APPLIED DIFFERENTIAL CALCULUS
LECTURE 2: Second-order ordinary differential equations.
PROBLEMS

Autores:
Manuel Carretero, Luis L. Bonilla, Filippo Terragni, Sergei Iakunin y Rocio Vega

---

Problem 1 Solve the following differential equation:

$$2y'' - 5y' - 3y = 0$$

SOLUTION:
The equation is a homogeneous linear second-order ODE with constant coefficients.
Solution:

$$y(x) = c_1 e^{-\frac{x}{2}} + c_2 e^{3x}\,; \quad c_1,\, c_2 \in \mathbb{R}.$$

Problem 2 Solve the following differential equation:

$$y'' - 10y' + 25y = 0$$

SOLUTION:
The equation is a homogeneous linear second-order ODE with constant coefficients.
Solution:

$$y(x) = c_1 e^{5x} + c_2 x e^{5x}\,; \quad c_1,\, c_2 \in \mathbb{R}.$$

Problem 3 Solve the following differential equation:

$$y'' + 4y' + 7y = 0$$

SOLUTION:
The equation is a homogeneous linear second-order ODE with constant coefficients.
Solution:

$$y(x) = c_1 e^{-2x} \sin(\sqrt{3}x) + c_2 e^{-2x} \cos(\sqrt{3}x)\,; \quad c_1,\, c_2 \in \mathbb{R}.$$

Problem 4 Solve the following differential equation:

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

SOLUTION:
The equation is a non-homogeneous linear second-order ODE with constant coefficients.
A particular solution can be obtained with the method of variation of parameters or with that of undetermined coefficients.
Solution:

$$y(x) = c_1 e^{2x} + c_2 x e^{2x} + (\frac{x^3}{6} + \frac{x^2}{2})e^{2x}\,; \quad c_1,\, c_2 \in \mathbb{R}.$$

Problem 5 Solve the following differential equation:

$$4y'' + 36y = \csc(3x)$$

SOLUTION:
The equation is a non-homogeneous linear second-order ODE with constant coefficients.
A particular solution can be obtained with the method of variation of parameters.
Solution:

$$y(x) = c_1 \cos(3x) + c_2 \sin(3x) - \frac{x}{12}\cos(3x) + \frac{1}{36}\sin(3x)\ln(\sin(3x))\,; \quad c_1,\, c_2 \in \mathbb{R}.$$

Problem 6 Solve the initial value problem (IVP):

$$\begin{cases} y'' - y' - 2y = 3e^{2x} \\ y(0) = 0\,,\ \ y'(0) = -2 \end{cases}.$$

SOLUTION:
The equation is a non-homogeneous linear second-order ODE with constant coefficients.
A particular solution can be obtained with the method of variation of parameters or with that of undetermined coefficients.
Solution:

$$y(x) = e^{-x} + e^{2x}(x-1).$$

Problem 7 Solve the following differential equation:

$$y'' - 4y = \sin x$$

SOLUTION:
The equation is a non-homogeneous linear second-order ODE with constant coefficients.
A particular solution can be obtained with the method of undetermined coefficients.
Solution:

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} - \frac{\sin(x)}{5}\,; \quad c_1,\, c_2 \in \mathbb{R}.$$

Problem 8 Solve the following differential equation:

$$y'' + 4y = 4\cos(2x)$$

SOLUTION:
The equation is a non-homogeneous linear second-order ODE with constant coefficients.
A particular solution can be obtained with the method of undetermined coefficients.
Solution:

$$y(x) = c_1\cos(2x) + c_2\sin(2x) + x\sin(2x)\,; \quad c_1,\, c_2 \in \mathbb{R}.$$

Problem 9 Solve the following differential equation:

$$y'' + 4y = -4x$$

SOLUTION:
The equation is a non-homogeneous linear second-order ODE with constant coefficients.
A particular solution can be obtained with the method of undetermined coefficients.
Solution:

$$y(x) = c_1\cos(2x) + c_2\sin(2x) - x\,; \quad c_1,\, c_2 \in \mathbb{R}.$$

Problem 10 Solve the following differential equation:

$$y'' + 4y = 4\cos(2x) - 4x$$

SOLUTION:
The equation is a non-homogeneous linear second-order ODE with constant coefficients.
A particular solution can be obtained with the method of undetermined coefficients and the superposition principle.
Solution:

$$y(x) = c_1\cos(2x) + c_2\sin(2x) - x + x\sin(2x)\,; \quad c_1,\, c_2 \in \mathbb{R}.$$

