APPLIED DIFFERENTIAL CALCULUS
LECTURE 1: First-order ordinary differential equations.
PROBLEMS

Authors:
Manuel Carretero, Luis L. Bonilla, Filippo Terragni, Sergei Iakunin y Rocio Vega

---

Problem 1 Solve the following differential equation:

$$y' + y = 2e^{-x} + x^2$$

SOLUTION:
The equation is a linear first-order ODE.
Solution:

$$y(x) = (c + 2x)e^{-x} + x^2 - 2x + 2\,; \quad c \in \mathbb{R}.$$

Problem 2 Solve the following differential equation:

$$y' + \frac{1}{x}y = x^2 - 1\,,\ x > 0$$

SOLUTION:
The equation is a linear first-order ODE.
Solution:

$$y(x) = \frac{c}{x} + \frac{x^3}{4} - \frac{x}{2}\,; \quad c \in \mathbb{R}.$$

Problem 3 Solve the following differential equation:

$$y' + y\cos x = \sin x \cos x$$

SOLUTION:
The equation is a linear first-order ODE.
Solution:

$$y(x) = ce^{-\sin x} + \sin x - 1\,; \quad c \in \mathbb{R}.$$

Problem 4 Solve the following differential equation:

$$y' = x^2/y$$

SOLUTION:
The equation is a separable first-order ODE.
Implicit solution:

$$y^2(x) = \frac{2x^3}{3} + c\,; \quad c \in \mathbb{R}.$$

Problem 5 Solve the following differential equation:

$$y' = \frac{x^2}{y(1+x^3)}$$

SOLUTION:
The equation is a separable first-order ODE.
Implicit solution:

$$y^2(x) = \frac{2}{3}\ln(1+x^3) + c\,; \quad c \in \mathbb{R}.$$

Problem 6 Solve the following differential equation:

$$y' + y^2 \sin x = 0$$

SOLUTION:
The equation is a separable first-order ODE.
Explicit solution:

$$y(x) = \frac{1}{c - \cos x}\,; \quad c \in \mathbb{R}.$$

Problem 7 Solve the initial value problem (IVP):

$$\text{(IVP)} \begin{cases} (1-x)(1-y)y' = \alpha \in \mathbb{R} \\ y(0) = 0 \end{cases} .$$

SOLUTION:
The equation is a separable first-order ODE.
Using the initial condition we get the implicit solution:

$$y(x) - \frac{y^2(x)}{2} + \alpha \ln|1-x| = 0\,.$$

Problem 8 Solve the following differential equation:

$$x^3 + xy^2 + (x^2 y + y^3)y' = 0$$

SOLUTION:
The equation is a exact first-order ODE.
Implicit solution:

$$x^4 + y^4(x) + 2x^2y^2(x) = c\,; \quad c \in \mathbb{R}.$$

Problem 9 Solve the following differential equation:

$$e^y + \big(xe^y + 2y\big)y' = 0$$

SOLUTION:
The equation is a exact first-order ODE.
Implicit solution:

$$x^4 + y^4(x) + 2x^2y^2(x) = c\,; \quad c \in \mathbb{R}.$$

Problem 10 Solve the following differential equation:

$$y^2e^{xy} + \cos x + \big(e^{xy} + xye^{xy}\big)y' = 0$$

SOLUTION:
The equation is a exact first-order ODE.
Implicit solution:

$$xe^{y(x)} + y^2(x) = c\,; \quad c \in \mathbb{R}.$$

Problem 11 Solve the following differential equation:

$$y' = (2x + y)/(x - y)$$

SOLUTION:
The equation is a homogeneous first-order ODE.
Implicit solution:

$$\frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}y(x)}{2x}\right) - \frac{1}{2}\ln\left(2 + \frac{y^2(x)}{x^2}\right) = \ln|x| + c\,; \quad c \in \mathbb{R}.$$

