Applied Differential Calculus
Lecture 3: Systems of differential equations

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Outline

- Systems of linear first order ODEs.
- Autonomous planar linear homogeneous systems.
- Inhomogeneous linear systems.

General properties

- Existence, uniqueness, extension and continuity: $f(t,x)$, $\partial f/\partial x_j$, continuous on a box B in $n+1$ space and $(t_0, x^0) \in B$. Then the IVP:

  $$x' = f(t,x), \quad x(t_0) = x^0,$$

  has a unique solution $x(t)$ on some t-interval containing t_0. We can extend the solution to any interval containing t_0 for which the time-state curve lies in B and extends to the boundary of B as t tends to either endpoint of the interval. The solution is continuous in the data x^0 and f.
- Autonomous systems: $f(x)$, $\partial f/\partial x_j$, continuous on a box S in n dimensional phase space and $(x^0) \in S$. Then the IVP:

  $$x' = f(x), \quad x(t_0) = x^0,$$

  has a unique solution $x(t)$ on some t-interval containing t_0. We can extend the solution to any interval containing t_0 for which the state curve lies in S. The solution is continuous in the data x^0 and f.

Linear systems

Assuming $a_{mn}(t)$, $F_n(t)$ continuous on $|t - t_0| \leq \delta$,

$$\frac{du_1}{dt} = a_{11}(t)u_1 + \ldots + a_{1n}(t)u_n + F_1(t),$$
$$\ldots$$
$$\frac{du_n}{dt} = a_{n1}(t)u_1 + \ldots + a_{nn}(t)u_n + F_n(t),$$

IVP with $\underline{u}(t_0) = \underline{u}_0$ has a unique solution. Equivalently, as a vector ODE:

$$\frac{d\underline{u}}{dt} = \underline{\underline{A}}(t)\,\underline{u} + \underline{F}(t),$$

$$\underline{u} = \begin{pmatrix} u_1 \\ \ldots \\ u_n \end{pmatrix}, \quad \underline{\underline{A}} = \begin{pmatrix} a_{11}(t) & \ldots & a_{1n}(t) \\ & \ldots & \\ a_{n1}(t) & \ldots & a_{nn}(t) \end{pmatrix}, \quad \underline{F}(t) = \begin{pmatrix} F_1(t) \\ \ldots \\ F_n(t) \end{pmatrix}.$$

Abel's formula. Autonomous homogeneous linear systems.

Wronskian:

$$W(\underline{u}_1, \ldots, \underline{u}_n) = \det(\underline{\underline{\Phi}}), \quad \underline{\underline{\Phi}}(t) = (\underline{u}_1(t), \ldots, \underline{u}_n(t)),$$

$\underline{\underline{\Phi}}(t)$ fundamental matrix of independent solutions. Abel's formula:

$$\frac{d}{dt} W(\underline{u}_1, \ldots, \underline{u}_n) = \mathrm{Tr}\underline{\underline{A}}(t)\, W(\underline{u}_1, \ldots, \underline{u}_n).$$

Autonomous homogeneous linear system: $\dot{\underline{u}}(t) = \underline{\underline{A}}\, \underline{u}(t)$.
Letting $\underline{u}(t) = \underline{U} e^{\lambda t}$, get eigenvalue problem:

$$\underline{\underline{A}}\, \underline{U} = \lambda \underline{U}.$$

If all eigenvalues λ_j are different,

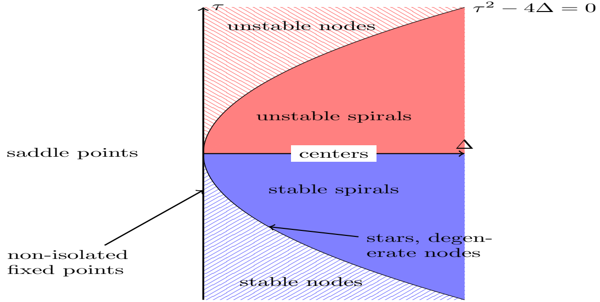
$$\underline{u}(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t} \underline{\Phi}_j, \quad \underline{\Phi}_j \text{ eigenvectors, } c_j \text{ are constant.}$$

