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Applied Differential Calculus

LECTURE 6: Fourier series and separation of variables: Wave equation.

Authors:

Manuel Carretero, Luis L. Bonilla, Filippo Terragni, Sergei Iakunin, Rocío Vega

Departamento de Ciencia e Ingeniería de Materiales e Ingeniería Química,

Universidad Carlos III de Madrid,

Bachelor's Degree in Computer Science and Engineering and

Dual Bachelor in Computer Science and Engineering and Business Administration.

I. 1D WAVE EQUATION

1. Derivation of the wave equation

Consider a perfectly inextensible string displaced a vertical distance $u(x, t)$ from its horizontal rest position on the x axis. Let $\theta(x, t)$ be the angle that the string forms with the x axis. The forces acting on the string segment between x and $x + dx$ are body forces and the tensions $T(x, t)$ and $T(x + \Delta x, t)$ that are tangent to the string.

These forces have a vertical component

$T(x + \Delta x, t)\sin\theta(x + \Delta x, t) - T(x, t)\sin\theta(x, t) + \rho(x)\Delta x Q(x, t)$

($\rho(x)\Delta x$ is the mass of the string segment), and therefore **Newton's second law** yields

$$\rho(x)\Delta x\frac{\partial^2 u}{\partial t^2} = T(x + \Delta x, t)\sin\theta(x + \Delta x, t) - T(x, t)\sin\theta(x, t) + \rho(x)\Delta x Q(x, t). \quad (1)$$

As $\Delta x \to 0$, (1) gives

$$\rho(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}[T(x, t)\sin\theta(x, t)] = \rho(x)Q(x, t). \quad (2)$$

We also have

$$\frac{\partial u}{\partial x} = \tan\theta \Longrightarrow \sin\theta = \frac{\frac{\partial u}{\partial x}}{\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}}, \quad (3)$$

which produces

$$\rho(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left[T(x, t)\frac{\frac{\partial u}{\partial x}}{\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}}\right] = \rho(x)Q(x, t). \quad (4)$$

when inserted in (2).

For small angles, this equation becomes linear:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left[T(x, t)\frac{\partial u}{\partial x}\right] = \rho(x)Q(x, t), \quad (5)$$

and we have an inhomogeneous wave equation.

For an inextensible string, T is constant and (5) becomes

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = Q(x, t), \quad (6)$$

where the $c = \sqrt{T/\rho}$ is the **wave velocity** (which is constant for a uniform string).

If the body forces are absent, we get **the one-dimensional wave equation** (which is linear and homogeneous):

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}. \quad (7)$$

2. Boundary conditions

This equation has two derivatives in space and two in time so that we need to specify two initial conditions and two boundary conditions at two different points to expect a unique solution.

For example, if the ends of the string are fixed, $u(0,t) = 0$ and $u(L,t) = 0$. Initially we may specify the string profile and its velocity at each point: $u(x,0) = u_0(x)$, $\partial u(x,0)/\partial t = v_0(x)$.

We have obtained the following **initial-boundary value problem (IBVP)**:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \qquad (8) \\ u(x,0) &= u_0(x), \quad 0 < x < L, \\ \frac{\partial u}{\partial t}(x,0) &= v_0(x), \quad 0 < x < L, \\ u(0,t) &= 0, \quad t > 0, \\ u(L,t) &= 0, \quad t > 0. \end{aligned}$$

We have gotten a **homogeneous Dirichlet problem**.

Other interesting boundary conditions follow when the ends of the string are attached to harmonic springs of negligible mass.

$$\begin{aligned} T\frac{\partial u}{\partial x}(0,t) - ku(0,t) &= -ku_0(t), \quad t > t_0, \\ -T\frac{\partial u}{\partial x}(L,t) - ku(L,t) &= -ku_L(t), \quad t > t_0. \end{aligned} \qquad (9)$$

These are **inhomogeneous Robin boundary conditions** when the rest positions of the springs, u_0 and u_L, are not zero.

Note that they have the same mathematical expressions as those given by Newton's cooling law.