Problem 11 Solve the initial value problem (IVP):

$$y'' - 4y' + 4y = 0\,,\ \ y(0) = 1\,,\ \ y'(0) = 1$$

SOLUTION:
The equation is a homogeneous linear second-order ODE with constant coefficients.
Applying initial data we get the solution:

$$y(x) = e^{2x}(1 - x).$$

Problem 12 Solve the initial value problem (IVP):

$$y'' - y' - 2y = 0\,, \;\; y(0) = 1\,, \;\; y'(0) = 0$$

SOLUTION:
The equation is a homogeneous linear second-order ODE with constant coefficients.
Applying initial data we get the solution:

$$y(x) = \frac{2}{3}e^{-x} + \frac{1}{3}e^{2x}.$$

Problem 13 Solve the initial value problem (IVP):

$$y'' + 3y' + 2y = 0\,, \;\; y(0) = 1\,, \;\; y'(0) = 0$$

SOLUTION:
The equation is a homogeneous linear second-order ODE with constant coefficients.
Applying initial data we get the solution:

$$y(x) = 2e^{-x} - e^{-2x}.$$

Problem 14 Solve the initial value problem (IVP):

$$y'' - 2y' + 2y = e^{-x}\,, \;\; y(0) = 0\,, \;\; y'(0) = 1$$

SOLUTION:
The equation is a non-homogeneous linear second-order ODE with constant coefficients.
A particular solution can be obtained with the method of undetermined coefficients or with that of variation of parameters.
Applying the initial conditions we get the solution:

$$y(x) = \frac{e^{-x}}{5} + \frac{7}{5}e^{x}\sin(x) - \frac{1}{5}e^{x}\cos(x).$$

Problem 15 Solve the initial value problem (IVP):

$$y'' - 2y' + 2y = \cos x\,, \;\; y(0) = 1\,, \;\; y'(0) = 0$$

SOLUTION:
The equation is a non-homogeneous linear second-order ODE with constant coefficients.
A particular solution can be obtained with the method of undetermined coefficients or with that of variation of parameters.
Applying the initial conditions we get the solution:

$$y(x) = \frac{\cos(x)}{5} - \frac{2}{5}\sin(x) - \frac{2}{5}e^{x}\sin(x) + \frac{4}{5}e^{x}\cos(x).$$

Problem 16 Given the following initial value problem (IVP):

$$y'' + 4y' + 4y = e^t\,; \quad y(0) = 1; \quad y'(0) = 0\,,$$

find the value of $y(2)$.

SOLUTION:
Solving the differential equation and applying the initial conditions we get:

$$y(t) = \frac{1}{9}e^t + \frac{8}{9}e^{-2t} + \frac{5}{3}te^{-2t} \Rightarrow y(2) = \frac{1}{9}(e^2 + 38e^{-4})\,.$$

Problem 17 Solve the initial value problem (IVP):

$$y'' + y' - 2y = e^{-t}, \quad y(0) = 1, \quad y'(0) = -1.$$

SOLUTION:
The equation is a non-homogeneous linear second-order ODE with constant coefficients.
A particular solution can be obtained with the method of undetermined coefficients or with that of variation of parameters.
Applying the initial conditions we get the solution: $y(t) = \frac{1}{2}(e^t - e^{-t}) + e^{-2t}$

Problem 18 Solve the initial value problem (IVP):

$$y'' - 2y' + 5y \ = \ t\,, \quad y(0) = 1\,, \quad y'(0) = 2\,.$$

SOLUTION:
The equation is a non-homogeneous linear second-order ODE with constant coefficients.
A particular solution can be obtained with the method of undetermined coefficients or with that of variation of parameters.
Applying the initial conditions we get the solution:

$$\boxed{y(t) \ = \ \frac{1}{25}\Big[\,2 + 5t + 23\,e^t\cos(2t) + 11\,e^t\sin(2t)\,\Big]}\,.$$

Problem 19 Consider the initial value problem (IVP)

$$y''' + 4y' = 4\,e^{2t}\ ; \quad y(0) = 1\,,\ \ y'(0) = 0\,,\ \ y''(0) = 1\,,$$

and solve it following the next steps:

(i) Apply the change of variable $v(t) = y'(t)$ to the differential equation of the IVP and find the general solution of the resulting second-order equation.