Problem 12 Solve the following differential equation:

$$y' = (x^2 + 3y^2)/2xy$$

SOLUTION:
The equation is a homogeneous first-order ODE.
Implicit solution:

$$1 + \frac{y^2(x)}{x^2} = c\,|x|\,; \quad c \in \mathbb{R}.$$

Problem 13 Solve the following differential equation:

$$y' = (y + \sqrt{x^2 - y^2})/x$$

SOLUTION:
The equation is a homogeneous first-order ODE.
Implicit solution:

$$y(x) = x\,\sin(\ln|x| + c)\,; \quad c \in \mathbb{R}.$$

Problem 14 Consider the following initial value problem (IVP)

$$\begin{cases} 2ty + (t^2 + y)\,y' \;=\; 0\,, \quad 0 < t \le 1\,, \\ y(0) = -2\,. \end{cases}$$

(i) Classify the given differential equation and prove that $y(t) = -t^2 - \sqrt{t^4 + 4}$ is the solution of the IVP.

(ii) Express the differential equation in the form $y' = f(t, y)$ and consider the numerical scheme

$$Y_{n+1} \;=\; Y_n + \frac{h}{2}\,\Big(f(t_{n+1}, \tilde{Y}_{n+1}) + f(t_n, Y_n)\Big)\,, \quad \text{with} \quad \tilde{Y}_{n+1} \;=\; Y_n + hf(t_n, Y_n)\,.$$

Prove that $Y_1 = 4/(h^2 - 2)$ for any step h. In addition, approximate $y(1)$ using the given scheme with $h_1 = 0.5$.

(iii) Estimate the order of the numerical method, knowing that $Y_{10}^{h_2} = -3.239$ is an approximation of $y(1)$ calculated with $h_2 = 0.1$.

SOLUTION:

(i) The given first-order ODE is exact, since it can be written as $M(t,y) + N(t,y)\,y' = 0$, where $M(t,y) = 2ty$ and $N(t,y) = t^2 + y$, and we have $\dfrac{\partial M}{\partial y} = 2t = \dfrac{\partial N}{\partial t}$. On the other hand, as the solution of the IVP is given, we can follow either of the next steps.

(A) – We can substitute the given solution into the IVP, showing it is verified by $y(t)$.

(B) – We can explicitly calculate the solution of the IVP as follows. As the ODE is exact, there exists a function $F = F(t,y)$ such that $\dfrac{\partial F}{\partial t} = 2ty$, $\dfrac{\partial F}{\partial y} = t^2 + y$, where $\dfrac{dF}{dt} = \dfrac{\partial F}{\partial t} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dt} = 0\,.$
Then, we can obtain F by integrating $\dfrac{\partial F}{\partial t}$, namely

$$F \;=\; \int (2ty)\,dt = t^2 y + \phi(y)\,.$$

In addition, if we calculate the derivative with respect to y of the previous expression and equate it to $\dfrac{\partial F}{\partial y} = t^2 + y$, we get $\phi'(y) = y$, hence $\phi(y) = \dfrac{y^2}{2} + C_1$. As a consequence

($C_1 = 0$), we can write $F(t,y) = \dfrac{y^2}{2} + t^2 y$. Finally, from $\dfrac{dF}{dt} = 0$ we get $\dfrac{y^2}{2} + t^2 y = C$, where the constant C is provided by the initial condition as

$$\frac{y(0)^2}{2} + 0^2 y(0) = C \implies C = 2\,.$$

The solution of the IVP is then $\boxed{y(t) = -t^2 - \sqrt{t^4 + 4}}$.