If $k = 0$, the ends of the string are free and we find the **Neumann boundary conditions**

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= 0, \quad t > 0, \\ \frac{\partial u}{\partial x}(L,t) &= 0, \quad t > 0. \end{aligned} \qquad (10)$$

A. Fourier series method for the wave equation

1. Separation of variables for the homogeneous wave equation

We start by trying to solve the homogeneous wave equation (7).
To this end, we look for special solutions of the form

$$u_p(x,t) = X(x)T(t), \tag{11}$$

that are products of functions of x and functions of t.
We insert (11) into (7) and divide the result by $c^2 u_p$, thereby obtaining

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda, \tag{12}$$

where λ is the **separation constant**.
The **Dirichlet problem** (8) produces the same **eigenvalue problem** as for the heat equation:

$$\begin{aligned} X''(x) + \lambda X(x) = 0, \quad 0 < x < L, \\ X(0) = 0, \quad X(L) = 0, \end{aligned} \tag{13}$$

whose solutions are

$$\lambda_n = n^2\pi^2, \quad X_n(x) = \sin(n\pi x), \quad n = 1, 2, \ldots. \tag{14}$$

For each of these eigenvalues, the general solution of the ODE for T in (12) is

$$T_n(t) = a_n \cos\frac{\pi nct}{L} + b_n \sin\frac{\pi nct}{L}. \tag{15}$$

Are there any other eigenfunctions?
You can check that the only solution (13) has for $\lambda \leq 0$ is $X(x) = 0$.
Now that we have found infinitely many particular solutions of the homogeneous wave equation with Dirichlet boundary conditions, we use the **superposition principle** to say that

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\frac{\pi nct}{L} + b_n \sin\frac{\pi nct}{L} \right) \sin\frac{n\pi x}{L}, \tag{16}$$

with constant coefficients a_n and b_n is a solution of the wave equation (whenever this series converges) that satisfies the Dirichlet boundary conditions.

To calculate the **coefficients** a_n, we use the initial conditions in (8)

$$u_0(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}. \tag{17}$$

$$v_0(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L} \sin\frac{n\pi x}{L}. \tag{18}$$

Using the **orthogonality** properties of the **Fourier sine series**, we find:

$$a_n = \frac{2}{L}\int_0^L u_0(x)\sin\frac{n\pi x}{L}dx, \tag{19}$$

$$b_n = \frac{2}{n\pi c}\int_0^L v_0(x)\sin\frac{n\pi x}{L}dx. \tag{20}$$

Putting together (16), (19) and (20), we find

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L u_0(s)\sin\frac{n\pi s}{l}\,ds\right)\cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L} \\ &+ \sum_{n=1}^{\infty}\left(\frac{2}{n\pi c}\int_0^L v_0(s)\sin\frac{n\pi s}{l}\,ds\right)\sin\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}. \end{aligned} \tag{21}$$

The product solutions in (16),

$$\left(a_n\cos\frac{\pi nct}{L} + b_n\sin\frac{\pi nct}{L}\right) = \sqrt{a_n^2+b_n^2}\,\sin\left(\frac{n\pi ct}{L} + \tan^{-1}\frac{a_n}{b_n}\right),$$

are called ***normal modes* of vibration** with **frequencies** $\omega_n = n\pi c/L$.

The **sound** is the superposition of these infinite number of **natural frequencies**.

The normal mode $n = 1$ is called the **first harmonic** or **fundamental**. It has an **angular frequency** $\omega_1 = \pi c/L$ or $f_1 = c/(2L)$ cycles per second. Larger natural frequency means higher sound pitch.

We can produce a desired frequency by changing $c = \sqrt{T/\rho}$, which can be achieved by varying the tension in the string. We can also change the frequency by changing the length of the string: shorter strings imply higher pitch.

For a vibrating string with fixed ends, the nth harmonic has a frequency n times higher than the fundamental one.

Note that the nth harmonic has $(n-1)$ zeros in $(0, L)$ called ***nodes***.

At each time, each mode looks like a simple oscillation in x. The **amplitude** at x varies periodically in time keeping the spatial profile. This is called a ***standing wave***.

Each standing wave is composed of **two traveling waves**; consider for example the term

$$\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L} = \frac{1}{2}\cos\frac{n\pi(x-ct)}{L} - \frac{1}{2}\cos\frac{n\pi(x+ct)}{L}.$$

Here the first term of the right-hand side is a wave traveling to the right with velocity c whereas the second term is a wave traveling to the left with the same velocity.

It is straightforward to analyze the wave equation with **Neumann** or **Robin** boundary conditions along the same lines.

2. ***Inhomogeneous wave equation***

Let us consider the inhomogeneous IBVP

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = Q(x,t), \quad 0 < x < L, \quad t > 0, \tag{22}$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = v_0(x), \quad 0 < x < L, \tag{23}$$

$$u(0,t) = T_0(t), \quad t > 0, \tag{24}$$

$$u(L,t) = T_1(t), \quad t > 0. \tag{25}$$

We expand the solution in a **Fourier sine series** (appropriate for Dirichlet boundary conditions) as:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{L}, \tag{26}$$

where now we do not know its time dependence.