(ii) Undo the change of variable applied in (i), integrate the result, and obtain the solution $y(t)$ of the IVP.

SOLUTION:

(i) We have $v(t) = y'(t) \implies v' = y'' \implies v'' = y'''$. Then, after substitution into the given equation, we obtain the following linear differential equation with constant coefficients, $v'' + 4v = 4e^{2t}$, whose solution is $v(t) = v_h(t) + v_p(t)$, where $v_h(t) = c_1\cos(2t) + c_2\sin(2t)$ is the general solution of the corresponding homogeneous equation and $v_p(t) = \frac{1}{2}e^{2t}$ is a particular solution of the nonhomogeneous one. Hence, the general solution of the above second-order equation is

$$\boxed{v(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{2}e^{2t}}\ .$$

(ii) We have $y'(t) = v(t) \implies y(t) = \displaystyle\int v(t)\,\mathrm{d}t = \frac{c_1}{2}\sin(2t) - \frac{c_2}{2}\cos(2t) + \frac{1}{4}e^{2t} + C\,.$

In addition, since $y(0) = 1\,, y'(0) = 0\,, y''(0) = 1\,,$ we get $c_1 = -1/2\,, c_2 = 0\,, C = 3/4\,,$ namely

$$\boxed{y(t) = \frac{3}{4} + \frac{1}{4}e^{2t} - \frac{1}{4}\sin(2t)}\ .$$

Problem 20 Consider the following ordinary differential equation (ODE)

$$y'' - 4xy' - 4y = e^{x}.$$

(a) Assuming that the solution of the ODE is given by the power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$, find the recurrence relation satisfied by the coefficients a_n.

(b) Supposing that $a_0 = 1$ and $a_1 = 0$, find an approximate value of the solution of the ODE at $x = 2$ by using only the first five terms of the power series in (a).

HINT. This result may be useful: $e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

SOLUTION:

(a) Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$, hence

$$y'(x) = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2}.$$

Substituting these series into the ODE yields

$$\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} - \sum_{n=1}^{\infty} 4n\, a_n\, x^n - \sum_{n=0}^{\infty} 4a_n\, x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

where the given *hint* has been used. In order to get the same power of x in each series, let us change the index in the first one, that is

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n - \sum_{n=1}^{\infty} 4n\, a_n\, x^n - \sum_{n=0}^{\infty} 4a_n\, x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

namely

$$(2a_2 - 4a_0 - 1) + \sum_{n=1}^{\infty} \left[(n+2)(n+1)\, a_{n+2} - 4(n+1)\, a_n - \frac{1}{n!} \right] x^n = 0.$$

Now, equating to zero the coefficients of each power of x, we get

$$2a_2 - 4a_0 - 1 = 0, \quad (n+2)(n+1)\, a_{n+2} - 4(n+1)\, a_n - \frac{1}{n!} = 0,$$

which can be expressed as

$$\boxed{a_2 = \frac{1}{2} + 2a_0, \quad a_{n+2} = \frac{1}{(n+2)!} + \frac{4}{n+2}\, a_n},$$

with $n = 1, 2, \ldots$.

(b) Using the recurrence relation obtained in (a), we get

$$a_2 = \frac{1}{2} + 2a_0\,, \qquad a_3 = \frac{1}{3!} + \frac{4}{3}a_1\,, \qquad a_4 = \frac{13}{4!} + 2a_0\,.$$

Since $a_0 = 1$ and $a_1 = 0$, we can write

$$a_2 = \frac{5}{2}\,, \qquad a_3 = \frac{1}{6}\,, \qquad a_4 = \frac{61}{24}\,.$$

Thus, the desired approximate value is

$$\boxed{y(2) \approx a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 \;=\; 1 + 0 + 10 + \frac{4}{3} + \frac{122}{3} \;=\; 53}\,.$$

Problem 21 Let $f(x) \;=\; 27 + (x^2+1)\,y'$, where y' is the first-order derivative of the function $y = y(x)$ with respect to the independent variable x. Knowing that y is sufficiently regular, do what follows.

