Applied Differential Calculus

LECTURE 2: Second-order ordinary differential equations.

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I. SECOND ORDER LINEAR ODES

This Lecture 2 deals with second-order linear ODEs and related IVPs. All notions can be immediately generalized to nth order ODEs. Later, we shall explain how a similar theory solves systems of first-order ODEs. Chapter 3 of the books by Boyce-Di Prima [1] and by Simmons [4] contain a detailed description and are good reading.

The general form of a second-order linear ODE is

$$\frac{d^2u}{dt^2} + a_1(t)\frac{du}{dt} + a_0(t)u = F(t), \tag{1}$$

where $a_j(t)$ are continuous functions in the interval $[t_0, T]$ where we want to solve the ODE. We now that if $u_p(t)$ is a particular solution of the inhomogeneous equation (1), then $v(t) = u(t) - u_1(t)$ solves the homogeneous ODE:

$$\frac{d^2u}{dt^2} + a_1(t)\frac{du}{dt} + a_0(t)u = 0. \tag{2}$$

The **superposition principle** says that if $u_1(t)$ and $u_2(t)$ are solutions of (2), then $u(t) = c_1u_1(t) + c_2u_2(t)$, where c_1 and c_2 are numbers, is also a solution.

To check these statements is immediate.

The calculation of u_1, u_2 and u_p in terms of elementary functions can be done only in particular cases, as we shall see later. However we can elaborate a general theory imagining that we have been able to find these functions.

A. Second-order linear homogeneous ODE

We shall see that the general solution of (2) is a superposition of two **linearly independent solutions** and therefore the dimension of the space of solutions is two. For this, we need a criterion to distinguish linearly independent solutions.

Definition. Given two differentiable functions $u_1(t)$ and $u_2(t)$, their **Wronskian determinant** is

$$W(u_1, u_2) = \begin{vmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{vmatrix}. \tag{3}$$

Lemma 1. Two differentiable functions, $u_1(t)$ and $u_2(t)$, are linearly dependent if and only if their Wronskian determinant is identically zero.

Proof. If $W(u_1, u_2) = 0$, we have $u_1 u_2' - u_1' u_2 = 0$, and therefore $u_1'/u_1 = u_2'/u_2$, which integrated produces $\ln|u_1| = \ln|u_2| + c$. This gives $u_1(t) = \pm e^c u_2(t)$ and therefore $u_1(t) = ku_2(t)$, with constant $k = \pm e^c$, and $u_1(t)$ depends linearly on $u_2(t)$. The reciprocal statement is immediate.

It turns out that the Wronskian determinant can be calculated directly from the ODE (2):

$$\begin{aligned}\frac{d}{dt}W(u_1,u_2) &= \frac{d}{dt}(u_1u_2' - u_1'u_2) = u_1u_2'' - u_1''u_2 = u_1(-a_1u_1' - a_0u_1) - u_2(-a_1u_2' - a_0u_2)\\ &= -a_1(u_1u_2' - u_1'u_2), \quad \text{and therefore,}\\ \frac{d}{dt}W(u_1,u_2) &= -a_1(t)W(u_1,u_2), \qquad (4)\end{aligned}$$

which is **Abel's formula**.

Abel's formula shows that the Wronskian determinant is either nonzero or identically zero. In fact, (4) is a first-order linear ODE which can be solved by **separation of variables**:

$$W(u_1,u_2)(t) = W(u_1,u_2)(t_0)\ \exp\left[-\int_{t_0}^{t} a_1(s)ds\right], \qquad (5)$$

and the only way the Wronskian be zero is that the constant of integration $W(u_1, u_2)(t_0) = 0$.

How do we calculate $u_1(t)$ and $u_2(t)$?