Note that the **orthogonality** of the sine functions give the formula

$$b_n(t) = \frac{2}{L}\int_0^L u(x,t)\sin\frac{n\pi x}{L}dx, \tag{27}$$

for the Fourier coefficients of the solution.

We now multiply (26) by $(2/L)\sin(n\pi x/L)$ and integrate the result from 0 to L, thereby obtaining

$$\frac{d^2 b_n}{dt^2} - \frac{2c^2}{L}\sin\frac{n\pi x}{L}\frac{\partial u}{\partial x}(x,t)\Big|_0^L + \frac{2n\pi c^2}{L^2}\cos\frac{n\pi x}{L}u(x,t)\Big|_0^L + \frac{2n^2\pi^2c^2}{L^3}\int_0^L u(x,t)\sin\frac{n\pi x}{L}dx$$
$$= \frac{2}{L}\int_0^L Q(x,t)\sin\frac{n\pi x}{L}dx,$$

after integrating twice by parts.

Using the boundary conditions (24) and (25) and the formula (27) in this equation, we find the ODE:

$$\frac{d^2 b_n}{dt^2} + \frac{n^2\pi^2c^2}{L^2}b_n = \frac{2n\pi}{L^2}[T_0(t) - (-1)^n T_1(t)] + \frac{2}{L}\int_0^L Q(x,t)\sin\frac{n\pi x}{L}dx \equiv f_n(t). \tag{28}$$

This is a **second-order linear ODE** to be solved with initial conditions

$$b_n(0) = \frac{2}{L}\int_0^L u_0(x)\sin\frac{n\pi x}{L}dx, \tag{29}$$

$$\frac{db_n(0)}{dt} = \frac{2}{n\pi}\int_0^L v_0(x)\sin\frac{n\pi x}{L}dx. \tag{30}$$

The solution of this IVP is

$$b_n(t) = b_n(0)\cos\frac{n\pi ct}{L} + \frac{L\dot{b}_n(0)}{n\pi c}\sin\frac{n\pi ct}{L} + \frac{L}{n\pi c}\int_0^t f_n(s)\sin\frac{n\pi c(t-s)}{L}ds. \tag{31}$$

Equations (26) and (31) are the solution of the IBVP (22)-(25), [1].

B. Resonance

Consider an inhomogeneous wave equation with a source term (or inhomogeneous boundary conditions) proportional to $\cos\omega t$, $f_n(t) = f_n\cos\omega t$.

Then (31) produces

$$b_n(t) = b_n(0)\cos\frac{n\pi ct}{L} + \frac{L\dot{b}_n(0)}{n\pi c}\sin\frac{n\pi ct}{L} + \frac{L}{2n\pi c}\left(\frac{1}{\omega - \frac{n\pi c}{L}} - \frac{1}{\omega + \frac{n\pi c}{L}}\right)\left(\cos\frac{n\pi ct}{L} - \cos\omega t\right) \tag{32}$$

If the forcing frequency approaches that of the nth mode, this equation becomes

$$b_n(t) = b_n(0)\cos\frac{n\pi ct}{L} + \frac{L\dot{b}_n(0)}{n\pi c}\sin\frac{n\pi ct}{L} + \frac{L}{2n\pi c}\frac{t}{2}\sin\frac{n\pi ct}{L}. \tag{33}$$

The **amplitude** of the last term in (33) **increases linearly with** t, so that the amplitude of the nth mode increases while all other modes remain bounded. After a while, **resonant modes** will dominate. Thus the spatial structure of a solution will be primarily due to the **eigenfunctions of the resonant modes**. The other modes are not significantly excited.

II. SUPPLEMENTARY MATERIAL: FINITE DIFFERENCE SOLUTION OF THE WAVE EQUATION

Let us consider the homogeneous Dirichlet problem for the homogeneous wave equation

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \\ u(x,0) &= u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = v_0(x), \quad 0 < x < L, \\ u(0,t) &= 0, \quad t > 0, \\ u(l,t) &= 0, \quad t > 0.\end{aligned} \tag{34}$$

A. Explicit scheme

The simplest **numerical scheme** consists of using **centered differences** in space and time:

$$\frac{\partial^2 u}{\partial t^2}(x_j,t_j) = \frac{u(x_j,t_j+\Delta t)+u(x_j,t_j-\Delta t)-2u(x_j,t_j)}{(\Delta t)^2} - \frac{(\Delta t)^2}{12}\frac{\partial^4 u}{\partial t^4}(x_j,\tilde{t}_j), \quad (35)$$

$$\frac{\partial^2 u}{\partial x^2}(x_j,t_j) = \frac{u(x_j+\Delta x,t_j)+u(x_j-\Delta x,t_j)-2u(x_j,t_j)}{(\Delta x)^2} - \frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(\tilde{x}_j,t_j), \quad (36)$$

where $t_j < \tilde{t}_j < t_j + \Delta t$, $x_j < \tilde{x}_j < x_j + \Delta x$, and we have discretized time and space according to:

$0 = t_0 < t_1 = t_0 + \Delta t < \ldots < t_M = t_0 + M\Delta t = T$, and

$0 = x_0 < x_1 = x_0 + \Delta x < \ldots < xx_N =_0 +N\Delta x = L$.