Two dimensional (planar) systems

$$\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{with } \tau = \text{Tr}\underline{\underline{A}} = a + d, \quad \Delta = \det\underline{\underline{A}} = ad - bc.$$

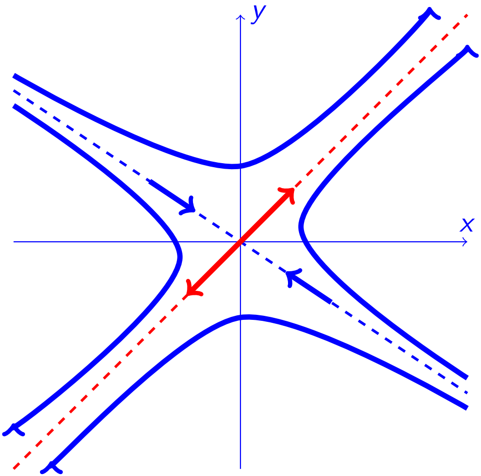
Characteristic equation for eigenvalues ($\lambda_1 + \lambda_2 = \tau$, $\lambda_1\lambda_2 = \Delta$)

$$\lambda^2 - \tau\lambda + \Delta = 0 \Longrightarrow \lambda_j = \frac{\tau + (-1)^j\sqrt{\tau^2 - 4\Delta}}{2}, \quad j = 1, 2.$$



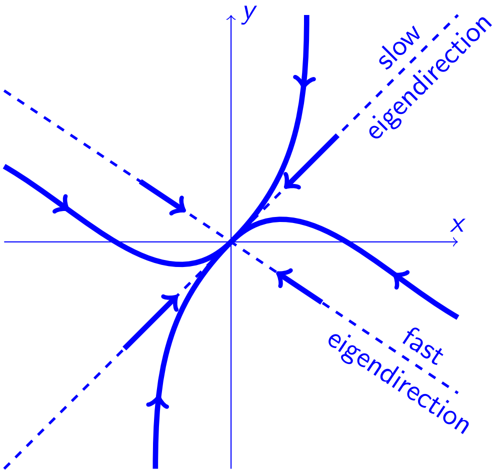
Saddle point

$\Delta < 0 \Longrightarrow \lambda_1 < 0 < \lambda_2$



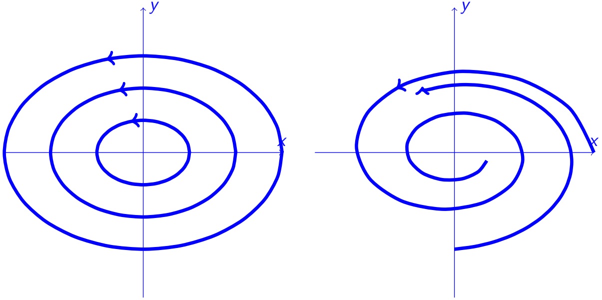
Asymptotically stable node

$\tau < 0,\ \Delta > 0,\ \tau^2 - 4\Delta > 0 \Longrightarrow \lambda_1 < \lambda_2 < 0.$



Centers or asymptotically stable spiral points

$\tau \leq 0$, $\Delta > 0$, $\tau^2 - 4\Delta = -4\Omega^2 < 0 \Longrightarrow \lambda_j = \frac{\tau}{2} \pm i\Omega$.

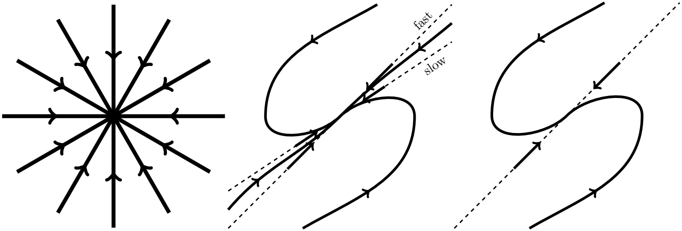


(a) center (b) spiral

Borderline cases

$\tau^2 - 4\Delta = 0.$

- $\underline{\underline{A}} = \frac{\tau}{2}\underline{\underline{I}}$, star point: all directions are eigendirections;
- otherwise degenerate node: only one eigendirection.



