(u)] = \frac{d}{dt} \left[\frac{u'^2}{2} + V(u) \right].$$

Then $\frac{u'^2}{2} + V(u) = C$ and

$$\pm \int \frac{du}{\sqrt{2[C - V(u)]}} = t - t_0.$$

Example: Pendulum : $ml\ddot{\theta} = -mg \sin \theta \implies \frac{1}{2}\dot{\theta}^2 + \frac{g}{l}(1 - \cos \theta) = C$.



Linear oscillations and resonance

Damped pendulum with force $-\gamma\dot{\theta} - mg/l \sin \theta$:

$$\ddot{\theta} + \frac{\gamma}{ml}\dot{\theta} + \frac{g}{l} \sin \theta = 0.$$

Use $\sin \theta \approx \theta$ and $t = \omega_0 \tilde{t}$, with $\omega_0 = \sqrt{\frac{g}{l}}$,

$$\ddot{\theta} + 2\beta\dot{\theta} + \theta = 0, \quad \beta = \frac{\gamma}{2m\sqrt{gl}}.$$

$\lambda^2 + 2\beta\lambda + 1 = 0$ gives $\lambda = -\beta \pm \sqrt{\beta^2 - 1}$.

- $\beta > 1$, overdamped pendulum: $\theta = ae^{\sqrt{\beta^2-1}t} + be^{-\sqrt{\beta^2-1}t}$.
- $0 \leq \beta < 1$, underdamped pendulum: $\theta(t) = ce^{-\beta t} \cos(\Omega t + \varphi)$, $\Omega = \sqrt{1 - \beta^2}$. Also $\theta = e^{-\beta t}(a \cos \Omega t + b \sin \Omega t)$, $a + ib = ce^{i\varphi}$.

Particular solution: transfer function

Add periodic forcing term (coefficient can be set equal to 1):

$$\ddot{\theta} + 2\beta\dot{\theta} + \theta = \cos \omega t.$$

For a force $e^{i\omega t}$

$$\theta(t) = \operatorname{Re} H(i\omega)e^{i\omega t} = \operatorname{Re} \frac{1}{1 - \omega^2 + 2i\beta\omega} e^{i\omega t},$$

where $H(r) = 1/(r^2 + 2\beta r + 1)$ is the transfer function.

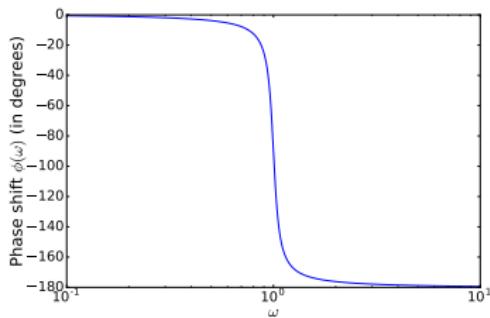
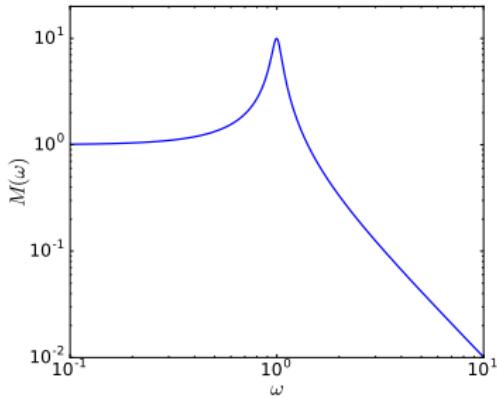
Amplitude and phase shift of transfer function:

$$M(\omega) = \frac{1}{\sqrt{(1 - \omega^2)^2 + 4\beta^2\omega^2}}, \quad \varphi(\omega) = -\arctan \frac{2\beta\omega}{1 - \omega^2}.$$

If $\omega < 1$, $-\pi < \varphi(\omega) < 0$. Steady solution is $\theta(t) = M(\omega) \cos[\omega t + \varphi(\omega)]$.

Gain and phase shift (Bodé plots)

Gain $M(\omega)$ (ratio of amplitude of response to force amplitude):



Resonance:

$$\max M(\omega) = \frac{1}{2\beta\sqrt{1-\beta^2}}, \quad \omega_{\max} = \sqrt{1-2\beta^2}.$$

As $\beta \rightarrow 0+$, $M(\omega) \rightarrow +\infty$.

Resonance

Spring driven by a periodic forcing term:

$$\ddot{\theta} + 2\beta\dot{\theta} + \theta = \cos \omega t.$$

General solution:

$$\theta(t) = M(\omega) \cos[\omega t + \varphi(\omega)] + c e^{-\beta t} \cos[\Omega t + \varphi_0], \quad \Omega = \sqrt{1 - \beta^2}$$

$$M(\omega) = \frac{1}{\sqrt{(1 - \omega^2)^2 + 4\beta^2\omega^2}}, \quad \varphi(\omega) = -\arctan \frac{2\beta\omega}{1 - \omega^2}.$$

Undamped oscillator with $\theta(0) = \dot{\theta}(0) = 0$:

$$\theta(t) = \frac{\cos \omega t - \cos t}{1 - \omega^2} = \frac{2 \sin \frac{(1-\omega)t}{2}}{1 - \omega^2} \sin \frac{(1+\omega)t}{2} \implies \theta \approx \frac{t}{2} \sin t (\omega \rightarrow 1).$$

Soft and hard springs

$$my'' = -ky \mp \alpha y^3 - cy'; \quad +: \text{soft spring}; -: \text{hard spring}.$$

$$V(y) = \frac{k}{2m}y^2 \pm \frac{\alpha}{4m}y^4.$$

For $c = 0$, $\frac{y'^2}{2} + V(y) = E$ and $V(y_{tp}) = E$, $y_{tp} > 0$, give the period

$$\pm \int^y \frac{du}{\sqrt{2[E - V(u)]}} = t - t_0 \implies P = 4 \int_0^{y_{tp}} \frac{du}{\sqrt{2[E - V(u)]}}.$$