(ii) We can write

$$y' = f(t,y) = -\frac{2ty}{t^2 + y} \quad \text{with} \quad y(t_0 = 0) = -2 = y_0 \equiv Y_0\,.$$

In order to show that $Y_1 = \dfrac{4}{h^2 - 2}$ for any h, let's first calculate $\tilde{Y}_1 = Y_0 + hf(t_0, Y_0) = -2$. Then, substituting into the numerical scheme, we have $f(t_1, \tilde{Y}_1) = f(h, -2) = \dfrac{4h}{h^2 - 2}$ together with $f(t_0, Y_0) = f(0, -2) = 0$, which yields $Y_1 = -2 + \dfrac{h}{2}\left(\dfrac{4h}{h^2 - 2}\right) = \dfrac{4}{h^2 - 2}$. The value $Y_2^{h_1}$ approximating $y(1)$ is obtained using $h_1 = 0.5$ in the previous expression for Y_1 and performing one more iteration with the scheme, which finally provides $\boxed{Y_2^{h_1} = -3.337}$.

(iii) From (i) we can calculate $y(1) = -3.236$. Furthermore, we have $E_{t=1}^{h_1} = \left|Y_2^{h_1} - y(1)\right| = 0.101$ and $E_{t=1}^{h_2} = \left|Y_{10}^{h_2} - y(1)\right| = 0.003$. As $h_2 = h_1/5$, we get

$$E_{t=1}^{h_2} \approx C h_2^p = C\left(\frac{h_1}{5}\right)^p \approx \frac{E_{t=1}^{h_1}}{5^p}\,,$$

where p is the order of the method. Taking the logarithm of both sides of the previous expression yields $p \approx 2.19$. Thus, the order of the given numerical scheme is $\boxed{p = 2}$.

Problem 15 Consider the following initial value problem (IVP)

$$\begin{cases} y' + ky = k \sin t + \cos t \\ y(0) = 1 \end{cases}$$

for $t \geq 0$, where k is a positive real parameter.

(a) Classify the differential equation of the IVP and calculate its solution.

(b) Take $k = 3$ in the IVP and find an approximated value of $y(\pi/4)$ by applying the method of explicit Euler with step $h = \pi/4$. Then, compare the result with that obtained considering the exact solution $y(t) = \sin t + e^{-3t}$.

(c) Is the approximation obtained in (b) with step $h = \pi/4$ acceptable? If yes, justify your answer. If no, find an upper bound for h yielding an acceptable approximation of $y(\pi/4)$.

SOLUTION:

(a) The equation is a first-order linear ODE, which can be solved by means of the integrating factor $\mu = e^{kt}$. Hence

$$(e^{kt}y)' = e^{kt}\,(\,k\sin t + \cos t\,) \implies e^{kt}y = k\int e^{kt}\sin t\,dt + \int e^{kt}\cos t\,dt + C\,.$$

As $\int e^{kt}\cos t\,dt = e^{kt}\sin t - k\int e^{kt}\sin t\,dt$, we directly get

$$y(t) = \sin t + Ce^{-kt}\,.$$

The initial condition gives $C = 1$, thus the solution of the IVP is

$$y(t) = \sin t + e^{-kt}\,.$$

(b) The approximated value of $y(\pi/4)$ obtained by the first iteration of the explicit Euler method is $Y_1^{h=\frac{\pi}{4}} = 1 + h(1-k) = 1 + \frac{\pi}{4}(1-3) = -0.571$. On the other hand, its exact value is $y(\pi/4) = 0.802$. Hence, the involved error is $|y(\pi/4) - Y_1^{h=\frac{\pi}{4}}| = 1.373$.

(c) As observed in (b), the exact value of $y(\pi/4)$ is positive, while its computed approximation is negative. Hence, the latter doesn't seem to be an acceptable approximation. In order to find an upper bound for h yielding an acceptable approximation of $y(\pi/4)$, let's perform the following analysis.
Upon examining the exact solution, namely $y(t) = \sin t + e^{-3t}$, we realize that the problem is stiff due to the term e^{-3t} that may perturb the stability of the used numerical scheme. Indeed, according to what obtained in (b), we can suspect that the explicit Euler method is unstable with $h = \pi/4$.
From the IVP we have $f(t,y) = -ky + k\sin t + \cos t$. For the sake of clarity, let's call $g_n = k\sin t_n + \cos t_n$, from which we can write $f(t_n, Y_n) = -kY_n + g_n$. Then

