Applied Differential Calculus

LECTURE 3: Systems of differential equations.

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I. SYSTEMS OF FIRST ORDER ODES

This Lecture 3 deals with systems of first-order linear ODEs and related IVPs.

The general form of a system of n first-order ODEs is

$$\frac{du_1}{dt} = f_1(u_1, \dots, u_n, t),$$
$$\dots$$
$$\frac{du_n}{dt} = f_n(u_1, \dots, u_n, t). \tag{1}$$

If the f_j and all their first partial derivatives with respect to the u_k are continuous functions in a region R containing $(u_1(t_0), \dots, u_n(t_0))$ and t_0, the **existence and uniqueness theorem** implies that there is a **unique solution** of (1) satisfying the initial condition on subregion of R.

If the f_j in (1) are independent of t, the ODE system is called ***autonomous***, otherwise it is ***non-autonomous***.

Any nth-order non-autonomous system is equivalent to a $(n+1)$st-order autonomous system given by (1) with the new unknown u_{n+1} instead of t in the $f_j(u_1, \dots, u_n, u_{n+1})$ and the additional ODE: $du_{n+1}/dt = 1$.

The **general form of a system of n first-order *linear* ODEs** is

$$\frac{du_1}{dt} = a_{11}(t)u_1 + \dots + a_{1n}(t)u_n + F_1(t),$$
$$\dots$$
$$\frac{du_n}{dt} = a_{n1}(t)u_1 + \dots + a_{nn}(t)u_n + F_n(t). \tag{2}$$

Eq. (2) can be written as a vector ODE:

$$\frac{d\underline{u}}{dt} = \underline{\underline{A}}(t)\,\underline{u} + \underline{F}(t), \tag{3}$$

$$\underline{u} = \begin{pmatrix} u_1 \\ \dots \\ u_n \end{pmatrix}, \quad \underline{\underline{A}} = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ & \dots & \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix}, \quad \underline{F}(t) = \begin{pmatrix} F_1(t) \\ \dots \\ F_n(t) \end{pmatrix}. \tag{4}$$

For a homogeneous system, $\underline{F}(t) = 0$, the following **Abel's formula** holds:

$$\frac{d}{dt}W(\underline{u}_1, \dots, \underline{u}_n) = \mathrm{Tr}\underline{\underline{A}}(t)\,W(\underline{u}_1, \dots, \underline{u}_n), \quad W(\underline{u}_1, \dots, \underline{u}_n) = \det(\underline{u}_1, \dots, \underline{u}_n). \tag{5}$$

To prove this, we first define a **fundamental matrix** whose columns are n independent solutions of the homogeneous system, $\underline{\underline{\Phi}} = (\underline{u}_1, \ldots, \underline{u}_n)$.
This matrix satisfies the equation: $\frac{d}{dt}\underline{\underline{\Phi}} = \underline{\underline{A}}\,\underline{\underline{\Phi}}$.
Now we differentiate the identity ln det $\underline{\underline{\Phi}}$ = Tr ln$\underline{\underline{\Phi}}$, with det $\underline{\underline{\Phi}} = W(\underline{u}_1, \ldots, \underline{u}_n)$, thereby getting

$$\frac{1}{W}\frac{d}{dt}W = \mathrm{Tr}\left[\underline{\underline{\Phi}}^{-1}\frac{d}{dt}\underline{\underline{\Phi}}\right] = \mathrm{Tr}[\underline{\underline{\Phi}}^{-1}\underline{\underline{A}}\,\underline{\underline{\Phi}}] = \mathrm{Tr}[\underline{\underline{A}}\,\underline{\underline{\Phi}}\,\underline{\underline{\Phi}}^{-1}] = \mathrm{Tr}\underline{\underline{A}},$$

