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Applied Differential Calculus

LECTURE 7: Fourier series and separation of variables: Laplace equation.

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I. 2D LAPLACE EQUATION

A. 2D Laplace equation

1. Laplace equation for a rectangular region

Consider heat conduction in a 2D rectangular, $\mathcal{R} = [0, L] \times [0, M]$, metal sheet without heat sources and fixed temperature at the boundaries.
As $t \to \infty$, the temperature distribution is independent of time and, therefore, it is a solution of the **homogeneous Laplace equation** with inhomogeneous Dirichlet boundary conditions:

$$\nabla^2 u = 0, \tag{1}$$

$$u(x, 0) = F_1(x), \quad u(x, M) = F_2(x), \quad u(0, y) = G_1(y), \quad u(L, y) = G_2(y). \tag{2}$$

It is easy to solve this boundary value problem (BVP) using the **superposition principle** and **separation of variables**.
Let us split the solution of this problem in four BVPs:

$$u(x, y) = U_1(x, y) + U_2(x, y) + U_3(x, y) + U_4(x, y), \tag{3}$$

where the functions U_j, $j = 1, 2, 3, 4$, solve (1) with the boundary conditions:

$$U_1(x, 0) = F_1(x), \quad U_1(x, M) = 0, \quad U_1(0, y) = 0, \quad U_1(L, y) = 0; \tag{4}$$

$$U_2(x, 0) = 0, \quad U_2(x, M) = F_2(x), \quad U_2(0, y) = 0, \quad U_2(L, y) = 0; \tag{5}$$

$$U_3(x, 0) = 0, \quad U_3(x, M) = 0, \quad U_3(0, y) = G_1(y), \quad U_3(L, y) = 0; \tag{6}$$

$$U_4(x, 0) = 0, \quad U_4(x, M) = 0, \quad U_4(0, y) = 0, \quad U_4(L, y) = G_2(y). \tag{7}$$

All these BVPs are solved in the same manner.
Separation of variables, $U_j = X(x)Y(y)$, yields

$$-\frac{Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)} = -\lambda, \tag{8}$$

where λ is the **separation constant**.
Then Dirichlet problem (4) produces the usual eigenvalue problem for the 1D **Laplacian**:

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < L, \tag{9}$$

$$X(0) = 0, \quad X(L) = 0,$$

whose solutions are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, \ldots. \tag{10}$$

For each of these eigenvalues, the general solution of the ODE for Y in (8) is

$$Y_n(y) = a_n e^{\pi ny/L} + b_n e^{-\pi ny/L}. \tag{11}$$

Using the boundary condition $U_1(x, M) = 0$, we find $Y_n(M) = 0$, so that

$$a_n e^{n\pi M/L} + b_n e^{-n\pi M/L} = 0.$$

Then we have

$$Y_n(y) = c_n \sinh\frac{\pi n(y - M)}{L}, \tag{12}$$

where $a_n = \frac{1}{2}c_n e^{-n\pi M/L}$, $b_n = -\frac{1}{2}c_n e^{n\pi M/L}$.

Superposition then gives the **Fourier sine series**

$$U_1(x, y) = \sum_{n=1}^{\infty} c_n \sinh\frac{\pi n(y - M)}{L} \sin\frac{n\pi x}{L}, \tag{13}$$

with constant coefficients c_n to be calculated so that

$$F_1(x) = U_1(x, 0) = \sum_{n=1}^{\infty} c_n \sinh\frac{\pi n(-M)}{L} \sin\frac{n\pi x}{L} = -\sum_{n=1}^{\infty} c_n \sinh\frac{\pi nM}{L} \sin\frac{n\pi x}{L}. \tag{14}$$

Then using the **orthogonality property** of the Fourier sine series, we find

$$c_n = -\frac{2}{L\sinh(n\pi M/L)} \int_0^L F_1(x) \sin\frac{n\pi x}{L} dx, \tag{15}$$

and therefore

$$U_1(x, y) = \frac{2}{L} \sum_{n=1}^{\infty} \frac{\int_0^L F_1(s) \sin\frac{n\pi s}{L} ds}{\sinh(n\pi M/L)} \sinh\frac{\pi n(M - y)}{L} \sin\frac{n\pi x}{L}. \tag{16}$$