We now substitute (35) and (36) into (34), drop the error terms and use the notation

$$u(x_j,t_m) = U_j^m \quad (37)$$

(indicating the elongation at the mesh point x_j at time t_m), in the result.

We obtain the following **difference scheme**:

$$U_j^{m+1} - 2U_j^m + U_j^{m-1} = \mu^2(U_{j+1}^m + U_{j-1}^m - 2U_j^m), \quad (38)$$

$$U_j^0 = u_0(x_j) \equiv u_{0j}\,(j=1,\ldots,N-1), \quad (39)$$

$$\frac{U_j^1 - U_j^{-1}}{2\Delta t} = v_0(x_j) \equiv v_{0j} \quad (40)$$

$$U_0^m = 0, \quad U_N^m = 0\,(m=1,\ldots,M-1), \quad (41)$$

in which μ in the **dimensionless ratio** between the **wave velocity** and the **numerical velocity**:

$$\mu = \frac{c}{(\Delta x/\Delta t)}. \quad (42)$$

Note that we have used a centered difference in time for $\partial u/\partial t(x,0)$ in (40) to maintain an $O((\Delta t)^2)$ truncation error.

Now we have to compute U_j^{-1}. We use the initial conditions, (39) and (40) and the partial difference equation for $m = 0$:

$$U_j^1 = 2U_j^0 - U_j^{-1} + \mu^2(U_{j+1}^0 + U_{j-1}^0 - 2U_j^0). \quad (43)$$

We can obtain U_j^{-1} by eliminating U_j^1 from (40) and (43). Once U_j^{-1} and U_j^0 are known, the latter values of u may be computed with equation (38).

B. Stability analysis

To determine stability of the discrete scheme (38), we substitute

$$U_j^m = e^{i\alpha x} R^{t/\Delta t} = e^{i\alpha j \Delta x} R^m, \tag{44}$$

in it.
Canceling $e^{i\alpha x} R^m$, we get

$$R + \frac{1}{R} - 2 = 2\mu^2[\cos(\alpha\Delta x) - 1] = -4\mu^2 \sin^2 \frac{\alpha\Delta x}{2}. \tag{45}$$

This is the same as

$$R^2 - 2\left(1 - 2\mu^2 \sin^2 \frac{\alpha\Delta x}{2}\right) R + 1 = 0, \tag{46}$$

whose solutions are

$$R = 1 - 2\mu^2 \sin^2 \frac{\alpha\Delta x}{2} \pm 2\mu \sin \frac{\alpha\Delta x}{2} \sqrt{\mu^2 \sin^2 \frac{\alpha\Delta x}{2} - 1}. \tag{47}$$

There are **two possibilities**:
If $\mu^2 \sin^2 \frac{\alpha\Delta x}{2} > 1$, the two roots are real and one of them has $|R| > 1$ and the other $|R| < 1$ so that their product is 1. In this case, **the numerical scheme is unstable**.
On the other hand, if $\mu^2 \sin^2 \frac{\alpha\Delta x}{2} < 1$, the two roots (47) are complex conjugate and their modulus is 1 because their product is 1 according to (46). In this case, the solution of the discrete scheme oscillates for fixed $x_j = j\Delta x$ as time $t_m = m\Delta t$ increases, which is similar to the vibrations of a string with fixed ends described by the wave equation.
Thus our **numerical scheme is *stable* if** $\mu^2 \leq 1$, i.e.,

$$\frac{c}{(\Delta x/\Delta t)} \leq 1, \quad c \leq \frac{\Delta x}{\Delta t}. \tag{48}$$

(48) is the ***Courant stability condition*** (also called **Courant-Friedrichs-Lewy** or **CFL condition**) for the wave equation:
the explicit scheme based on centered differences is stable if the wave velocity is kept below the numerical velocity $\Delta x/\Delta t$.
It can be shown that the error of the scheme at fixed x and t is of $O(\Delta t)$. Since it is usual to compute with fixed $\Delta x/\Delta t$, to reduce the error by half, we need to decrease Δx and Δt by a factor 2 without changing the numerical velocity.

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