APPLIED DIFFERENTIAL CALCULUS
LECTURE 1: First-order ordinary differential equations.
PROBLEMS

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Problem 1 Solve the following differential equation:

$$y' + y = 2e^{-x} + x^2$$

Problem 2 Solve the following differential equation:

$$y' + \frac{1}{x}y = x^2 - 1\,,\ \ x > 0$$

Problem 3 Solve the following differential equation:

$$y' + y\cos x = \sin x \cos x$$

Problem 4 Solve the following differential equation:

$$y' = x^2/y$$

Problem 5 Solve the following differential equation:

$$y' = \frac{x^2}{y(1+x^3)}$$

Problem 6 Solve the following differential equation:

$$y' + y^2 \sin x = 0$$

Problem 7 Solve the initial value problem (IVP):

$$\text{(IVP)} \begin{cases} (1-x)(1-y)y' = \alpha \in \mathbb{R} \\ y(0) = 0 \end{cases} .$$

Problem 8 Solve the following differential equation:

$$x^3 + xy^2 + (x^2 y + y^3)y' = 0$$

Problem 9 Solve the following differential equation:

$$e^y + \big(xe^y + 2y\big)y' = 0$$

Problem 10 Solve the following differential equation:

$$y^2 e^{xy} + \cos x + \big(e^{xy} + xye^{xy}\big)y' = 0$$

Problem 11 Solve the following differential equation:

$$y' = (2x + y)/(x - y)$$

Problem 12 Solve the following differential equation:

$$y' = (x^2 + 3y^2)/2xy$$

Problem 13 Solve the following differential equation:

$$y' = (y + \sqrt{x^2 - y^2})/x$$

Problem 14 Consider the following initial value problem (IVP)

$$\begin{cases} 2ty + (t^2 + y)\, y' = 0\,, & 0 < t \leq 1\,, \\ y(0) = -2\,. \end{cases}$$

(i) Classify the given differential equation and prove that $y(t) = -t^2 - \sqrt{t^4 + 4}$ is the solution of the IVP.

(ii) Express the differential equation in the form $y' = f(t, y)$ and consider the numerical scheme

$$Y_{n+1} = Y_n + \frac{h}{2}\left(f(t_{n+1}, \tilde{Y}_{n+1}) + f(t_n, Y_n) \right), \quad \text{with} \quad \tilde{Y}_{n+1} = Y_n + hf(t_n, Y_n)\,.$$

Prove that $Y_1 = 4/(h^2 - 2)$ for any step h. In addition, approximate $y(1)$ using the given scheme with $h_1 = 0.5$.

(iii) Estimate the order of the numerical method, knowing that $Y_{10}^{h_2} = -3.239$ is an approximation of $y(1)$ calculated with $h_2 = 0.1$.

Problem 15 Consider the following initial value problem (IVP)

$$\begin{cases} y' + ky = k \sin t + \cos t \\ y(0) = 1 \end{cases}$$

for $t \geq 0$, where k is a positive real parameter.

(a) Classify the differential equation of the IVP and calculate its solution.

(b) Take $k = 3$ in the IVP and find an approximated value of $y(\pi/4)$ by applying the method of explicit Euler with step $h = \pi/4$. Then, compare the result with that obtained considering the exact solution $y(t) = \sin t + e^{-3t}$.

(c) Is the approximation obtained in (b) with step $h = \pi/4$ acceptable? If yes, justify your answer. If no, find an upper bound for h yielding an acceptable approximation of $y(\pi/4)$.

Problem 16 Solve the following initial value problem, writing the solution in explicit form.

$$\begin{cases} (1-\ln x)\, y' \;=\; 1+\ln x + \dfrac{y}{x}\,, \quad \text{for } \; 0 < x < e\,, \\ y(1) \;=\; 1\,. \end{cases}$$

Problem 17 Consider the following initial value problem

$$\begin{cases} y' + y \;=\; 2\,t^2 \\ y(0) \;=\; 5\,. \end{cases}$$

(i) Verify that $y(t) \;=\; e^{-t} + 2\,t^2 - 4\,t + 4$ is the exact solution.

(ii) Use the following Runge-Kutta method

$$Y_{n+1} \;=\; Y_n + \frac{1}{2}\,(\,K_1 \,+\, K_2\,)\,, \quad \text{with } K_1 \;=\; h\,f(t_n, Y_n)\,,\;\; K_2 \;=\; h\,f(t_{n+1}, Y_n + K_1)\,,$$

for $n = 0, 1, 2, \ldots$, to approximate the value $y(0.2)$ with $h = h_1 = 0.1$.

(iii) Knowing that $Y_{20}^{h_2} = 4.09875$ is an approximation of $y(0.2)$ calculated with $h = h_2 = 0.01$, estimate the order of the numerical method in (ii).

Problem 18 Consider the differential equation $xy^2y' + x^3 \;=\; y^3$, with $0 < x < 2$.

(a) Classify it, justifying your answer.

(b) Solve it together with the condition $y(1) = 2$.

Problem 19 Consider the following initial value problem

$$\begin{cases} y' + 6\,y = 0 \\ y(0) = 1\,. \end{cases}$$

(a) Apply to the problem one iteration of the explicit Euler method with step $h_1 = 0.05$. Then, say whether the method is stable with the suggested step.

(b) Use the value Y_1 computed in (a) and the following Adams–Moulton method of order 2

$$Y_{n+2} = Y_{n+1} + \frac{h}{2}\Big[\, f(t_{n+1}, Y_{n+1}) + f(t_{n+2}, Y_{n+2})\,\Big],$$

for $n = 0, 1, 2, \ldots$, to approximate the value $y(0.1)$ with $h = h_1 = 0.05$.

(c) Knowing that $E^{h_2}_{t=0.1} = 0.00112$ is the error of approximating $y(0.1)$ using the method in (b) with step $h_2 = h_1/q$, calculate the value of h_2 (note that $y(0.1) = 0.54881$ and $q \in \mathbb{N}$ is the step reduction factor).

Problem 20 Can you use the explicit Euler method to solve approximately the initial value problem: $y' = 1 + y^2$ in $x \in [0, 3]$, with $y(0) = 0$? *Hint: Find the exact solution and discuss.*

Problem 21 A new European football league is planned so that the yearly sales of tickets grow at a speed proportional to the difference between sales at time t and a higher bound of 300 million euros. Assume no ticket has been sold at $t = 0$ and that sales should be 40 million euro after 3 years (otherwise the competition is cancelled). Based on this assumption, how long it should take for the yearly ticket sales to have reached 220 million euro?

Problem 22 The population of a large cluster of atoms decreases to one third its initial size in one year, at a rate proportional to the instantaneous number of atoms. (a) Model the evolution of the atom population by means of a differential equation and calculate its growth rate proportionality constant r. (b) Use the explicit Euler method to solve the ODE and indicate the maximum step size for which the Euler scheme is stable.

Problem 23 Given the Ordinary Differential Equation (ODE):

$$y' = e^{x+y} \quad \text{with} \quad x > 0,$$

i) Classify this ODE.

ii) Solve the ODE with initial condition $y(1) = 1$.

Problem 24 A cauldron of boiling soup is placed in a room at 0^oC and its temperature becomes 20^oC after 30 minutes. (a) Model the evolution of the soup temperature by means of a differential equation (Newton cooling law) and calculate the cooling rate proportionality constant $-r$. (b) Use the explicit Euler method to solve the ODE and indicate the maximum step size for which the Euler scheme is stable.