which is **Abel's identity**.
The formula ln(det $\underline{\underline{\Phi}}$) = Tr(ln$\underline{\underline{\Phi}}$) follows from the identities det($\underline{\underline{A}}\,\underline{\underline{B}}$) = det$\underline{\underline{A}}$ det$\underline{\underline{B}}$ and Tr($\underline{\underline{A}}\,\underline{\underline{B}}$) = Tr($\underline{\underline{B}}\,\underline{\underline{A}}$) valid for any matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$. Defining $\underline{\underline{A}} = \underline{\underline{R}}^{-1}$, $\underline{\underline{B}} = \underline{\underline{\Phi}}\,\underline{\underline{R}}$, $\underline{\underline{R}}$ the matrix of **eigenvectors** of $\underline{\underline{\Phi}}$, and $\underline{\underline{D}} = \underline{\underline{R}}^{-1}\underline{\underline{\Phi}}\,\underline{\underline{R}}$, the **diagonalized matrix** with the **eigenvalues** of $\underline{\underline{\Phi}}$, we can show that ln(det $\underline{\underline{\Phi}}$) $= \ln \Pi_{j=1}^{n}\lambda_i = \sum_{i=1}^{n}\ln\lambda_i$ = Tr(ln$\underline{\underline{\Phi}}$), where λ_i are the eigenvalues of $\underline{\underline{\Phi}}$.

A. Autonomous linear homogeneous system

The general form of an autonomous linear homogeneous system is

$$\begin{aligned} \frac{du_1}{dt} &= a_{11}u_1 + \ldots + a_{1n}u_n, \\ &\ldots \\ \frac{du_n}{dt} &= a_{n1}u_1 + \ldots + a_{nn}u_n. \end{aligned} \tag{6}$$

Equation (6) can be written as a vector ODE:

$$\frac{d\underline{u}}{dt} = \underline{\underline{A}}\,\underline{u}, \tag{7}$$

$$\underline{u} = \begin{pmatrix} u_1 \\ \ldots \\ u_n \end{pmatrix}, \quad \underline{\underline{A}} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ & \ldots & \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}. \tag{8}$$

We seek a solution of this equation as $\underline{u}(t) = \underline{U}e^{\lambda t}$ which, inserted in (7) yields the **eigenvalue problem**:

$$\underline{\underline{A}}\,\underline{U} = \lambda\underline{U}. \tag{9}$$

If the matrix $\underline{\underline{A}}$ has n **different eigenvalues** λ_j with eigenvectors $\underline{\Phi}_j$, $j = 1, \ldots, n$, the **superposition principle** implies that the **general solution** of (7) is the linear combination

$$\underline{u}(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t} \underline{\Phi}_j, \tag{10}$$

where the c_j are arbitrary.

B. Two-dimensional linear homogeneous system

In this case, the **coefficient matrix** is

$$\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{with } \tau = \text{Tr}\underline{\underline{A}} = a + d, \quad \Delta = \det\underline{\underline{A}} = ad - bc. \tag{11}$$

The **eigenvalue problem** leads to the equation

$$\lambda^2 - \tau\lambda + \Delta = 0 \Longrightarrow \lambda_j = \frac{\tau + (-1)^j \sqrt{\tau^2 - 4\Delta}}{2}, \quad j = 1, 2, \tag{12}$$

$$\lambda_1 + \lambda_2 = \tau, \quad \lambda_1\lambda_2 = \Delta. \tag{13}$$

There several **cases** to consider:

*1. **Case:*** $\lambda_1 < 0 < \lambda_2$***, equivalently,*** $\Delta < 0$***.***

Let $\underline{\Phi}_j$ be the eigenvector corresponding to the eigenvalue λ_j. The general solution (10) (with $n = 2$) becomes $c_2\underline{\Phi}_2 e^{\lambda_2 t}$ as $t \to \infty$ provided $c_2 \neq 0$ and it becomes $c_1\underline{\Phi}_2 e^{\lambda_1 t}$ as $t \to -\infty$ provided $c_1 \neq 0$.

The special solutions with $c_1 = 0$, $c_2 \neq 0$ tend to $(0, 0)$ as $t \to -\infty$ and escape away from the origin as $t \to \infty$. Similarly, solutions with $c_2 = 0$, $c_1 \neq 0$ tend to $(0, 0)$ as $t \to \infty$ and escape away from the origin as $t \to -\infty$.