$$\begin{aligned}
Y_{n+1} &= Y_n + hf(t_n, Y_n)\\
&= (1-hk)Y_n + h\,g_n\\
&= (1-hk)^2 Y_{n-1} + h\,[g_n + (1-hk)g_{n-1}]\\
&= (1-hk)^3 Y_{n-2} + h\,\left[g_n + (1-hk)g_{n-1} + (1-hk)^2 g_{n-2}\right]\\
&= \ldots\\
&= (1-hk)^{n+1}Y_0 + h\sum_{p=0}^{n}(1-hk)^p g_{n-p}\,.
\end{aligned}$$

The powers $(1-hk)^{n+1}$ vanish as $n \to \infty$ only if $|1-hk| < 1$. As $k > 0$, we get $h < \frac{2}{k} = \frac{2}{3}$ (note that $h = \pi/4 > 2/3$, which justifies the unacceptable approximation obtained above). Thus, taking for instance $h = \frac{\pi}{8} < \frac{2}{3}$ and performing two iterations with the explicit Euler scheme, we get the value $Y_2^{h=\frac{\pi}{8}} = 0.775$, which is a reasonable approximation of $y(\pi/4)$.

Problem 16 Solve the following initial value problem, writing the solution in explicit form.

$$\begin{cases} (1-\ln x)\, y' \;=\; 1+\ln x+\dfrac{y}{x}\,, & \text{for } \; 0<x<e\,, \\ y(1) \;=\; 1\,. \end{cases}$$

SOLUTION:
The differential equation is exact. Indeed, it can be written in the form $M(x,y)+N(x,y)\,y' \;=\; 0$ as

$$\left(1+\ln x+\frac{y}{x}\right)+(\ln x-1)\,y' \;=\; 0$$

with

$$\frac{\partial M(x,y)}{\partial y} \;=\; \frac{\partial N(x,y)}{\partial x} \;=\; \frac{1}{x}\,.$$

Then, its solution is of the form $F(x,y(x)) \;=\; C$, where C is a constant and F is a function satisfying

$$\frac{\partial F}{\partial x} \;=\; M(x,y)\,, \qquad \frac{\partial F}{\partial y} \;=\; N(x,y)\,.$$

The desired function F can be obtained as

$$F(x,y) \;=\; \int M(x,y)\,dx \;=\; \int \left(1+\ln x+\frac{y}{x}\right)\,dx \;=\; x+x\ln x-x+y\ln x+h(y) \;=\; x\ln x+y\ln x+h(y)\,,$$

where $h(y)$ has to be found. On the other hand, since $\dfrac{\partial F}{\partial y} \;=\; N(x,y)$, we can write

$$\ln x-1 \;=\; \ln x+h'(y) \;\Longrightarrow\; h'(y) \;=\; -1 \;\Longrightarrow\; h(y) \;=\; -y\,,$$

where the integration constant has been taken equal to zero. Hence, the general solution of the given differential equation is

$$F(x,y(x)) \;=\; x\,\ln x+y(x)\,\ln x-y(x) \;=\; C\,.$$

Moreover, the initial condition $y(1) \;=\; 1$ yields $C \;=\; -1$.

Finally, the solution of the given initial value problem can be written in explicit form as

$$\boxed{y(x) \;=\; \frac{x\,\ln x+1}{1-\ln x}}$$

with $0<x<e$.

Problem 17 Consider the following initial value problem

$$\begin{cases} y'+y \;=\; 2\,t^2 \\ y(0) \;=\; 5\,. \end{cases}$$

(i) Verify that $y(t) = e^{-t} + 2\,t^2 - 4\,t + 4$ is the exact solution.

(ii) Use the following Runge-Kutta method

$$Y_{n+1} = Y_n + \frac{1}{2}\,(\,K_1 + K_2\,)\,, \quad \text{with } K_1 = h\,f(t_n, Y_n)\,,\ K_2 = h\,f(t_{n+1}, Y_n + K_1)\,,$$

for $n = 0, 1, 2, \ldots$, to approximate the value $y(0.2)$ with $h = h_1 = 0.1$.