(a) Prove that the equation $f'(x) = 0$ is equivalent to the equation $(x^2+1)y'' + 2xy' = 0$.

(b) Find the solution of the equation in (a) in the form of a power series $\displaystyle\sum_{n=0}^{\infty} a_n\, x^n$.

(c) Impose the initial conditions $y(0) = \beta\,,\; y'(0) = 1$. Then, find the value of the parameter $\beta \in \mathbb{R}$ that yields an odd solution, namely such that $y = y(x)$ satisfies $y(-x) = -y(x)$.

SOLUTION:

(a) We have

$$f'(x) \;=\; \frac{d}{dx}\Big[\,y'\,(x^2+1) + 27\,\Big] \;=\; (x^2+1)y'' + 2xy'\,.$$

Hence, the equation $f'(x) = 0$ is equivalent to the differential equation $(x^2+1)y''+2xy' \;=\; 0$.

(b) Supposing that $y(x) = \sum_{n=0}^{\infty} a_n\, x^n$, we have

$$y'(x) = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1}\,, \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2}\,.$$

After substituting these expressions into the differential equation in (a), we get

$$\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^n + \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} + \sum_{n=1}^{\infty} 2n\, a_n\, x^n \;=\; 0\,.$$

In order to obtain the same power of x in each series, let's change index in the second one as

$$\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^n + \sum_{n=0}^{\infty}(n+2)(n+1)\, a_{n+2}\, x^n + \sum_{n=1}^{\infty} 2n\, a_n\, x^n \;=\; 0\,,$$

which is equivalent to

$$2a_2 + (6a_3 + 2a_1)x + \sum_{n=2}^{\infty} \Big[n(n+1)\, a_n + (n+2)(n+1)\, a_{n+2} \Big] x^n \;=\; 0\,.$$

Now, equating to zero the coefficients of each power of x, we find that

$$2a_2 = 0\,, \quad 6a_3 + 2a_1 = 0\,, \quad n(n+1)\, a_n + (n+2)(n+1)\, a_{n+2} = 0\,,$$

namely

$$a_2 = 0\,, \quad a_3 = -\frac{1}{3}\, a_1\,, \quad a_{n+2} = -\frac{n}{n+2}\, a_n \;\text{ for }\; n = 2, 3, \dots\,.$$

Finally, calculating the first coefficients in terms of a_0 and a_1, we get

$$y(x) \;=\; a_0 \;+\; a_1 \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \right) \;=\; a_0 \;+\; a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, x^{2n+1}\,,$$

where the coefficients of all even powers of x are equal to zero.

(c) By applying the initial conditions we get $a_0 = \beta$ and $a_1 = 1$. Then, we can easily conclude that the value $\beta = 0$ makes the function $y = y(x)$ odd.

Problem 22 Solve the following initial value problem by applying the change of variable $x = e^z \iff z = \ln(x)$.

$$\begin{cases} x^2 y'' + 2xy' + \dfrac{5}{2} y \;=\; 0\,, \quad \text{for } x > 0\,, \\ y(1) \;=\; -1\,, \\ y'(1) \;=\; 1\,. \end{cases}$$

SOLUTION:

The given differential equation is a Cauchy-Euler equation, which can be solved by using the suggested change of variable. First, the chain rule yields

$$\frac{dy}{dx} \;=\; \frac{dy}{dz}\frac{dz}{dx} \;=\; \frac{1}{x}\frac{dy}{dz}\,, \qquad \frac{d^2y}{dx^2} \;=\; \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\,.$$

Then, after substitution, we get the following differential equation with constant coefficients

$$\frac{d^2y}{dz^2} + \frac{dy}{dz} + \frac{5}{2}\, y \;=\; 0\,,$$

whose characteristic equation is $r^2 + r + 5/2 = 0$, with complex conjugate roots $r_1 = -1/2 + 3/2\,i$ and $r_2 = -1/2 - 3/2\,i$. Hence, its general solution is

$$y(z) = e^{-z/2}\left[a\cos\left(\frac{3}{2}z\right) + b\sin\left(\frac{3}{2}z\right)\right],$$

which can be written in terms of the original independent variable x as

$$y(x) = x^{-1/2}\left[a\cos\left(\frac{3}{2}\ln(x)\right) + b\sin\left(\frac{3}{2}\ln(x)\right)\right].$$