These special solutions are called **separatrices** and they are straight lines with slopes $u_2/u_1 = \Phi_{22}/\Phi_{21}$ ($c_1 = 0$) and $u_2/u_1 = \Phi_{12}/\Phi_{11}$ ($c_2 = 0$). In this case, **the origin** is a solution corresponding to $c_1 = c_2 = 0$ called a ***saddle point***.

In the ***phase plane*** (u_1, u_2), the solutions (10) (with $n = 2$ and for different c_1 and c_2) are curves called ***trajectories***.

Except for the separatrices, all trajectories are hyperbolas whose asymptotes are the separatrices.

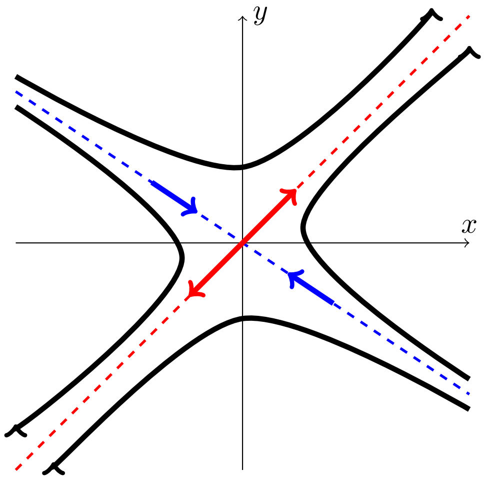


FIG. 1: Phase portrait of a saddle point.

Note that in any neighborhood of the origin there are points such that the trajectories passing through them leave the neighborhood after some positive time. This means that the origin is an ***unstable fixed point*** (constant solutions of the system of ODEs are called fixed points: for (7) the origin is the only **fixed point**). A picture comprising the saddle point, the separatrices and a few trajectories is a *phase portrait*. See figure 1.

2. ***Case:*** $\tau^2 > 4\Delta > 0$.

In this case, the eigenvalues are real and they are both positive or both negative.
Consider first that $\lambda_1 < \lambda_2 < 0$. Then $e^{\lambda_1 t} \ll e^{\lambda_2 t}$ for sufficiently large $t > 0$. This means that $\underline{u}(t) \sim c_2 \underline{\Phi}_2 e^{\lambda t}$ if $c_2 \neq 0$. Eventually all trajectories tend to the origin as $t \to \infty$ and, for $c_2 \neq 0$, they approach the straight line $u_2/u_1 = \Phi_{22}/\Phi_{21}$ as $t \to \infty$. This line is the normal direction of approach to the origin (**slow eigendirection**). For special initial conditions such that $c_2 = 0$, the trajectories are straight lines $u_2/u_1 = \Phi_{12}/\Phi_{11}$ entering the origin as $t \to \infty$ and constitute the exceptional direction of approach to the origin (**fast eigendirection**).
Note that, as $t \to -\infty$, all trajectories escape from the origin and approach the exceptional direction. All points in a neighborhood of the origin tend to it as $t \to \infty$. The origin is called an ***asymptotically stable node***. See Fig. 2.

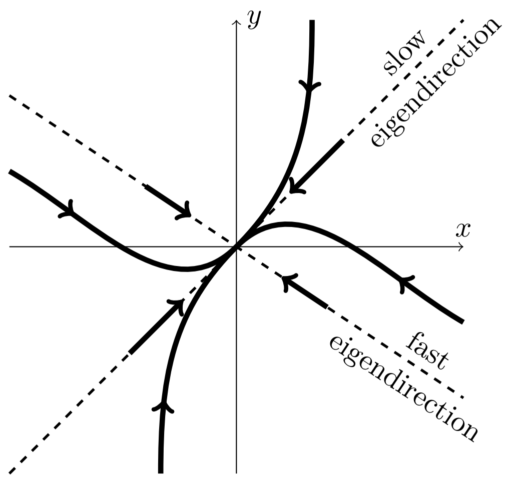


FIG. 2: Phase portrait of an asymptotically stable node.