The other parts of the solution are similarly calculated, thereby producing the results:

$$U_2(x, y) = \frac{2}{L} \sum_{n=1}^{\infty} \frac{\int_0^L F_2(s) \sin\frac{n\pi s}{L} ds}{\sinh(n\pi M/L)} \sinh\frac{\pi ny}{L} \sin\frac{n\pi x}{L}, \tag{17}$$

$$U_3(x, y) = \frac{2}{M} \sum_{n=1}^{\infty} \frac{\int_0^M G_1(s) \sin\frac{n\pi s}{M} ds}{\sinh(n\pi L/M)} \sinh\frac{\pi n(L - x)}{M} \sin\frac{n\pi y}{M}, \tag{18}$$

$$U_4(x, y) = \frac{2}{M} \sum_{n=1}^{\infty} \frac{\int_0^M G_2(s) \sin\frac{n\pi s}{M} ds}{\sinh(n\pi L/M)} \sinh\frac{\pi nx}{M} \sin\frac{n\pi y}{M}. \tag{19}$$

2. Laplace equation for a circular disk

Consider now the Laplace equation for a circular disk with known temperature at the border $r = a$:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \tag{20}$$

$$u(a, \theta) = f(\theta), \quad -\pi \le \theta \le \pi. \tag{21}$$

Separation of variables in the form:

$$u(r, \theta) = R(r)\Theta(\theta), \tag{22}$$

yields the following equations when inserted into (20):

$$\Theta(\theta)\left(R''(r) + \frac{R'(r)}{r}\right) + \frac{R(r)}{r^2}\Theta''(\theta) = 0 \Longrightarrow \frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda, \tag{23}$$

where λ is the **separation constant**.
Thus we find the **eigenvalue problem**

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \tag{24}$$

$$\Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi). \tag{25}$$

The solutions of this eigenvalue problem are

$$\lambda_n = n^2, \quad n = 0, 1, \dots, \tag{26}$$

$$\Theta_n^{(1)}(\theta) = \cos n\theta, \quad \Theta_n^{(2)}(\theta) = \sin n\theta, \tag{27}$$

The **radial part** $R_n(r)$ obeys the equation:

$$r^2 \frac{d^2 R_n}{dr^2} + r\frac{dR_n}{dr} - n^2 R_n = 0. \tag{28}$$

This is an **Euler equation** to be solved assuming $R_n = r^q$, so that $q(q-1) + q - n^2 = 0$, which gives $q^2 = n^2$ for $n \neq 0$, thereby yielding $q = \pm n$. Then $R_n = a_n r^n + b_n r^{-n}$.
For $n = 0$, $0 = rR_0'' + R_0' = (rR_0')'$. This gives $rR_0' = b_0 \Longrightarrow R_0(r) = b_0 \int dr/r = b_0 \ln r + a_0$.
We have found

$$R_n(r) = \begin{cases} a_n r^n + b_n r^{-n}, & n \neq 0, \\ a_0 + b_0 \ln r, & n = 0. \end{cases} \tag{29}$$

At the center of the disk, the temperature should be finite, so that the condition $|R_n(0)| < \infty$ implies $b_n = 0$.

Then the **superposition principle** yields

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta). \tag{30}$$

The boundary condition (21) gives

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta, \quad \text{for } n \neq 0 \text{ and} \tag{31}$$

$$a_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta, \tag{32}$$

$$b_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta. \tag{33}$$

Then (30)-(33) produce

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta')d\theta' + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\int_{-\pi}^{\pi}(\cos n\theta' \cos n\theta + \sin n\theta' \sin n\theta)d\theta' \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta')d\theta' + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\int_{-\pi}^{\pi} f(\theta')\cos[n(\theta-\theta')]d\theta'. \end{aligned} \tag{34}$$

Note that $(r/a)^n \cos[n(\theta-\theta')]$ is the real part of $(re^{i(\theta-\theta')}/a)^n$ and that we can sum the **geometric progression**

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\theta')}\right)^n = \frac{1}{1-re^{i(\theta-\theta')}/a} - 1.$$

Then (34) can be written as

$$\begin{aligned} u(r,\theta) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta')\left[-\frac{1}{2} + \mathrm{Re}\,\frac{1}{1-re^{i(\theta-\theta')}/a}\right]d\theta' \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta')\left[-\frac{1}{2} + \frac{1-\frac{r}{a}\cos(\theta-\theta')}{(1-\frac{r}{a}\cos(\theta-\theta'))^2 + \frac{r^2}{a^2}\sin^2(\theta-\theta')}\right]d\theta' \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta')\left[-\frac{1}{2} + \frac{1-\frac{r}{a}\cos(\theta-\theta')}{1-\frac{2r}{a}\cos(\theta-\theta')+\frac{r^2}{a^2}}\right]d\theta' \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta')\frac{1-\frac{r^2}{a^2}}{1-\frac{2r}{a}\cos(\theta-\theta')+\frac{r^2}{a^2}}d\theta', \end{aligned}$$

i.e.,

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(a^2-r^2)f(\theta')}{a^2-2ar\cos(\theta-\theta')+r^2}d\theta', \tag{35}$$

which is ***the Poisson formula***.