In general, we can use the existence and uniqueness theorem and define $u_1(t)$ as the unique solution of the IVP:

$$\begin{cases} u'' + a_1(t)u' + a_0(t)u = 0,\\ u(t_0) = 1,\ u'(t_0) = 0, \end{cases} \qquad (6)$$

and $u_2(t)$ as the unique solution of the IVP:

$$\begin{cases} u'' + a_1(t)u' + a_0(t)u = 0,\\ u(t_0) = 0,\ u'(t_0) = 1. \end{cases} \qquad (7)$$

To use the existence and uniqueness theorem, you need to write the second-order ODE as a system of two first-order ODEs and check that the vector function in its right-hand side is Lipschitz.

The solutions $u_1(t)$ and $u_2(t)$ defined by (6) and (7) are independent because of Abel's formula (5) and

$$W(u_1,u_2)(t_0) = \begin{vmatrix} u_1(t_0) & u_2(t_0) \\ u_1'(t_0) & u_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0, \tag{8}$$

according to the initial conditions.

We can also show that any other solution of (2) is a linear combination of u_1 and u_2.
To do so, we construct the Wronskian

$$W(u_1,u_2,u) = \begin{vmatrix} u_1(t) & u_2(t) & u(t) \\ u_1'(t) & u_2'(t) & u'(t) \\ u_1''(t) & u_2''(t) & u''(t) \end{vmatrix},$$

and use (2) to show that this Wronskian is identically zero.
This proves that u_1, u_2 and u are linearly dependent and therefore any of them can be written as a linear combination of the other two.
In other words, we have proved that the dimension of the space of solutions of (2) is two.

Example 1.

$$u'' + 3u' + 2u = 1. \tag{9}$$

Clearly, $u_p = 1/2$, is a particular solution of (9).
The solutions of the homogeneous ODE are of the form: $u = e^{\lambda t}$.
Inserting this in the homogeneous ODE

$$u'' + 3u' + 2u = 0,$$

we find $\lambda^2 + 3\lambda + 2 = 0$. Its solutions are $\lambda_1 = -1$ and $\lambda_2 = -2$.
The Wronskian determinant is

$$W(e^{-t}, e^{-2t}) = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = -2e^{-3t} + e^{-3t} = -e^{-3t} \neq 0, \tag{10}$$

which proves that e^{-t} and e^{-2t} are independent.
As it should be, (10) agrees with Abel's formula (5).
Then the general solution of the inhomogeneous problem is

$$u(t) = \frac{1}{2} + c_1 e^{-t} + c_2 e^{-2t}. \tag{11}$$

Note that the solutions of (6) and (7) with $t_0 = 0$ are $u_1(t) = 2e^{-t} - e^{-2t}$ and $u_2(t) = e^{-t} - e^{-2t}$, respectively.

B. Variation of parameters and inhomogeneous ODE

What happens if we do not spot a particular solution of (1) but we are able to find the solutions of the homogeneous equation (2)?

In general, we can use **variation of parameters** for known solutions $u_1(t)$ and $u_2(t)$ of (2). The idea is to replace the constants of integration c_1 and c_2 in the general solution for unknown functions $y_1(t)$ and $y_2(t)$:

$$u(t) = y_1(t)u_1(t) + y_2(t)u_2(t). \tag{12}$$

We have introduced two unknowns y_1 and y_2 instead of the original one, $u(t)$. So we can impose one additional restriction to get a unique solution.

If we take one derivative of (12),

$$u'(t) = y_1'(t)u_1(t) + y_2'(t)u_2(t) + y_1(t)u_1'(t) + y_2(t)u_2'(t), \tag{13}$$

and impose that

$$y_1'(t)u_1(t) + y_2'(t)u_2(t) = 0, \tag{14}$$

we guarantee that the second derivative of $u(t)$ does not contain second derivatives of the y_j.

Equations (13) and (14) yield

$$u'' = y_1'(t)u_1'(t) + y_2'(t)u_2'(t) + y_1(t)u_1''(t) + y_2(t)u_2''(t). \tag{15}$$

Inserting (13), (14) and (15) in (1), we obtain

$$y_1(u_1'' + a_1u_1' + a_0u_1) + y_2(u_2'' + a_1u_2' + a_0u_2) + u_1'y_1' + u_2'y_2' = F(t). \tag{16}$$

The coefficients of y_1 and y_2 in this equation are zero because u_1 and u_2 are solutions of the homogeneous equation.