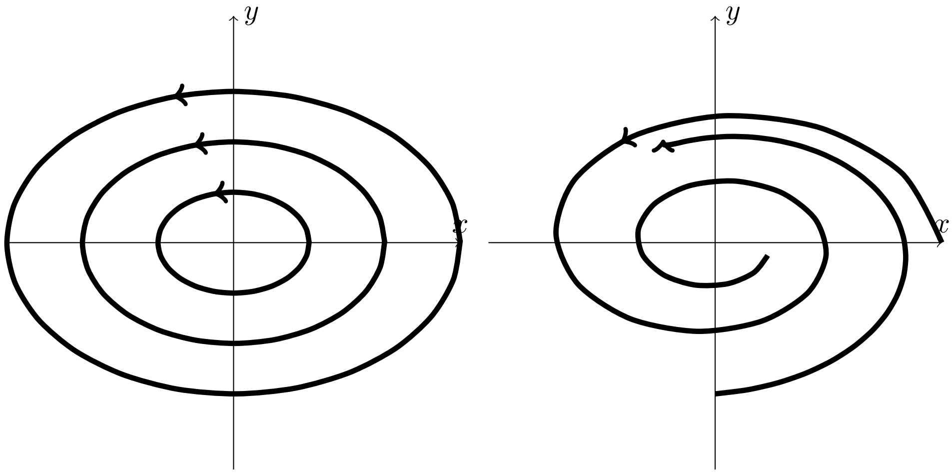
The case $\lambda_1 > \lambda_2 > 0$ is obtained from the previous one by replacing $-t$ instead of t. All trajectories escape the origin which is an ***unstable node***.

3. ***Case:*** $0 < \tau^2 < 4\Delta$.

The eigenvalues are complex: $\lambda = \tau/2 + i\Omega$ and $\overline{\lambda} = \tau/2 - i\Omega$, where $\Omega = \sqrt{\Delta - \tau^2/4}$, with associated complex eigenvectors $\underline{\Phi}$ and $\overline{\underline{\Phi}}$.
The general solution of (7) (with $n = 2$) is now

$$\underline{u}(t) = e^{\tau t/2}\mathrm{Re}\left(c\underline{\Phi}e^{i\Omega t}\right), \tag{14}$$

where c is an arbitrary complex constant.
If the eigenvalues have negative real parts ($\tau < 0$), all trajectories are **spirals** that tend to the origin as $t \to \infty$.
The origin is an ***asymptotically stable spiral point*** (also called **stable focus**); see Fig. 3.
If the eigenvalues have positive real parts ($\tau > 0$), all trajectories escape from the origin as time increases (they tend to the origin as $t \to -\infty$) and the origin is an **unstable focus** or **spiral point**.



(a) center (b) spiral

FIG. 3: Phase portrait of (a) center, (b) stable spiral point.

4. ***Case:*** $\tau = 0$, $\Delta > 0$.

The eigenvalues are pure imaginary, all trajectories are closed (**ellipses** or **circles**) and the origin is a ***center***, also seen in Fig. 3.
A center is stable but not asymptotically stable because the trajectories do not tend to it as $t \to \infty$.
If $c = \rho e^{i\alpha}$ and the eigenvector is

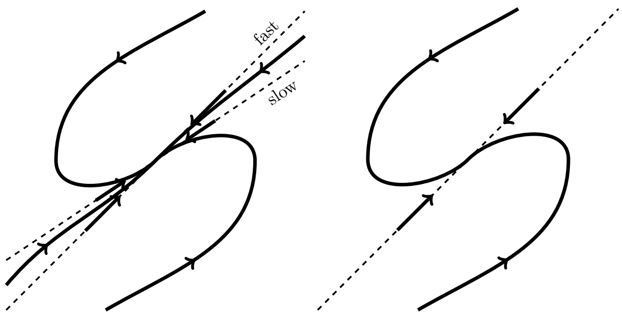
$$\underline{\Phi} = \begin{pmatrix} 1 \\ re^{i\theta} \end{pmatrix}, \tag{15}$$

the general solution (14) is

$$\underline{u}(t) = \rho \begin{pmatrix} \cos(\Omega t + \alpha) \\ r\cos(\Omega t + \alpha + \theta) \end{pmatrix}. \tag{16}$$

A little exercise in trigonometry shows that the ellipses corresponding to (16) are

$$\frac{u_1^2}{\rho^2} + \left(\frac{u_1 \cot\theta}{\rho} - \frac{u_2}{r\rho\sin\theta}\right)^2 = 1. \tag{17}$$



(a) node (b) degenerate node

FIG. 4: Borderline cases: degenerate node.