(a) star (b) node (c) degenerate node

Variation of parameters $\underline{\dot{u}} = \underline{\underline{A}}(t)\underline{u} + \underline{F}(t)$

Fundamental matrix $\underline{\underline{\Psi}}(t) = (\underline{\psi}_1(t), \ldots, \underline{\psi}_n(t))$, s.t., $\underline{\underline{\dot{\Psi}}}(t) = \underline{\underline{A}}(t)\,\underline{\underline{\Psi}}(t)$. Then we insert a solution of the homogeneous system,

$$\underline{u}(t) = \sum_{j=1}^{n} y_j \underline{\psi}_j(t) = \underline{\underline{\Psi}}(t)\underline{y}$$

for $y_j = y_j(t)$ in the inhomogeneous system:

$$\underline{\underline{\Psi}}(t)\underline{\dot{y}}(t) + \cancel{\underline{\underline{\dot{\Psi}}}(t)\underline{y}(t)} - \cancel{\underline{\underline{A}}(t)\underline{\underline{\Psi}}(t)\underline{y}(t)} = \underline{F}(t) \Longrightarrow \underline{\dot{y}}(t) = \underline{\underline{\Psi}}^{-1}(t)\underline{F}(t).$$

Integrating and substituting in $\underline{u}(t)$, we find the solution of the IVP $\underline{\dot{u}}(t) - \underline{\underline{A}}(t)\underline{u}(t) = \underline{F}(t)$, $\underline{u}(0) = \underline{u}_0$:

$$\underline{u}(t) = \underline{\underline{\Psi}}(t) \int_0^t \underline{\underline{\Psi}}^{-1}(s)\underline{F}(s)ds + \underline{\underline{\Psi}}(t)\underline{u}_0.$$

Example: $\dot{\underline{u}} = \underline{\underline{A}}\,\underline{u} + \underline{F}$

$$\underline{F}(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}, \quad \underline{\underline{A}} = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}, \quad \text{with } \tau = -3, \quad \Delta = -4. \qquad (1)$$

Eigenvalues of $\underline{\underline{A}}$ are $\lambda_1 = -4$, with eigenvector $\underline{\psi}_1 = (2, -3)$ and $\lambda_2 = 1$ with $\underline{\psi}_2 = (1, 1)$. The fundamental matrix is

$$\underline{\underline{\Psi}}(t) = \begin{pmatrix} 2e^{-4t} & e^t \\ -3e^{-4t} & e^t \end{pmatrix}, \text{ and its inverse is } \underline{\underline{\Psi}}^{-1}(t) = \frac{1}{5}\begin{pmatrix} e^{4t} & -e^{4t} \\ 3e^{-t} & 2e^{-t} \end{pmatrix}$$

For $\underline{F}(t)$ as in (1), we find

$$\frac{d\underline{y}}{dt} = \underline{\underline{\Psi}}^{-1}\underline{F} = \frac{1}{5}\begin{pmatrix} e^{5t} \\ 3 \end{pmatrix} \Longrightarrow \underline{y} = \frac{1}{5}\begin{pmatrix} (e^{5t} - 1)/5 \\ 3t \end{pmatrix}, \qquad (3)$$

which satisfies $\underline{y}(0) = \underline{0}$ and yields the particular solution

$$\underline{u}(t) = \underline{\underline{\Psi}}\,\underline{y} = \frac{e^t}{5}\begin{pmatrix} \frac{2}{5}(1 - e^{-5t}) + 3t \\ -\frac{3}{5}(1 - e^{-5t}) + 3t \end{pmatrix}. \qquad (4)$$

Undetermined coefficients for the example

We use $\underline{u}(t) = (\underline{b} + \underline{c}t)e^t$, because $\lambda_2 = 1$. We get

$$(\underline{b} + \underline{c} + \underline{c}t)\cancel{e^t} = \underline{\underline{A}}(\underline{b} + \underline{c}t)\cancel{e^t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cancel{e^t}.$$

