Applied Differential Calculus

LECTURE 4: Boundary value problems.

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I. BOUNDARY VALUE PROBLEMS

A. Existence of solutions of a BVP

Until now, we have studied **initial value problems** for which the existence and uniqueness theorem establishes that they can be solved uniquely under rather general conditions.
There are also problems in which the solutions of an ODE of order higher than one or a system of at least two ODEs have to satisfy conditions at two or more different values of the independent variable t. These **boundary value problems (BVP)** are more difficult to study because continuity of the functions appearing in the ODEs and of their derivatives do not guarantee the existence and uniqueness of a solution.

Example 1.
Consider the BVP $u''(x) + u(x) = 0$ with $u'(0) = 0$, $u(\pi/2) = 1$.
The **general solution** of the ODE is $u(x) = c_1 \cos x + c_2 \sin x$.
The coefficient functions of the ODE are continuous with continuous derivatives and $W(\cos x, \sin x) = 1 \neq 0$. However the BVP **has no solutions** because $u'(0) = 0$ implies $c_2 = 0$ and $u(\pi/2) = c_1 \cos(\pi/2) = 0$ cannot satisfy the other boundary condition for any possible c_1.

Most BVP have unique solutions.
Consider

$$u'' + a_1(x)u' + a_0(x)u = 0 \tag{1}$$

with **boundary conditions** $u'(x_0) = a$, $u(x_1) = b$.
If $u_1(x)$ and $u_2(x)$ are two independent solutions of the ODE (1), its general solution is $u(x) = Au_1(x) + Bu_2(x)$ which inserted into the boundary conditions produces the system

$$\begin{aligned} Au_1'(x_0) + Bu_2'(x_0) &= a, \\ Au_1(x_1) + Bu_2(x_1) &= b. \end{aligned} \tag{2}$$

These equations can be solved uniquely for A and B provided the determinant

$$\begin{vmatrix} u_1'(x_0) & u_2'(x_0) \\ u_1(x_1) & u_2(x_1) \end{vmatrix} \neq 0. \tag{3}$$

This condition generalizes the nonzero Wronskian condition of the IVP and is usually satisfied by BVP, but not by that in **example 1**.

In that case, the determinant in (3) vanishes and the BVP either has no solution as in example 1, or it has infinitely many solutions.

The latter would be the case in example 1 if the boundary condition at $x = \pi/2$ had been $u(\pi/2) = 0$: Then $u(x) = c_1 \cos x$ with arbitrary values of c_1 solve the BVP.

This latter situation is typical of **eigenvalue problems** expressed as BVP of ODEs, as we shall see later in the course.

Boundary value problems for nonlinear ODEs.

In general, nonlinear BVPs are much harder to solve than linear ones.

Here **we just give a few examples**.

The idea is to first find the **phase portrait** of the nonlinear system, identify the boundary conditions with appropriate curves and then decide which trajectories (if any) may satisfy the BVP taking into account that the 'time' it takes a trajectory to join the 'boundary' curves should be the 'length' L.

It could be that the BVP has several solutions corresponding to different trajectories or no solution because no such trajectories exist.

Example 2.

Consider the BVP:

$$u''(x) = -\alpha \sin u(x), \quad u(0) = 0,\, u(L) = a > 0, \tag{4}$$

to be solved in the interval $0 \leq x \leq L$.

We can show that this problem **has a unique solution** for $a > \pi$ whereas it may have **several solutions** for sufficiently large L.

The idea is to consider the associated **phase plane** obtained from (4):

$$u' = v, \quad v' = -\alpha \sin u \Longrightarrow v^2 - 2\alpha \cos u = h, \tag{5}$$

which is that of the **nonlinear pendulum** in Fig. 1.

We observe that $h > -2\alpha$ for u'^2 to be non-negative.

In this case, the **critical points** are $v = 0$, $u = n\pi$ $(n = 0, \pm 1, \ldots)$.

Points with even n have $h = -2\alpha$ and points with odd n have $h = 2\alpha$.