5. ***Borderline cases:*** $\tau^2 - 4\Delta = 0$.

In these cases, borderline between nodes and spiral points, the eigenvalue $\lambda = \tau/2$ has multiplicity two.
There are two possibilities, either there is a single eigenvector associated to the eigenvalue or there are two independent eigenvectors.
An example of the first case is the coefficient matrix

$$\underline{\underline{A}} = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}.$$

There is a single eigendirection in which trajectories approach a **stable degenerate node** and trajectories try to wind about the fixed point, but they are parallel to the eigendirection both as $t \to \infty$ and as $t \to -\infty$. See Figure 4; the idea is that the degenerate node corresponds to a node in which the fast and slow eigendirections approach each other and become the same.
We can use the **method of undetermined coefficients** to find the solution:
$\underline{u}(t) = (\underline{V}_1 + \underline{V}_2 t)e^{\tau t/2}$.
Inserting this into (7), we find

$$\frac{\tau}{2}(\underline{V}_1 + \underline{V}_2 t)e^{\tau t/2} + \underline{V}_2 e^{\tau t/2} = \underline{\underline{A}}(\underline{V}_1 + \underline{V}_2 t)e^{\tau t/2},$$

from which

$$\underline{\underline{A}}\,\underline{V}_2 = \frac{\tau}{2}\underline{V}_2, \quad \left(\underline{\underline{A}} - \frac{\tau}{2}\right)\underline{V}_1 = \underline{V}_2. \tag{18}$$

Then $\underline{V}_2$ is the eigenvector corresponding to the double eigenvalue $\lambda = \tau/2$ and $\underline{V}_1$ is a generalized eigenvector corresponding to the same eigenvalue.

Another method to determine the other independent solution is to use **Abel's formula**. Let $\underline{u}_1 = \underline{V}_1 e^{\tau t/2}$ be the solution corresponding to the double eigenvalue $\tau/2$.

By Abel's formula, the Wronskian with the other independent solution is $W(\underline{V}_1 e^{\tau t/2}, \underline{u}(t)) = e^{\tau t}$. Then $V_{11}u_2(t) - V_{12}u_1(t) = e^{\tau t/2}$ is a relation between the components of $\underline{u}$.

Let us assume that $V_{11} \neq 0$. Inserting $u_2(t) = [e^{\tau t/2} + V_{12}u_1(t)]/V_{11}$ in the ODE for $u_1(t)$, we obtain a first-order linear ODE for u_1 which, when solved, provides the other independent solution $\underline{u}$.

Example 1.

The coefficient matrix in the following system of ODEs has trace $\tau = 4$ and a double eigenvalue $\lambda = 2$,

$$\underline{u}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \underline{u}.$$

One solution is

$$\underline{u}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

From Abel's formula

$$\begin{vmatrix} e^{2t} & u \\ -e^{2t} & v \end{vmatrix} = e^{4t} \Longrightarrow u + v = e^{2t}.$$

The first ODE of the system is $u' = u - v = 2u - e^{2t}$.

This yields $(e^{-2t}u)' = -1$ and therefore $u = -te^{2t}$, $v = (t+1)e^{2t}$ (we have ignored an arbitrary constant).

The same solutions are obtained by the method of undetermined coefficients.

The diagonal matrix

$$\underline{\underline{A}} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

is an example of the second case.
Any direction is an eigendirection and we have a **star node**, in which initial points approach the origin (or escape from it) along the straight line joining them to the origin.

6. Classification of fixed points.

Figure 5 visualizes the classification of fixed points depending on the values of trace and determinant of the coefficient matrix.
We have used the formulas

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right), \quad \lambda_1 + \lambda_2 = \tau. \quad \lambda_1\lambda_2 = \Delta.$$

-If $\Delta < 0$, one eigenvalue is positive and the other negative, the fixed point is a **saddle**.

