APPLIED DIFFERENTIAL CALCULUS
LECTURE 2: Second-order ordinary differential equations.
PROBLEMS

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Problem 1 Solve the following differential equation:

$$2y'' - 5y' - 3y = 0$$

Problem 2 Solve the following differential equation:

$$y'' - 10y' + 25y = 0$$

Problem 3 Solve the following differential equation:

$$y'' + 4y' + 7y = 0$$

Problem 4 Solve the following differential equation:

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Problem 5 Solve the following differential equation:

$$4y'' + 36y = \csc(3x)$$

Problem 6 Solve the initial value problem (IVP):

$$\begin{cases} y'' - y' - 2y = 3e^{2x} \\ y(0) = 0\,, \ \ y'(0) = -2 \end{cases} .$$

Problem 7 Solve the following differential equation:

$$y'' - 4y = \sin x$$

Problem 8 Solve the following differential equation:

$$y'' + 4y = 4\cos(2x)$$

Problem 9 Solve the following differential equation:

$$y'' + 4y = -4x$$

Problem 10 Solve the following differential equation:

$$y'' + 4y = 4\cos(2x) - 4x$$

Problem 11 Solve the initial value problem (IVP):

$$y'' - 4y' + 4y = 0\,, \ \ y(0) = 1\,, \ \ y'(0) = 1$$

Problem 12 Solve the initial value problem (IVP):

$$y'' - y' - 2y = 0\,, \ \ y(0) = 1\,, \ \ y'(0) = 0$$

Problem 13 Solve the initial value problem (IVP):

$$y'' + 3y' + 2y = 0\,, \ \ y(0) = 1\,, \ \ y'(0) = 0$$

Problem 14 Solve the initial value problem (IVP):

$$y'' - 2y' + 2y = e^{-x}\,, \ \ y(0) = 0\,, \ \ y'(0) = 1$$

Problem 15 Solve the initial value problem (IVP):

$$y'' - 2y' + 2y = \cos x\,, \;\; y(0) = 1\,, \;\; y'(0) = 0$$

Problem 16 Given the following initial value problem (IVP):

$$y'' + 4y' + 4y = e^t\,; \quad y(0) = 1; \quad y'(0) = 0\,,$$

find the value of $y(2)$.

Problem 17 Solve the initial value problem (IVP):

$$y'' + y' - 2y = e^{-t}, \quad y(0) = 1, \quad y'(0) = -1.$$

Problem 18 Solve the initial value problem (IVP):

$$y'' - 2y' + 5y \;=\; t\,, \quad y(0) = 1\,, \quad y'(0) = 2\,.$$

Problem 19 Consider the initial value problem (IVP)

$$y''' + 4y' \;=\; 4\,e^{2t}\;; \quad y(0) = 1\,, \;\; y'(0) = 0\,, \;\; y''(0) = 1\,,$$

and solve it following the next steps:

(i) Apply the change of variable $v(t) = y'(t)$ to the differential equation of the IVP and find the general solution of the resulting second-order equation.

(ii) Undo the change of variable applied in (i), integrate the result, and obtain the solution $y(t)$ of the IVP.

Problem 20 Consider the following ordinary differential equation (ODE)

$$y'' - 4xy' - 4y = e^{x} .$$

(a) Assuming that the solution of the ODE is given by the power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$, find the recurrence relation satisfied by the coefficients a_n.

(b) Supposing that $a_0 = 1$ and $a_1 = 0$, find an approximate value of the solution of the ODE at $x = 2$ by using only the first five terms of the power series in (a).

HINT. This result may be useful: $e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Problem 21 Let $f(x) = 27 + (x^2 + 1)\, y'$, where y' is the first-order derivative of the function $y = y(x)$ with respect to the independent variable x . Knowing that y is sufficiently regular, do what follows.

(a) Prove that the equation $f'(x) = 0$ is equivalent to the equation $(x^2 + 1)y'' + 2xy' = 0$.

(b) Find the solution of the equation in (a) in the form of a power series $\sum_{n=0}^{\infty} a_n x^n$.

(c) Impose the initial conditions $y(0) = \beta$, $y'(0) = 1$. Then, find the value of the parameter $\beta \in \mathbb{R}$ that yields an odd solution, namely such that $y = y(x)$ satisfies $y(-x) = -y(x)$.

Problem 22 Solve the following initial value problem by applying the change of variable $x = e^z \iff z = \ln(x)$.

$$\begin{cases} x^2 y'' + 2xy' + \frac{5}{2} y = 0 , & \text{for } x > 0 , \\ y(1) = -1 , \\ y'(1) = 1 . \end{cases}$$

Problem 23 Verify that the functions $y_1(t) = e^t$ and $y_2(t) = t$ are solutions of the homogeneous equation associated with

$$(1 - t)y'' + ty' - y = 2(t - 1)^2 e^{-t}$$

with $0 < t < 1$. In addition, using y_1 and y_2, calculate a particular solution of the given nonhomogeneous equation by the method of variation of parameters.

Problem 24 Solve the initial value problem: $y'' + 4y = \cos(2x)$, $y(0) = 0$, $y'(0) = 1$.

Problem 25 A mechanical fish is trying to extract a bar from a wall by applying a force per unit mass $\cos(\omega t)$ to it. The bar returns to its rest state with a restoring coefficient 34 per unit mass and the friction force per unit mass is minus 10 times the fish velocity. Assuming that the bar cannot be extracted, what is the oscillatory motion of the fish as time goes to infinity? For which value of ω is the displacement maximum? What is this maximum displacement?

Problem 26 Find the solution of the following initial boundary value problem:

$$y'' - 2y' + 5y = \cos t, \quad y(0) = 0, \quad y'(0) = 0.$$

Problem 27 The vertical motion of a mass attached to a spring (suspended from a rigid support) is driven by an external forcing $f(t)$ and is described by the following second-order differential equation

$$y'' + 3y \;=\; f(t)\,,$$

neglecting any damping effect. Solve the equation by means of the Laplace transform, when the initial displacement and velocity are $y(0) = 1$ and $y'(0) = -2$, respectively, and the forcing is modelled by

$$f(t) \;=\; \frac{7}{2}\int_0^t e^{-2\tau}\,\sin\left(\,2(t-\tau)\,\right)\,d\tau\,.$$

Problem 28 Solve the following initial value problem (IVP):

$$y'' - 2y' + 5y \;=\; e^{-3t}\,, \quad y(0) = 1\,, \;\; y'(0) = 0\,.$$

Problem 29 Solve the following initial value problem (IVP):

$$y'' - 2y' + 10y \;=\; 1\,, \quad y(0) = 0\,, \quad y'(0) = 1\,.$$

Problem 30 Solve the initial value problem (IVP)

$$(IVP)\begin{cases} y'' - y' - 2y = 3e^{2x}\,, & x \in \mathbb{R} \\ y(0) = 0\,, \;\; y'(0) = 4 \end{cases}$$

Problem 31 Solve the initial value problem

$$(IVP)\begin{cases} y'' + y' - 6y = 6\,, & x \in \mathbb{R} \\ y(0) = 1\,, & y'(0) = 1 \end{cases}$$