(iii) Knowing that $Y_{20}^{h_2} = 4.09875$ is an approximation of $y(0.2)$ calculated with $h = h_2 = 0.01$, estimate the order of the numerical method in (ii).

SOLUTION:

(i) Solving the given linear differential equation (by means of the integrating factor $\mu(t) = e^{\,t}$) together with the initial condition $y(0) = 5$ yields the suggested solution. Alternatively, the validity of the proposed solution can be checked by substituting its expression in both the differential equation and the initial condition of the given problem.

(ii) We can write the given differential equation as $y' = f(t, y) = 2\,t^2 - y$. Then, applying the formula of the numerical method, with $h = h_1 = 0.1$, for $n = 0$ and $n = 1$ provides $Y_1 = 4.52600$ and the desired approximation

$$\boxed{y(0.2) \approx Y_2 \equiv Y_2^{h_1} = 4.10093}\,,$$

respectively.

(iii) Using the exact solution suggested in (i), we can calculate $y(0.2) = 4.09873$. Furthermore, we have $E_{t=0.2}^{h_1} = \left| Y_2^{h_1} - y(0.2) \right| = 0.0022$ and $E_{t=0.2}^{h_2} = \left| Y_{20}^{h_2} - y(0.2) \right| = 0.00002$. As $h_2 = h_1/10$, we get

$$E_{t=0.2}^{h_2} \approx C\,h_2^p = C\left(\frac{h_1}{10}\right)^p \approx \frac{E_{t=0.2}^{h_1}}{10^p}\,,$$

where p is the order of the method (C is a constant). The previous expression yields $p \approx 2.04$. Thus, we can conclude that the order of the numerical method in (ii) is

$$\boxed{p = 2}\,.$$

Problem 18 Consider the differential equation $xy^2y' + x^3 = y^3$, with $0 < x < 2$.

(a) Classify it, justifying your answer.

(b) Solve it together with the condition $y(1) = 2$.

SOLUTION:

(a) The given first-order differential equation is nonlinear and homogeneous, since dividing by xy^2 (supposing $y(x) \neq 0$ for $0 < x < 2$) and isolating y' yield

$$y' = \frac{y}{x} - \frac{x^2}{y^2} = \frac{y}{x} - \left(\frac{y}{x}\right)^{-2},$$

where the right-hand side is a function of y/x. Another way to verify that the equation is homogeneous consists in writing $y' = \frac{y}{x} - \frac{x^2}{y^2} \equiv F(x,y)$ and observing that ($\alpha \in \mathbb{R}$)

$$F(\alpha x, \alpha y) = \frac{\alpha y}{\alpha x} - \frac{(\alpha x)^2}{(\alpha y)^2} = \frac{\alpha y}{\alpha x} - \frac{\alpha^2 x^2}{\alpha^2 y^2} = \frac{y}{x} - \frac{x^2}{y^2} = F(x,y).$$

(b) Upon the change of variable $v = \frac{y}{x}$, which implies $y' = v'x + v$, the equation becomes separable, namely

$$v'x + v = v - v^{-2} \implies v^2 dv = -\frac{dx}{x}.$$

Then, integration yields $\frac{v^3}{3} = -\ln x + C$ and, undoing the change of variable, we get $\frac{y^3}{3x^3} = -\ln x + C$, where C is a constant. Finally, using the given condition $y(1) = 2$, we obtain $C = 8/3$. Thus, the desired solution is

$$\frac{y^3}{3x^3} = -\ln x + \frac{8}{3} \implies \boxed{y^3 = x^3(8 - 3\ln x)}.$$

Problem 19 Consider the following initial value problem

$$\begin{cases} y' + 6y = 0 \\ y(0) = 1. \end{cases}$$

(a) Apply to the problem one iteration of the explicit Euler method with step $h_1 = 0.05$. Then, say whether the method is stable with the suggested step.