Thus $\underline{\underline{A}}\underline{c} = \underline{c}$ (so $\underline{c} = \mu\underline{\psi}_1$ is the eigenvector corresponding to $\lambda_2 = 1$), and

$$\underline{c} = \mu\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (\underline{\underline{A}} - \underline{\underline{I}})\underline{b} = \underline{c} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The latter equation gives $2(c_2 - c_1) = \mu - 1$, $-3(c_2 - c_1) = \mu$. Then $-\frac{2}{3} = \frac{\mu-1}{\mu} \Longrightarrow \mu = \frac{3}{5}$ and $c_1 - c_2 = 1/5$. Thus

$$\underline{u}(t) = e^t\begin{pmatrix} \frac{3t}{5} + c_2 + \frac{1}{5} \\ \frac{3t}{5} + c_2 \end{pmatrix},$$

and the general solution of the system is ($c_2 = -\frac{3}{5}$, $k_1 = -\frac{1}{5}$ previously)

$$\underline{u}(t) = e^t\begin{pmatrix} \frac{3t}{5} + \frac{1}{5} \\ \frac{3t}{5} \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix} + k_1\begin{pmatrix} 2e^{-4t} \\ -3e^{-4t} \end{pmatrix},$$ c_2, k_1 arbitrary const.

Supplementary material: Reduction to normal modes

Let $\underline{\underline{A}} = \underline{\underline{R}}^{-1}\underline{\underline{D}}\,\underline{\underline{R}}$, where $\underline{\underline{D}}$ is a diagonal matrix with entries $d_1, \ldots, d_n$. $\underline{u} = \underline{\underline{R}}\,\underline{v}$ yields

$$\dot{\underline{u}} = \underline{\underline{R}}\dot{\underline{v}} = \underline{\underline{A}}\,\underline{\underline{R}}\,\underline{v} + \underline{F}(t) \Longrightarrow \dot{\underline{v}} = \underline{\underline{R}}^{-1}\underline{\underline{A}}\,\underline{\underline{R}}\,\underline{v} + \underline{\underline{R}}^{-1}\underline{F}(t) = \underline{\underline{D}}\,\underline{v} + \underline{\underline{R}}^{-1}\underline{F}(t),$$

that is

$$\dot{\underline{v}}(t) = \underline{\underline{D}}\,\underline{v} + \underline{\underline{R}}^{-1}\underline{F}(t), \quad \dot{v}_i = d_i v_i + \sum_{j=1}^{n}(\underline{\underline{R}}^{-1})_{ij}F_j(t),$$

for $i = 1, \ldots, n$. The solutions are

$$v_i(t) = c_i e^{d_i t} + \sum_{j=1}^{n}(\underline{\underline{R}}^{-1})_{ij}\int_0^t e^{d_i(t-s)}F_j(s)\,ds, \tag{5}$$

$$u_i(t) = \sum_{k=1}^{n}(\underline{\underline{R}})_{ik}c_k e^{d_k t} + \sum_{j,k=1}^{n}(\underline{\underline{R}})_{ik}(\underline{\underline{R}}^{-1})_{kj}\int_0^t e^{d_k(t-s)}F_j(s)\,ds. \tag{6}$$

Supplementary material: Reduction to normal modes for the example

Transformation matrices: $\underline{\underline{R}} = \underline{\underline{\Psi}}(0)$ and $\underline{\underline{R}}^{-1} = \underline{\underline{\Psi}}^{-1}(0)$, the diagonal components of $\underline{\underline{D}}$ are $d_1 = -4$, $d_2 = 1$. The components of the vector $\underline{\underline{R}}^{-1}\underline{F}(t)$ are $\frac{1}{5}e^t$ and $\frac{3}{5}e^t$. Then normal mode equations are

$$\frac{dv_1}{dt} = -4v_1 + \frac{1}{5}e^t, \quad \frac{dv_2}{dt} = v_2 + \frac{3}{5}e^t.$$

Solving these equations, we find the normal modes

$$v_1 = e^{-4t}c_1 + \frac{1}{25}e^t, \quad v_2 = \left(c_2 + \frac{3t}{5}\right)e^t, \quad \text{thereby yielding}$$

$$u_1 = 2e^{-4t}c_1 + e^t c_2 + \frac{e^t}{5}\left(\frac{2}{5} + 3t\right), \; u_2 = -3c_1 e^{-4t} + \left(c_2 + \frac{3t}{5} - \frac{3}{25}\right)e^t.$$

With $c_1 = -\frac{1}{25}$, $c_2 = 0$, this equation becomes (4).