-If $\Delta > 0$, we can have nodes for $\tau^2 - 4\Delta > 0$ (**stable** for $\tau < 0$, **unstable** for $\tau > 0$), **spiral points** for $\tau^2 - 4\Delta < 0$ and $\tau \neq 0$ (**stable** for $\tau < 0$, **unstable** for $\tau > 0$) and **centers** for $\tau = 0$.
For $\tau^2 - 4\Delta = 0$, we have the **borderline cases** of **degenerate nodes** or **stars**.

-If $\Delta = 0$, at least one of the eigenvalues is zero and this means that the corresponding eigenvector defines a straight line all whose points are fixed points. This is a **line of non-isolated fixed points**.

II. INHOMOGENEOUS LINEAR SYSTEMS. VARIATION OF PARAMETERS. UNDETERMINED COEFFICIENTS.

Let us consider the general nth-order system

$$\frac{d\underline{u}}{dt} - \underline{\underline{A}}(t)\,\underline{u} = \underline{F}(t), \tag{19}$$

where $\underline{F}$ and $\underline{u}$ are n-dimensional column vectors and the coefficient matrix is time dependent.
Assume that we know n linearly independent solutions $\underline{\psi}_j(t)$ $(j = 1, \ldots, n)$ of the homogeneous system with $\underline{F}(t) = 0$.

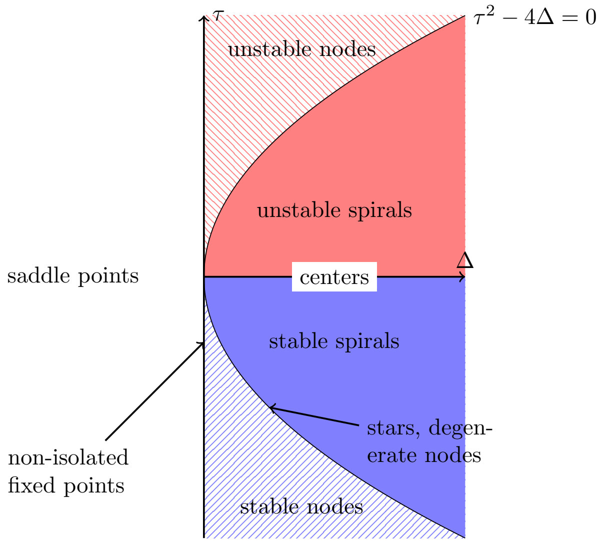


FIG. 5: Classification of fixed points depending on the values of τ and Δ.

The **variation of parameters solution** is

$$\underline{u}(t) = \sum_{j=1}^{n} y_j(t)\underline{\psi}_j(t) \equiv (\underline{\psi}_1(t), \ldots, \underline{\psi}_n(t)) \begin{pmatrix} y_1(t) \\ \ldots \\ y_n(t) \end{pmatrix} \equiv \underline{\underline{\psi}}(t)\underline{y}(t). \tag{20}$$

Substituting (20) into (19), we find

$$\sum_{j=1}^{n} \frac{dy_j}{dt}\underline{\psi}_j(t) = \underline{F}(t). \tag{21}$$

This equation can be written in matrix form as

$$\underline{\underline{\psi}}\frac{d}{dt}\underline{y} = \underline{F}(t). \tag{22}$$

Since the $\underline{\psi}_j(t)$ are independent, $\det(\underline{\underline{\psi}}) \neq 0$, and there exist the inverse matrix $\underline{\underline{\psi}}^{-1}$. Then (22) yields

$$\underline{y}(t) = \int_{t_0}^{t} \underline{\underline{\psi}}^{-1}(s)\underline{F}(s)ds, \tag{23}$$

$$\underline{u}(t) = \underline{\underline{\psi}}(t)\int_{t_0}^{t} \underline{\underline{\psi}}^{-1}(s)\underline{F}(s)ds. \tag{24}$$

Is this solution unique?

The $\underline{\psi}_j(t)$ form a set of independent solutions of the homogeneous problem. Any other set of independent solutions, $\underline{\Psi}_j(t)$, $j = 1, \ldots, n$ will be related by $\underline{\underline{\Psi}}(t) = \underline{\underline{\psi}}(t)\,\underline{\underline{C}}$, where $\underline{\underline{C}}$ is an invertible matrix of constant coefficients.