Thus we have obtained

$$u_1'y_1' + u_2'y_2' = F(t). \tag{17}$$

Equations (14) and (17) are two first-order ODEs for the two unknown functions y_1 and y_2. We get

$$\begin{cases} y_1' = -\frac{u_2 F}{W(u_1,u_2)}, \\ y_2' = \frac{u_1 F}{W(u_1,u_2)}, \end{cases} \tag{18}$$

after some algebra.

The right hand side (RHS) of these equations are known functions of t and therefore we can find y_1 and y_2 by integration.

Inserting the result in (12), we obtain

$$u(t) = \left[c_1 - \int_{t_0}^{t} \frac{u_2(s)F(s)}{W(u_1,u_2)(s)} ds\right] u_1(t) + \left[c_2 + \int_{t_0}^{t} \frac{u_1(s)F(s)}{W(u_1,u_2)(s)} ds\right] u_2(t), \tag{19}$$

which is the general solution of (1).

Let us check (19) for Example 1.

We have $F = 1$, $u_1 = e^{-t}$, $u_2 = e^{-2t}$ and $W(u_1, u_2) = -e^{-3t}$ according to (10).

Then (18) is

$$\begin{cases} y_1' = -\frac{e^{-2t}}{-e^{-3t}} = e^t, \\ y_2' = \frac{e^{-t}}{-e^{-3t}} = -e^{2t}. \end{cases} \tag{20}$$

Then $y_1 = e^t$ and $y_2 = -e^{2t}/2$.

The particular solution is $u_p = y_1 u_1 + y_2 u_2 = 1 - 1/2 = 1/2$, the same one we got by inspection.

C. Method of undetermined coefficients

This method is really organized guesswork, but it is faster than variation of parameters when it works. It is applicable to second or higher order ODEs with constant coefficients when the source term is an additive or multiplicative combination of e^t, $\sin t$, $\cos t$ and polynomials in t, or equidimensional equations where $F(t)$ is a polynomial in t.

1. Linear ODEs with constant coefficients

The idea is to try a particular solution with undetermined parameters of the same form as the source term $F(t)$.

We illustrate the method by **a few examples**:

Example 2.

(a) To solve $u'' + u = e^t \sin t$, we try a particular solution of the form $u_p = ae^t \sin t + be^t \cos t$ and determine the ***undetermined coefficients*** a and b by substituting into the ODE.
Result: $u_p'' + u_p = e^t[(a - 2b)\sin t + (2a + b)\cos t] = e^t \sin t$.
This implies $a - 2bb = 1$ and $2a + b = 0$, from which $a = 1/5$, $b = -2/5$.

(b) To solve $u'' - u = e^t$, we cannot try $u_p = ae^t$ because e^t is a solution of the homogeneous ODE.
We try instead $u_p = ate^t$ that gives $u_p'' - u_p = 2ae^t = e^t$, hence $a = 1/2$.
Thus in the similar example $u'' + u = \cos t$, we try $u_p = at \sin t$, thereby obtaining again $a = 1/2$.

(c) To solve $u'' + u = t^3 - 2t$, we try $u_p = at^3 + bt^2 + ct + d$, thereby getting $u_p'' + u_p = at^3 + bt^2 + (c + 6a)t + d + 2b = t^3 - 2t$, so that $a = 1$, $b = d = 0$, $c = -8$.

Example 3.