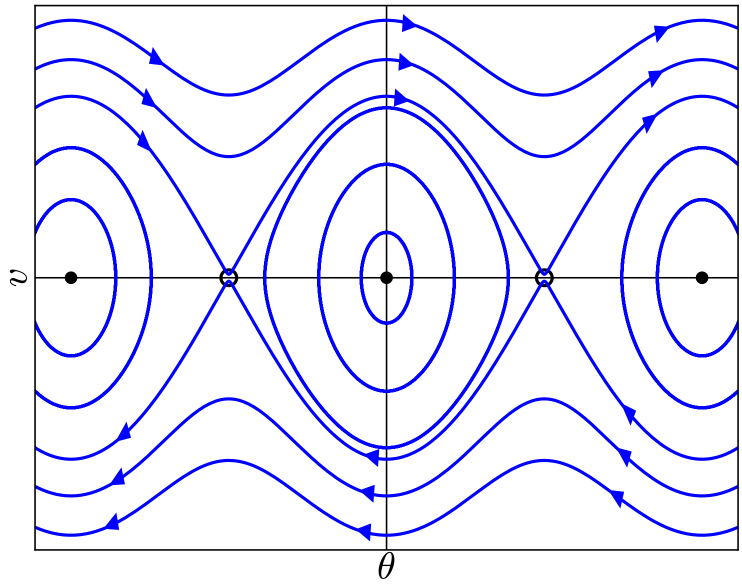

FIG. 1: Phase portrait of the undamped pendulum $x = u$, $v = u'$.

In the latter case, **separatrices** satisfying $v = \pm\sqrt{2\alpha}\sqrt{1+\cos u}$ connect these critical (**saddle**) points above and below the u axis in the phase plane (u, v).

If $|h| < 2\alpha$, the curves (5) are closed curves encircling the critical points $v = 0$, $u = 2n\pi$ $(n = 0, \pm 1, \ldots)$ and the **amplitude** A satisfies $2\alpha \cos A + h = 0$.

The **period of the oscillations** associated with these closed curves is

$$P = 4\int_0^A \frac{dx}{\sqrt{2\alpha \cos x + h}} = \frac{4}{\sqrt{2\alpha}}\int_0^A \frac{dx}{\sqrt{\cos x - \cos A}}. \tag{6}$$

Replacing x by means of $\sin\frac{x}{2} = \sin\frac{A}{2}\sin\theta$, we obtain

$$P = \frac{4}{\sqrt{\alpha}}\int_0^{\pi/2} \frac{dx}{\sqrt{1 - \sin^2\frac{A}{2}\sin^2\theta}} = \frac{4}{\sqrt{\alpha}}K(\sin^2 A),$$

and the period increases with the amplitude A.

$K(m)$ is the complete elliptic integral of the first kind [1]

If $h > 2\alpha$, (5) indicates that u' never becomes zero, and the trajectories are open curves as shown in Fig. 1.

The solutions of the BVP are trajectories that start at $x = 0$ on the v axis ($u = 0$) and end at the vertical line $u = a$ for $x = L$. If $a > \pi$, only one trajectory satisfies these requirements, that having

$$L = \int_0^a \frac{dx}{\sqrt{2\alpha \cos x + h}}, \tag{7}$$

with an appropriate $h > 2\alpha$ given by (7). If $0 < a < \pi$, there may be several trajectories corresponding to the solution of the BVP, namely those satisfying

$$L = nP(h_n) + \int_0^a \frac{dx}{\sqrt{2\alpha \cos x + h_n}}, \tag{8}$$

for $n = 0, 1, \dots N$, where $P(h)$ is the period (6) and h_n is an appropriate number between 0 and 2α.

n is the number of times the trajectory encircles the origin along the closed orbit corresponding to h_n. The longer L is, the larger the integer number N may be.

Similarly, there are trajectories that reach $u = a$ with negative slope and satisfy

$$L = \left(n + \frac{1}{2}\right) P(h'_n) - \int_0^a \frac{dx}{\sqrt{2\alpha \cos x + h'_n}}, \tag{9}$$

for an appropriate h'_n.