Then in (23), $\underline{\underline{\Psi}}(t)\underline{\underline{\Psi}}^{-1}(s) = \underline{\underline{\psi}}(t)\underline{\underline{C}}\,\underline{\underline{C}}^{-1}\underline{\underline{\psi}}(s) = \underline{\underline{\psi}}(t)\underline{\underline{\psi}}(s)$, and (23) with $\underline{\underline{\Psi}}$ instead of $\underline{\underline{\psi}}$ produces the same formula.

Example 2.

Let

$$\underline{F}(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}, \quad \underline{\underline{A}} = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}, \quad \text{with } \tau = -3, \quad \Delta = -4. \tag{25}$$

The eigenvalues of $\underline{\underline{A}}$ are $(-3 \pm 5)/2$, i.e., 1 and -4.

The eigenvector corresponding to $\lambda_1 = -4$ is $\underline{\Phi}_1 = (2, -3)$ and that corresponding to $\lambda_2 = 1$ is $\underline{\Phi}_2 = (1, 1)$.

The solution matrix is

$$\underline{\underline{\psi}}(t) = \begin{pmatrix} 2e^{-4t} & e^t \\ -3e^{-4t} & e^t \end{pmatrix}, \quad \text{and its inverse is} \quad \underline{\underline{\psi}}^{-1}(t) = \frac{1}{5}\begin{pmatrix} e^{4t} & -e^{4t} \\ 3e^{-t} & 2e^{-t} \end{pmatrix}. \tag{26}$$

For $\underline{F}(t)$ as in (25), we find

$$\frac{d\underline{y}}{dt} = \underline{\underline{\psi}}^{-1}\underline{F} = \frac{1}{5}\begin{pmatrix} e^{5t} \\ 3 \end{pmatrix} \Longrightarrow \underline{y} = \frac{1}{5}\begin{pmatrix} (e^{5t}-1)/5 \\ 3t \end{pmatrix}, \tag{27}$$

which satisfies $\underline{y}(0) = \underline{0}$ and yields the particular solution

$$\underline{u}(t) = \underline{\underline{\psi}}\,\underline{y} = \frac{e^t}{5}\begin{pmatrix} \frac{2}{5}(1-e^{-5t}) + 3t \\ -\frac{3}{5}(1-e^{-5t}) + 3t \end{pmatrix}. \tag{28}$$

We can check this particular solution with the **method of undetermined coefficients**. Since one of the solutions of the homogeneous system is proportional to e^t, we insert

$$\underline{u}(t) = (\underline{b} + \underline{c}t)e^t, \tag{29}$$

into (19) thereby getting

$$\underline{b} + \underline{c} + \underline{c}t = \underline{\underline{A}}(\underline{b} + \underline{c}t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then $\underline{c}$ is an eigenvector corresponding to the eigenvalue 1 so that

$$\underline{c} = \mu \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (\underline{\underline{A}} - \underline{\underline{I}})\underline{b} = \underline{c} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{30}$$

The latter equation gives

$$2(c_2 - c_1) = \mu - 1, \quad -3(c_2 - c_1) = \mu \Longrightarrow -\frac{2}{3} = \frac{\mu - 1}{\mu} \Longrightarrow \mu = \frac{3}{5}, \tag{31}$$

and $c_1 - c_2 = 1/5$.

Then (29) becomes

$$\underline{u}(t) = e^t \begin{pmatrix} \frac{3t}{5} + c_2 + \frac{1}{5} \\ \frac{3t}{5} + c_2 \end{pmatrix},$$

and the **general solution of the system** is

$$\underline{u}(t) = e^t \begin{pmatrix} \frac{3t}{5} + \frac{1}{5} \\ \frac{3t}{5} \end{pmatrix} + c_2 \begin{pmatrix} e^t \\ e^t \end{pmatrix} + k_1 \begin{pmatrix} 2e^{-4t} \\ -3e^{-4t} \end{pmatrix}, \quad c_2,\ k_1 \text{ are arbitrary numbers.} \tag{32}$$

For $c_2 = -3/25$ and $k_1 = -1/25$, (32) becomes (28).