Consider $u'' + 2u' + u = e^{-t}$.
A particular solution of the form $u_p = ae^t$ does not work because e^{-t} is a solution of the homogeneous ODE.
What about $u_p = ate^{-t}$**?**.
It turns out this is also a solution of the homogeneous equation.
Let us define $Lu = u'' + 2u' + u$. We have $L(te^{-t}) = te^{-t} - 2e^{-t} + 2e^{-t} - 2te^{-t} + te^{-t} = 0$.
What is going on? $Le^{\lambda t} = (\lambda^2 + 2\lambda + 1)e^{\lambda t} = (\lambda + 1)^2 e^{\lambda t}$, and we find that there is only one solution $\lambda = -1$ of the equation $(\lambda + 1)^2 = 0$. When we find a double zero of the characteristic polynomial, then the two independent solutions of the homogeneous ODE are $e^{\lambda t}$ and $te^{\lambda t}$.
This is a general fact, as we will show after we finish this example. Anyway, in this case, we have to try at^2e^{-t} as a solution:

$$L(at^2e^{-t}) = (2a + at^2 - 4at + 4at - 2at^2 + at^2)e^{-t} = 2ae^{-t}$$

and therefore $a = 1/2$.

The general solution of $u'' + 2u' + u = e^{-t}$ is $u(t) = (\frac{1}{2}t^2 + c_1 t + c_2)e^{-t}$.

To find our what happens when $\lambda = \lambda_0$ is a **double zero of the characteristic polynomial**, we consider the operator $L = (d/dt - \lambda_0)^2$.
We know that $e^{\lambda_0 t}$ is a solution, so we try $u = v(t)e^{\lambda_0 t}$.
We find

$$Lu = \left(\frac{d}{dt} - \lambda_0\right)(v' e^{\lambda_0 t}) = v'' e^{\lambda_0 t},$$

and therefore $L(ve^{\lambda_0 t}) = 0$ implies $v'' = 0$, so that $v' = c$ and $v = ct + d$.
Then $u = ve^{\lambda_0 t} = (ct+d)e^{\lambda_0 t}$ which is the general solution of the homogeneous ODE $Lu = 0$.

2. Linear ODEs with constant coefficients

Consider the nth-order homogeneous linear ODE:

$$u^{(n)} + a_{n-1}u^{(n-1)} + \ldots + a_0 u = 0, \tag{21}$$

which can be written as

$$p_n(D)u \equiv [D^n + a_{n-1}D^{n-1} + \ldots + a_0]u = 0, \tag{22}$$

where $D = d/dt$ and $p_n(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n.
We can factorize the polynomial using its n roots (real or complex) as
$p_n(x) = (x - \alpha_1)\ldots(x - \alpha_n)$.
Then (22) can be written as

$$(D - \alpha_1)\ldots(D - \alpha_n)u = 0. \tag{23}$$

If **all roots** α_j **are different and real**, commutativity of the factors in (23) implies that u must satisfy $(D - \alpha_j)u = 0$ for each j. This means $u' = \alpha_j u$ whose solution is $u(t) = c_j e^{\alpha_j t}$.
The general solution is thus $u(t) = \sum_{j=1}^{n} c_j e^{\alpha_j t}$.
If α_j **is complex**, say for $j = 1$ with $\alpha_1 = \mu + i\nu$, then there is another root which is its complex conjugate, so that we shall have two solutions $e^{(\mu \pm i\nu)t} = e^{\mu t}(\cos \nu t \pm i \sin \nu t)$.
This pair of complex conjugate roots will contribute $e^{\mu t}(c_1 \cos \nu t + c_2 \sin \nu t)$ to the general solution of the ODE.
What happens if there are multiple roots? Let α be a root with multiplicity l. Then

$p_n(D)$ will contain a factor $(D-\alpha)^l$ in its factorization. The equation $(D-\alpha)^l u = 0$ has one obvious solution $u = e^{\alpha t}$. We seek its other solutions as $u = e^{\alpha t}v(t)$. We have $(D-\alpha)e^{\alpha t}v = e^{\alpha t}v'$, and therefore $0 = (D-\alpha)^l u = e^{\alpha t}D^l v$, and we get $v^{(l)} = 0$. This implies that $v(t) = c_0 + c_1 t + \ldots c_{l-1}t^{l-1}$ (polynomial of degree $(l-1)$) and therefore

$$u(t) = (c_0 + c_1 t + \ldots c_{l-1}t^{l-1})e^{\alpha t} + \ldots \qquad \text{if } \alpha \text{ is real, and}$$
$$u(t) = (c_0 + c_1 t + \ldots c_{l-1}t^{l-1})e^{\mu t}\cos\nu t + (d_0 + d_1 t + \ldots d_{l-1}t^{l-1})e^{\mu t}\sin\nu t + \ldots, \quad (24)$$

if $\alpha = \mu + i\nu$ and its complex conjugate have multiplicity l.