3. Qualitative properties of the Laplace equation

As a consequence of the Poisson formula (35) or of (30) and (31), we find

$$u(r=0,\theta) = a_0 = \frac{1}{2\pi a}\int_{-\pi}^{\pi} f(\theta')ad\theta'. \tag{36}$$

The temperature at the center of a disk of radius a is equal to the average value of the temperature at the edges of the disk.

Consider now a region $\mathcal{R}$, an interior point p and a circle of radius a and center p entirely contained in $\mathcal{R}$. The previous analysis holds and therefore ***the solution of the Laplace equation at the point $p \in \mathcal{R}$ is equal to the average of the solution along any circle of radius a lying inside $\mathcal{R}$ centered at that point.***

This is **the mean value theorem for the Laplace equation**.

We can use the mean value theorem to prove that **the solution of the Laplace equation inside a finite region $\mathcal{R}$ attain its maximum and minimum values on the boundary of $\mathcal{R}$ unless the solution is constant everywhere**.

These are the ***maximum and minimum principles for the Laplace equation***.

We can do the proof of these principles by contradiction:

Suppose that the maximum value is attained at an interior point $\tilde{p}$. Due to the mean value theorem, this value is the average of the solution along the edges of any interior circle centered at $\tilde{p}$. But this is not possible unless the solution is a constant.

Suppose now that we vary the boundary data of the BVP:

$$\nabla^2 u = 0 \quad \text{with } u = f(\underline{x}) \text{ for } \underline{x} \in \partial\mathcal{R}, \tag{37}$$

from the function f to a **close** function g:

$$\nabla^2 v = 0 \quad \text{with } v = g(\underline{x}) \text{ for } \underline{x} \in \partial\mathcal{R}. \tag{38}$$

Consider now $\omega = u - v$, so that

$$\nabla^2 \omega = 0 \quad \text{with } \omega(\underline{x}) = f(\underline{x}) - g(\underline{x}) \text{ for } \underline{x} \in \partial\mathcal{R}. \tag{39}$$

According to the maximum and the minimum principles, we get

$$\min[f(\underline{x}) - g(\underline{x})] \leq \omega(\underline{x}) \leq \max[f(\underline{x}) - g(\underline{x})]. \tag{40}$$

Thus if the difference between f and g on $\partial\mathcal{R}$ is everywhere small, so is the difference between the solutions of the corresponding BVPs.

That **the solution of the Laplace equation is unique** follows from (40): if u and v are two different solutions corresponding to the same boundary data, $f = g$, (40) yields $0 \leq w \leq 0$ everywhere in $\mathcal{R}$. Thus, if the solution of the Laplace equation exists, it is unique and it depends continuously on the solution specified at the boundary.

We then say that Laplace equation with u specified on the boundary is a ***well-posed problem***.

B. Supplementary material: Poisson equation

Let us consider the Poisson equation on a rectangle $\Omega = (0.L) \times (0, M)$ with inhomogeneous boundary conditions:

$$\nabla^2 u = Q(\underline{x}), \quad \underline{x} \in \Omega, \tag{41}$$

$$u = \alpha(\underline{x}), \quad \underline{x} \in \partial\Omega. \tag{42}$$

We assume that the source terms $Q(\underline{x})$ and $\alpha(\underline{x})$ are continuous with continuous first derivatives on Ω and $\partial\Omega$, respectively.

1. *Solution by 2D eigenfunction expansions*

We will use the eigenfunctions of the Laplacian operator:

$$\nabla^2 \phi_i = -\lambda_i \phi_i, \quad \underline{x} \in \Omega, \tag{43}$$

$$\phi_i(\underline{x}) = 0, \quad \underline{x} \in \partial\Omega. \tag{44}$$

These eigenfunctions for the rectangle $\Omega = (0, L) \times (0, M)$ are

$$\phi_i(\underline{x}) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{M}, \tag{45}$$

$$n, m = 1, 2, \ldots, \quad i = (n, m), \quad \lambda_i = \pi^2\left(\frac{n^2}{L^2} + \frac{m^2}{M^2}\right). \tag{46}$$

We now multiply (41) by ϕ_i and integrate over Ω. The result is

$$\int \phi_i Q = \int \phi_i \nabla^2 u = \int [u\nabla^2\phi_i + \nabla\cdot(\phi_i\nabla u - u\nabla\phi_i)] = -\lambda_i \int u\phi_i - \oint u\underline{n}\cdot\nabla\phi_i,$$

in which $\underline{n}$ is the unit outer normal to the rectangle and we have used the identity $v\nabla^2 u - u\nabla^2 v = \nabla\cdot(v\nabla u - u\nabla v)$, (43) and (44). Using now (42), we obtain

$$\int u\phi_i = -\frac{\int \phi_i Q + \oint \alpha\underline{n}\cdot\nabla\phi_i}{\lambda_i}. \tag{47}$$