(b) Use the value Y_1 computed in (a) and the following Adams–Moulton method of order 2

$$Y_{n+2} = Y_{n+1} + \frac{h}{2}\Big[\, f(t_{n+1}, Y_{n+1}) + f(t_{n+2}, Y_{n+2}) \,\Big],$$

for $n = 0, 1, 2, \ldots$, to approximate the value $y(0.1)$ with $h = h_1 = 0.05$.

(c) Knowing that $E^{h_2}_{t=0.1} = 0.00112$ is the error of approximating $y(0.1)$ using the method in (b) with step $h_2 = h_1/q$, calculate the value of h_2 (note that $y(0.1) = 0.54881$ and $q \in \mathbb{N}$ is the step reduction factor).

SOLUTION:

(a) One iteration of the explicit Euler method (for $n = 0$) with step $h_1 = 0.05$ reads $Y_1 = Y_0 - 6\,h_1\,Y_0 = 1 - 0.3 = 0.7$. Despite the given linear differential equation is *stiff*, the used numerical method is stable, as $h_1 = 0.05 < 2\,/\,6 \approx 0.33$.

(b) Applying the formula of the given numerical method, with $h = h_1 = 0.05$, for $n = 0$ we get $Y_2 = Y_1 + (h_1/2)\,[-6\,Y_1 - 6\,Y_2]$, namely $Y_2 = Y_1\,(1 - 3\,h_1)\,/\,(1 + 3\,h_1) = 0.51739$. Thus, $\boxed{Y_2 = Y_2^{h_1} = 0.51739}$ is the desired approximation of $y(0.1)$.

(c) Using the suggested value $y(0.1) = 0.54881$, we can calculate $E^{h_1}_{t=0.1} = \left| Y_2^{h_1} - y(0.1) \right| = 0.03142$. Then, being $p = 2$ the order of the method in (b), we have

$$E^{h_2}_{t=0.1} \approx C\,h_2^2 = C\left(\frac{h_1}{q}\right)^2 \approx \frac{E^{h_1}_{t=0.1}}{q^2},$$

where $q \in \mathbb{N}$ is the step reduction factor. Thus, the previous expression yields $q \approx 5$ and we can conclude that $\boxed{h_2 = h_1/5 = 0.01}$.

Problem 20 Can you use the explicit Euler method to solve approximately the initial value problem: $y' = 1 + y^2$ in $x \in [0, 3]$, with $y(0) = 0$? *Hint: Find the exact solution and discuss.*

SOLUTION:
The exact solution follows from separation of variables:

$$\int \frac{dy}{1 + y^2} = \int dx + C \Longrightarrow \arctan y = x + C \Longrightarrow y = \tan(x + C).$$

The initial condition gives $0 = \tan C$, so that $C = 0$ and $y(x) = \tan x$. The tangent function has a vertical asymptote at $x = \pi/2 < 3$, which cannot be captured by the explicit Euler method. This method will not be able to find the asymptote, as the slopes used to calculate the approximate solution will fall below $y(x)$ and surpass the vertical line $x = \pi/2$.

Problem 21 A new European football league is planned so that the yearly sales of tickets grow at a speed proportional to the difference between sales at time t and a higher bound of 300 million euros. Assume no ticket has been sold at $t = 0$ and that sales should be 40 million euro after 3 years (otherwise the competition is cancelled). Based on this assumption, how long it should take for the yearly ticket sales to have reached 220 million euro?