There are cases in which we can guarantee that the BVP has a unique solution:

Theorem. *Let $f(x, u, v)$, $\partial f/\partial u$ and $\partial f/\partial v$ be continuous in the region $R = \{(x, u, v) : x_0 \le x \le x_1, -\infty < u < \infty, -\infty < v < \infty\}$. If*

$$\frac{\partial f}{\partial u}(x, u, v) > 0, \quad (x, u, v) \in R, \tag{10}$$

$$\left| \frac{\partial f}{\partial v}(x, u, v) \right| \le M, \quad (x, u, v) \in R, \tag{11}$$

for a positive number M, then the BVP

$$u'' = f(x, u, u'), \quad u(x_0) = a,\ u(x_1) = b \tag{12}$$

has a unique solution $u = u(x)$ in $x_0 \le x \le x_1$.

For the linear BVP given by the ODE (1) with continuous $a_1(x)$, $a_0(x)$ and boundary conditions $u(x_0) = a$, $u(x_1) = b$, there is a unique solution provided $a_0 < 0$ (which is ***not*** satisfied in example 1, in which $a_0 = 1$).

In example 2, $\partial f/\partial u = -\alpha \cos u$ is negative for $|u| < \pi/2$ and therefore the conclusions of the theorem do not apply.

B. Supplementary material: Shooting method for linear BVP

We shall restrict ourselves to BVP for second-order linear ODEs such as

$$u'' = p(x)u'(x) + q(x)u + r(x), \quad u(a) = \alpha,\, u(b) = \beta, \tag{13}$$

with $q(x) > 0$.

To solve **numerically** the BVP (13), we use the **superposition principle** a BVP to calculate numerically the unique solutions of two different IVPs:

$$v'' = p(x)v'(x) + q(x)v + r(x), \quad v(a) = \alpha,\, v'(a) = 0, \tag{14}$$

and

$$w'' = p(x)w'(x) + q(x)w, \quad w(a) = 0,\, w'(a) = 1, \tag{15}$$

Then

$$u(x) = v(x) + Kw(x) \tag{16}$$

is a solution of the ODE in (13) that satisfies the **boundary conditions**

$$\begin{aligned} u(a) = v(a) + Kw(a) = \alpha + 0 = \alpha, \\ u(b) = v(b) + Kw(b). \end{aligned} \tag{17}$$

We now use the other boundary condition to calculate K: $v(b) + Cw(b) = \beta$ gives $K = [\beta - v(b)]/w(b)$ provided $w(b) \neq 0$, and therefore

$$u(x) = v(x) + \frac{\beta - v(b)}{w(b)} w(x). \tag{18}$$

Since $q(x) > 0$, the BVP (13) fulfills the conditions of the theorem and the problematic case $w(b) = 0$ does not appear.

Now, $v(x)$ and $w(x)$ are IVP whose numerical solutions we can find using, for example, **RK4**. We can then calculate the coefficient K in (18) and obtain the numerical solution using the latter formula.

This is the ***shooting method*** for linear BVPs.

For nonlinear second-order ODEs with the boundary conditions in (13), the shooting method consists of solving the IVP associated to the same ODE with initial conditions

$u(a) = \alpha$, $u'(a) = \mu$ for a given μ. Let us call $U(x;\mu)$ the resulting solution which is calculated numerically using any method (e.g., RK4). Then $U(b;\mu)$ is calculated numerically and compared to the required value β. If it is not the same, μ is changed, and the process repeated until $U(b;\mu) = \beta$ is satisfied.

[1] M. Abramowitz & I. A. Stegun, Handbook of Mathematical Functions. Dover, N. Y., 1965.

[2] W.E. Boyce and R.C. Di Prima, Elementary differential equations and boundary value problems. 9th ed. John Wiley & Sons, N.Y. 2009. Chapter 10.

[3] J. H. Mathews & K. D. Fink, Métodos numéricos con MATLAB. 3rd ed. Prentice Hall, 2000. Chapter 9.