If we expand the solution of (41)-(42) as $u(\underline{x}) = \sum_i u_i\phi_i(\underline{x})$, the coefficients u_i are

$$u_i = \frac{\int u\phi_i}{\int \phi_i^2} = -\frac{\int \phi_i Q + \oint \alpha\underline{n}\cdot\nabla\phi_i}{\lambda_i\int\phi_i^2}. \tag{48}$$

In the case of the rectangle with boundary conditions,

$$u(x,y) = \begin{cases} F_1(x), & y=0, \quad 0<x<L, \quad \underline{n} = -(0,1), \\ F_2(x), & y=M, \quad 0<x<L, \quad \underline{n} = (0,1), \\ G_1(y), & x=0, \quad 0<y<M, \quad \underline{n} = -(1,0), \\ G_2(y), & x=L, \quad 0<y<M, \quad \underline{n} = (1,0), \end{cases} \tag{49}$$

this general formula becomes

$$u(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} u_{n,m}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{M}, \tag{50}$$

$$\begin{aligned} u_{n,m} &= -\frac{4LM}{\pi^2(n^2M^2+m^2L^2)}\int_0^L\int_0^M Q(x',y')\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{M}dxdy \\ &+ \frac{4mL}{\pi(n^2M^2+m^2L^2)}\int_0^L [F_1(x) - (-1)^m F_2(x)]\sin\frac{n\pi x}{L}dx \\ &+ \frac{4nM}{\pi(n^2M^2+m^2L^2)}\int_0^M [G_1(y) - (-1)^n G_2(y)]\sin\frac{m\pi y}{M}dy \\ &= -\frac{Q_{nm}}{\pi^2\left(\frac{n^2}{L^2}+\frac{m^2}{M^2}\right)} + \frac{\frac{2m}{M^2}[F_{1,n} - (-1)^m F_{2,n}] + \frac{2n}{L^2}[G_{1,m} - (-1)^n G_{2,m}]}{\pi\left(\frac{n^2}{L^2}+\frac{m^2}{M^2}\right)}. \end{aligned} \tag{51}$$

Note that the outer normal determines the sign of the corresponding contribution to $\oint \alpha\underline{n}\cdot\nabla\phi_i$ as written in (51). The latter formula holds assuming that the source term $Q(x,y)$ is continuous and that the boundary terms have continuous first derivatives and are zero at the intersection points $x=0$, $x=L$, $y=0$ and $y=M$ (so that the odd periodic extensions of the $F_i(x)$ and $G_i(y)$, with $i=1,2$, are continuous at the corners of the rectangle).

2. ***Solution by 1D eigenfunctions***

Let us solve the Poisson equation with homogeneous Dirichlet boundary conditions on the rectangle by a different method.

We use the **1D eigenfunction expansion**

$$u(x,y) \;=\; \sum_{n=1}^{\infty} B_n(y)\sin\frac{n\pi x}{L}, \tag{52}$$

where the sine coefficients $B_n(y)$ are functions of y.

Differentiating (52) with respect to y and substituting this into the Poisson equation, we get

$$\sum_{n=1}^{\infty}\left(\frac{d^2B_n}{dy^2} - \frac{n^2\pi^2}{L^2}B_n\right)\sin\frac{n\pi x}{L} = Q. \tag{53}$$

We have differentiated term by term because the boundary conditions are homogeneous. Thus the sine coefficients satisfy the inhomogeneous ODE:

$$\frac{d^2B_n}{dy^2} - \frac{n^2\pi^2}{L^2}B_n = \frac{2}{L}\int_0^L Q(x,y)\sin\frac{n\pi x}{L}dx \equiv Q_n(y). \tag{54}$$

The homogeneous Dirichlet boundary conditions for u produce the following boundary conditions for $B_n(y)$: $B_n(0) = 0$ and $B_n(M) = 0$. The solution of this BVP is

$$\begin{aligned} B_n(y) = -\frac{L}{n\pi\sinh\frac{n\pi M}{L}}\Bigg[&\sinh\frac{n\pi(M-y)}{L}\int_0^y Q_n(\eta)\sinh\frac{n\pi\eta}{L}d\eta \\ &+ \sinh\frac{n\pi y}{L}\int_y^M Q_n(y)\sinh\frac{n\pi(M-\eta)}{L}d\eta\Bigg]. \end{aligned} \tag{55}$$

[1] R. Haberman, Elementary applied partial differential equations. 3rd ed. Prentice Hall, 1998.