In the **case of the inhomogeneous ODE and distinct real roots** α_j, we can formally write the solution as

$$u = \frac{1}{p_n(D)}F(t) = \frac{1}{\prod_{j=1}^n (D-\alpha_j)}F(t) = \sum_{j=1}^n \frac{A_j}{D-\alpha_j}F(t) = \sum_{j=1}^n A_j \int_{t_0}^t e^{\alpha_j(t-s)}F(s)ds. \quad (25)$$

Each $(D-\alpha_j)^{-1}F(t)$ is the solution of the first order inhomogeneous ODE $(D-\alpha_j)w = F(t)$, so that $(D-\alpha_j)^{-1}F(t) = \int_{t_0}^t e^{\alpha_j(t-s)}F(s)ds$, as written above.

The cases of complex and multiple roots are left for the reader to do.

3. Equidimensional ODEs

Equidimensional (or Euler) linear ODEs are so named because they are invariant under the scale change $t \to at$.

They have the form:

$$\frac{d^n u}{dt^n} + \frac{a_{n-1}}{t}\frac{d^{n-1}u}{dt^{n-1}} + \ldots + \frac{a_1}{t^{n-1}}\frac{du}{dt} + \frac{a_0}{t^n}u = F(t). \quad (26)$$

Let us consider the homogeneous case first. The homogeneous equidimensional ODE can be transformed into a constant-coefficient ODE by **changing variables**:

$$t = e^x, \quad t\frac{d}{dt} = \frac{d}{dx}. \quad (27)$$

Alternatively, they can be solved by **direct substitution** $u = t^\lambda$ in (26) with $F = 0$:

$$[\lambda(\lambda-1)\ldots(\lambda-n+1) + a_{n-1}\lambda(\lambda-1)\ldots(\lambda-n+2) + \ldots + a_1\lambda + a_0]t^{\lambda-n} = 0. \quad (28)$$

As in the case of the constant coefficient ODE, λ is one of the roots of the resulting polynomial.

Example 4.

Inserting $u = t^\lambda$ in $u'' + u'/t - u/t^2 = 0$, we find $\lambda(\lambda - 1) + \lambda - 1 = 0$. Then $\lambda^2 = 1$ and $\lambda = \pm 1$.

The general solution of this homogeneous ODE is $u(t) = c_1 t + c_2/t$.

The same results are obtained after **changing variables** $t = e^x$ or $x = \ln t$. $du/dt = t^{-1}du/dx$, $d^2u/dt^2 = (d/dt)[t^{-1}du/dx] = -t^{-2}du/dx + t^{-2}d^2u/dx^2$. Then $t^2u'' + tu' - u = d^2u/dx^2 - du/dx + du/dx - u = d^2u/dx^2 - u$, and the equation $d^2u/dx^2 - u = 0$ has the solutions $u = c_1 e^x + c_2 e^{-x}$ which becomes $u = c_1 t + c_2/t$ once we go back to the variable t.

Example 5.

The same procedure applied to the equation $u'' + u/(4t^2) = 0$ gives $\lambda(\lambda - 1) + 1/4 = 0$ which is $(\lambda - 1/2)^2 = 0$. In this case $\lambda = 1/2$ is a **double root**.

One of the solutions has the form $u_1 = t^{1/2}$ but, what is the other one?