SOLUTION:
Let $x(t)$ be the ticket sales at time t. The corresponding initial value problem is

$$\frac{dx}{dt} = \alpha(300 - x), \quad x(0) = 0,$$

where α is a constant. The solution of the IVP is $x(t) = 300(1 - e^{-\alpha t})$. We know that $x(3) = 40$, and therefore

$$40 = 300(1 - e^{-3\alpha}) \Longrightarrow e^{-3\alpha} = 1 - \frac{40}{300} = \frac{13}{15} \Longrightarrow \alpha = \frac{1}{3}\ln\frac{15}{13} = \frac{1}{3}(\ln 15 - \ln 13).$$

We now seek a time T such that $x(T) = 220$. Thus

$$220 = 300(1 - e^{-\alpha T}) \Longrightarrow e^{-\alpha T} = 1 - \frac{11}{15} \Longrightarrow T = \frac{1}{\alpha}\ln\frac{15}{4} \Longrightarrow T = 3\frac{\ln 15 - \ln 4}{\ln 15 - \ln 13} \approx 27.71 \text{ years.}$$

Problem 22 The population of a large cluster of atoms decreases to one third its initial size in one year, at a rate proportional to the instantaneous number of atoms. (a) Model the evolution of the atom population by means of a differential equation and calculate its growth rate proportionality constant r. (b) Use the explicit Euler method to solve the ODE and indicate the maximum step size for which the Euler scheme is stable.

SOLUTION:
(a) The ODE for the atom population $y(t)$ is $\boxed{y' = -ry}$ which, solved for $y(0) = y_0$, yields $y(t) = y_0 e^{-rt}$. For $t = 1$ yr, $\frac{1}{3} = e^{-r}$ and therefore $\boxed{r = \ln 3/\text{yr}}$.
(b) The explicit Euler method gives $y_{j+1} = y_y - hry_j = (1 - rh)y_j$. The solution is

$$y_j = (1 - rh)^j y_0,$$

and stability requires

$$-1 \le 1 - rh \le 1 \Longrightarrow r \ge 0 \; \& -1 + rh \le 1 \Longrightarrow r \ge 0 \; \& \boxed{h \le \frac{2}{r} = \frac{2}{\ln 3}}.$$

The step size should be less or equal than $2/\ln 3 \approx 1.82$ yr.

Problem 23 Given the Ordinary Differential Equation (ODE):

$$y' = e^{x+y} \quad \text{with} \quad x > 0,$$

i) Classify this ODE.

ii) Solve the ODE with initial condition $y(1) = 1$.

SOLUTION:

i) First order separable ODE.

ii) We multiply it by e^{-y} to obtain $e^{-y}y' = e^x$. Then $\int_1^y e^{-y}dy = \int_1^x e^x dx$ solves the initial value problem. We find

$$-e^y|_1^y = e^x|_1^x \Longrightarrow e^{-1} - e^{-y} = e^x - e \Longrightarrow e^{-y} = (e + e^{-1} - e^x).$$

$$y = -\ln(e + e^{-1} - e^x).$$

Problem 24 A cauldron of boiling soup is placed in a room at 0^oC and its temperature becomes 20^oC after 30 minutes. (a) Model the evolution of the soup temperature by means of a differential equation (Newton cooling law) and calculate the cooling rate proportionality constant $-r$. (b) Use the explicit Euler method to solve the ODE and indicate the maximum step size for which the Euler scheme is stable.

SOLUTION:
(a) The ODE for the soup temperature $y(t)$ is $y' = -ry$ which, solved for $y(0) = 100$ yields $y(t) = 100e^{-rt}$. For $t = 30$ min, $20 = 100e^{-30r}$ and therefore $e^{30r} = 100/20 = 5$, so that $r = \frac{1}{30}\ln 5$ min^{-1}.
(b) The explicit Euler method gives $y_{j+1} = y_j - hry_j = (1 - rh)y_j$. As $y_0 = 100$, the solution is

$$y_j = 100\,(1 - rh)^j,$$

and stability requires

$$-1 \le 1 - rh \le 1 \Longrightarrow r \ge 0 \;\&\; -1 + rh \le 1 \Longrightarrow r \ge 0 \;\&\; h \le \frac{2}{r} = \frac{60}{\ln 5}.$$

The step size should be less or equal than $60/\ln 5 = 37.28$ min.