III. SUPPLEMENTARY MATERIAL: REDUCTION TO NORMAL MODES

A linear system with a constant coefficient matrix $\underline{\underline{A}}$ can be reduced to a system of n independent first-order linear ODEs provided $\underline{\underline{A}}$ is **diagonalizable**, $\underline{\underline{A}} = \underline{\underline{R}}^{-1}\underline{\underline{D}}\,\underline{\underline{R}}$, where $\underline{\underline{D}}$ is a **diagonal matrix** with nonzero entries $d_1, \ldots, d_n$.

In this case, the **change of variable** $\underline{u} = \underline{\underline{R}}\,\underline{v}$ yields

$$\frac{d}{dt}\underline{u}(t) = \underline{\underline{R}}\frac{d}{dt}\underline{v} = \underline{\underline{A}}\,\underline{\underline{R}}\,\underline{v} + \underline{F}(t) \Longrightarrow \frac{d}{dt}\underline{v} = \underline{\underline{R}}^{-1}\underline{\underline{A}}\,\underline{\underline{R}}\,\underline{v} + \underline{\underline{R}}^{-1}\underline{F}(t) = \underline{\underline{D}}\,\underline{v} + \underline{\underline{R}}^{-1}\underline{F}(t),$$

that is

$$\frac{d}{dt}\underline{v}(t) = \underline{\underline{D}}\,\underline{v} + \underline{\underline{R}}^{-1}\underline{F}(t), \quad \frac{dv_i}{dt} = d_i v_i + \sum_{j=1}^{n} (\underline{\underline{R}}^{-1})_{ij} F_j(t), \tag{33}$$

for $i = 1, \ldots, n$.

The **solutions** are

$$v_i(t) = c_i e^{d_i t} + \sum_{j=1}^{n} (\underline{\underline{R}}^{-1})_{ij} \int_0^t e^{d_i(t-s)} F_j(s)\, ds, \tag{34}$$

$$u_i(t) = \sum_{k=1}^{n} (\underline{\underline{R}})_{ik} c_k e^{d_k t} + \sum_{j,k=1}^{n} (\underline{\underline{R}})_{ik} (\underline{\underline{R}}^{-1})_{kj} \int_0^t e^{d_k(t-s)} F_j(s)\, ds. \tag{35}$$

Example 3.

Let us solve Example 2 by reduction to normal modes.

The transformation matrices are $\underline{\underline{R}} = \underline{\underline{\psi}}(0)$ and $\underline{\underline{R}}^{-1} = \underline{\underline{\psi}}^{-1}(0)$, and the diagonal components of the matrix $\underline{\underline{D}}$ are the eigenvalues of $\underline{\underline{A}}$, $d_1 = -4$, $d_2 = 1$.

The components of the vector $\underline{\underline{R}}^{-1}\underline{F}(t)$ are $\frac{1}{5}e^t$ and $\frac{3}{5}e^t$.

Then (33) gives the normal mode equations

$$\frac{dv_1}{dt} = -4v_1 + \frac{1}{5}e^t, \quad \frac{dv_2}{dt} = v_2 + \frac{3}{5}e^t.$$

Solving these equations, we find the **normal modes**

$$v_1 = e^{-4t}c_1 + \frac{1}{25}e^t, \quad \frac{dv_2}{dt} = \left(c_2 + \frac{3t}{5}\right)e^t,$$

thereby yielding

$$u_1 = 2e^{-4t}c_1 + e^t c_2 + \frac{e^t}{5}\left(\frac{2}{5} + 3t\right), \quad u_2 = -3c_1e^{-4t} + \left(c_2 + \frac{3t}{5} - \frac{3}{25}\right)e^t. \tag{36}$$

With $c_1 = -\frac{1}{25}$, $c_2 = 0$, Eq. (36) becomes (28).

[1] W.E. Boyce and R.C. Di Prima, Elementary differential equations and boundary value problems. 9th ed. John Wiley & Sons, N.Y. 2009. Chapter 7.

[2] G. F. Simmons, Differential equations with applications and historical notes. 2nd ed. MacGraw Hill, N.Y. 1991. Chapter 7.