When we know one solution $u_1(t)$ of a higher order linear homogeneous ODE, the change $u = u_1 v$ gives a simpler first order ODE for v':

$$u'' - \frac{u}{4t^2} = v''u_1 + 2v'u_1' + vu_1'' - \frac{vu_1}{4t^2} = 0.$$

The two last terms of this equation cancel and therefore we get a first order linear ODE for $z = v'$: $u_1 z' + 2u_1' z = 0$. Multiplying this by u_1 we have $u_1^2 z' + z(u_1^2)' = 0$, which gives $u_1^2 z = c$. Then we have $z = v' = c/u_1^2 = c/t$. Integrating again, we find $v = c \ln t$. Then the other solution is $u = u_1 v = t^{1/2} \ln t$, and the general solution of the ODE is $u(t) = c_1 t^{1/2} + c_2 t^{1/2} \ln t$
The same result could be obtained by means of the change of variable $x = e^t$.

Example 6.

(a) To solve $u'' - u/t^2 = t^4 + t^3$, we guess a particular polynomial solution $u_p = at^6 + bt^5$. Then $u_p'' - u_p/t^2 = (30a - a)t^4 + (20b - b)t^3 = t^4 + t^3$. We get $a = 1/29$ and $b = 1/19$.

(b) To solve $u'' + tu' + 2u = 1$, we guess $u_p = a$ which gives $a = 1/2$, whereas to solve $u'' + tu' + 2u = t^4$, we guess $u_p = at^4 + bt^2 + c$, so that $u_p'' + tu_p' + 2u_p = (4a + 2a)t^4 + (12a + 2b + 2b)t^2 + (2b + 2c) = t^4$ and we find $6a = 1$, $12a + 4b = 0$, $c = -b$. Thus $a = 1/6$, $b = -1/2$, $c = 1/2$; $u_p = \frac{1}{6}t^4 - \frac{1}{2}t^2 + \frac{1}{2}$.

Example 7.

To solve the ODE $t^2u'' + u/4 = t^{1/2}$, we try $u_p = at^{1/2}(\ln t)^2$. We get $a = 1/2$. The reason is that the homogeneous ODE is equidimensional with a double zero $\lambda = 1/2$ so that both $t^{1/2}$ and $t^{1/2}\ln t$ are solutions of the homogeneous ODE.

The proposed guess solution increases in one the degree of the polynomial in $\ln t$ multiplying $t^{1/2}$.

D. Reduction of order

When we know a solution of a linear ODE (homogeneous or not), we can reduce its order by factoring off this solution.

Let $u_1(t)$ be a solution of $Lu = 0$. Then we seek further linearly independent solutions of the form

$$u(t) = u_1(t)v(t). \tag{29}$$

Substituting this expression for $u(t)$ into $Lu = 0$ gives a new equation for v of the form $Mv = 0$ which does not have a term of the form $a_0(t)v$.

Thus $Mv = 0$ is a linear homogeneous ODE of order $(n-1)$ for $w(t) = v'(t)$ if $Lu = 0$ is of order n.

We have used reduction of order in the examples of the method of undetermined coefficients.

Example 8.

We observe that the sum of the coefficients of the ODE $u'' - u'(1+t)/t + u/t = 0$ is 0.

It follows that one solution is $u = e^t$. Then we try $u = e^t v$ and get $[v'' + 2v' - v'(1+t)/t]e^t = 0$ which gives $v'' + (1 - 1/t)v' = 0$. Then $z'/z = -1 + 1/t$ for $z = v'$ and $\ln z = \ln t - t + \ln c$ which gives $z = v' = te^{-t}$.

One more integration yields $v = (1+t)e^{-t}$, and the other solution of the original ODE is $u = 1 + t$.

The general solution is a linear combination: $u(t) = c_1(1+t) + c_2e^t$.

Example 9.

To reduce the order of the inhomogeneous ODE $u'' - u'(1+t)/t + u/t = te^t$, we substitute

$u = e^t v$ as before. We get $v'' + (1-1/t)v' = t$, and therefore $z' + (1-1/t)z = t$. Multiplying this ODE by the integrating factor e^t/t, we get $(e^t z/t)' = e^t$, therefore $e^t z/t = e^t$. Then $z = v' = t$ and $v = t^2/2$.

We have found the particular solution $u_p = t^2 e^t/2$, so that the general solution is $u = \frac{1}{2}t^2 e^t + c_1 e^t + c_2(1+t)$.

II. SUPLEMENTARY MATERIAL: LINEAR OSCILLATOR AND RESONANCE

A. Unforced oscillator

Let us consider a **damped pendulum**. The force acting on its mass is $-mg\sin\theta - \gamma\dot{\theta}$, where $\dot{\theta} = d\theta/dt$ and γ is the damping coefficient. The acceleration is $ml\ddot{\theta}$, where l is the pendulum length.

Newton's second law then yields the equation

$$\ddot{\theta} + \frac{\gamma}{ml}\dot{\theta} + \frac{g}{l}\sin\theta = 0. \tag{30}$$

If we are interested in small oscillations about the equilibrium $\theta = 0$, then we can approximate $\sin\theta \approx \theta$ and (30) becomes

$$\ddot{\theta} + \frac{\gamma}{ml}\dot{\theta} + \frac{g}{l}\theta = 0. \tag{31}$$

As we shall see below, $\omega_0 = \sqrt{g/l}$ is the **natural frequency of the harmonic oscillator** in (31).

It is convenient to render time dimensionless by defining a new time $\tilde{t} = \omega_0 t$.

Then (31) becomes

$$\frac{d^2\theta}{d\tilde{t}^2} + 2\beta\frac{d\theta}{d\tilde{t}} + \theta = 0, \tag{32}$$

$$\beta = \frac{\gamma}{2m\sqrt{gl}}, \tag{33}$$

where $\beta > 0$ is a dimensionless parameter.

From now on, we shall study (32) omitting the tilde over the nondimensional time. To recover the dimensional time, we replace $\omega_0 t$ instead of t in the resulting formulas.

The solution of (32) is found by inserting $\theta = e^{\lambda t}$ in it, thereby obtaining

$$\lambda^2 + 2\beta\lambda + 1 = 0, \tag{34}$$

whose solutions are

$$\lambda = -\beta \pm \sqrt{\beta^2 - 1}. \tag{35}$$

We have **three possible cases**:

(i) Overdamped oscillator: $\beta > 1$,

(ii) damped oscillator: $\beta < 1$,

(iii) undamped oscillator: $\beta = 0$.

1. **Overdamped oscillator** $\beta > 1$

The two solutions (35) are both real and negative.

The general solution of the ODE (32) is

$$\theta(t) = ae^{-(\beta+\sqrt{\beta^2-1})t} + be^{-(\beta-\sqrt{\beta^2-1})t}. \tag{36}$$

For any initial condition, $\theta \to 0$ as $t \to \infty$. Since $\beta - \sqrt{\beta^2 - 1} < \beta + \sqrt{\beta^2 - 1}$, the second term in (36) goes to 0 more slowly than the first one provided a and b are both different from 0.

In the limiting case $\beta = 1$ both solutions in (35) are equal to -1.

The general solution is now

$$\theta(t) = (a + bt)e^{-t}. \tag{37}$$

2. **Damped oscillator** $\beta < 1$

Now the solutions (35) are complex conjugate of each other.

If we define $\Omega = \sqrt{1 - \beta^2}$, $\lambda = -\beta \pm i\Omega$. Since $e^{-\beta t \pm i\Omega t} = e^{-\beta t}[\cos(\Omega t) \pm i \sin(\Omega t)]$, we can write the general solution of (32) as

$$\theta(t) = e^{-\beta t}[a \cos(\Omega t) + b \sin(\Omega t)] = ce^{-\beta t} \cos(\Omega t + \phi), \tag{38}$$

where a, b, c and ϕ are real numbers.

The solution (38) represents damped oscillations with **frequency** Ω, **phase shift** ϕ and **relaxation time** $1/\beta$.

3. Undamped oscillator

For $\beta = 0$, the general solution of (32) as

$$\theta(t) = a\cos t + b\sin t = c\cos(t+\phi). \tag{39}$$

These are undamped oscillations of **amplitude** c, unit frequency and period 2π.

B. Resonance

Let us assume now that there is a force acting periodically on the oscillator with frequency ω (we continue using nondimensional units).
Instead of (32), we have now

$$\frac{d^2\theta}{dt^2} + 2\beta\frac{d\theta}{dt} + \theta = \cos(\omega t). \tag{40}$$

We can write the coefficient of $\cos(\omega t)$ in (40) as 1 without losing generality. Using the **method of undetermined coefficients**, we look for a **particular solution**

$$\theta_p = a\cos(\omega t) + b\sin(\omega t), \tag{41}$$

which inserted in (40) yields

$$[(1-\omega^2)a + 2\beta\omega b]\cos(\omega t) + [(1-\omega^2)b - 2\beta\omega a]\sin(\omega t) = \cos(\omega t). \tag{42}$$

Provided $\omega^2 \neq 1$, we then find

$$a = \frac{1-\omega^2}{(1-\omega)^2 + 4\beta^2\omega^2} = \frac{\cos\varphi}{\sqrt{(1-\omega)^2+4\beta^2\omega^2}}, \quad b = \frac{2\beta\omega}{(1-\omega^2)^2+4\beta^2\omega^2} = \frac{\sin\varphi}{\sqrt{(1-\omega^2)^2+4\beta^2\omega^2}} \tag{43}$$

Inserting this in (41), we obtain

$$\theta_p = \frac{\cos(\omega t - \varphi)}{\sqrt{(1-\omega^2)^2 + 4\beta^2\omega^2}}. \tag{44}$$

The same result is obtained by taking the real part of the solution $Ae^{i\omega t}$, with $A = e^{-i\varphi}/\sqrt{(1-\omega^2)^2+4\beta^2\omega^2}$ of (40) with right hand side $e^{i\omega t}$.
Therefore the response of the oscillator to a **harmonic force** is **harmonic** with an **amplitude** $1/\sqrt{(1-\omega^2)^2+4\beta^2\omega^2}$, the same **frequency** and a **phase shift** φ given by (43).

The amplitude has a maximum value $1/(2\beta)$ for $\omega^2 = 1$, which becomes infinite in the undamped case $\beta = 0$. **What happens then?**

Clearly, for $\omega = 1$ and $\beta = 0$, we have to solve the ODE $\ddot{\theta} + \theta = \cos t$ and $\cos t$ is a solution of the corresponding homogeneous ODE.

According to the **method of undetermined coefficients**, we guess $\theta_p = at\sin t$, and obtain $a = 1/2$ by insertion in the equation.

The amplitude of the oscillations described by this solution is $t/2$, which increases linearly with time.

This phenomenon is called ***resonance*** between the frequency of the force, ω, and the natural frequency of the unforced oscillator, 1.

It is convenient to visualize the resonance phenomenon as $\omega \to 1$. For $\beta = 0$ and $\omega \neq 1$, the general solution is

$$\theta(t) = \frac{\cos(\omega t)}{1-\omega^2} + a\cos t + b\sin t. \tag{45}$$

If initially $\theta = 0$ and $\dot{\theta} = 0$, (45) becomes

$$\theta(t) = \frac{\cos(\omega t) - \cos t}{1-\omega^2} = \frac{2\sin[(1-\omega)t/2]}{1-\omega^2}\sin\left(\frac{\omega+1}{2}t\right). \tag{46}$$

This is a "**fast**" oscillation of period $P_+ = 4\pi/(\omega+1)$ modulated by a "**slow**" oscillation of period $P_- = 4\pi/|1-\omega|$ and "**amplitude**" $\frac{2\sin[(1-\omega)t/2]}{1-\omega^2}$.

As $\omega \to 1$, this amplitude tends to its slope at the origin, that is, $t/2$ and (46) becomes the resonant solution $\theta = (t/2)\sin t$.

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