In addition, the given initial conditions provide the values for the two constants, namely $a = -1$ and $b = 1/3$. Finally, the solution of the given initial value problem is

$$\boxed{y(x) = x^{-1/2}\left[\frac{1}{3}\sin\left(\frac{3}{2}\ln(x)\right) - \cos\left(\frac{3}{2}\ln(x)\right)\right]}.$$

Problem 23 Verify that the functions $y_1(t) = e^t$ and $y_2(t) = t$ are solutions of the homogeneous equation associated with

$$(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t}$$

with $0 < t < 1$. In addition, using y_1 and y_2, calculate a particular solution of the given nonhomogeneous equation by the method of variation of parameters.

SOLUTION:
Indeed y_1, y_2 are solutions of the homogeneous equation associated with the given differential equation (proof is omitted). In order to calculate a particular solution of the nonhomogeneous equation, let us rewrite it as

$$y'' + \frac{t}{1-t}y' - \frac{1}{1-t}y = 2(1-t)e^{-t}.$$

Then, according to the method of variation of parameters, a solution in the form $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ is sought, where u_1 and u_2 are functions to be determined. Moreover, we know that the derivatives of such functions satisfy the system

$$\begin{aligned} u_1' e^t + u_2' t &= 0 \\ u_1' e^t + u_2' &= 2(1-t)e^{-t}, \end{aligned}$$

which yields $u_1' = -2t\,e^{-2t}$, $u_2' = 2\,e^{-t}$. Finally, after integration, we get $u_1 = (t + \frac{1}{2})\,e^{-2t}$, $u_2 = -2\,e^{-t}$. Hence, a particular solution is

$$\boxed{y(t) = \frac{1}{2}(1-2t)e^{-t}}.$$

Problem 24 Solve the initial value problem: $y'' + 4y = \cos(2x)$, $y(0) = 0$, $y'(0) = 1$.

SOLUTION:
Homogeneous and particular solutions are:

$$\begin{aligned} y_h(x) &= A\cos 2x + B\sin 2x, \\ y_p(x) &= Cx\cos 2x + Dx\sin 2x. \end{aligned}$$

We use the x multiplier in the particular solution because the right hand side of the ODE is a solution of the homogeneous ODE. We will obtain the coefficients C and D by using *undetermined coefficients* and then we will use the initial conditions to determine A and B. Plugging y_p into the ODE produces:

$$\begin{aligned} & y_p(x) = Cx\cos 2x + Dx\sin 2x, \\ & y_p'(x) = C\cos 2x + D\sin 2x - 2Cx\sin 2x + 2Dx\cos 2x, \\ & y_p''(x) = -4C\sin 2x + 4D\cos 2x - 4Cx\cos 2x - 4Dx\sin 2x, \\ & y_p'' + 4y_p = \left(-4C\sin 2x + 4D\cos 2x - \underline{4Cx\cos 2x} - \underline{\underline{4Dx\sin 2x}}\right) + 4\left(\underline{Cx\cos 2x} + \underline{\underline{Dx\sin 2x}}\right) \\ & \qquad = -4C\sin 2x + 4D\cos 2x = \cos 2x \text{ for all } x, \\ & \qquad \Rightarrow C = 0, \quad D = 1/4. \end{aligned}$$

Thus the solution of the ODE is:

$$y(x) = A\cos 2x + B\sin 2x + \frac{1}{4}x\sin 2x$$

Using the initial conditions, we obtain:

$$\begin{aligned} & y(0) = A = 0, \\ & y'(0) = 2B + \left(\frac{1}{4}\sin 2x + \frac{1}{2}x\cos 2x\right)\Bigg|_{x=0} = 2B = 1 \Rightarrow B = \frac{1}{2}. \end{aligned}$$

The solution of the IVP is:

$$y(x) = \frac{x+2}{4}\sin 2x$$

Another method is variation of parameters.
Insertion of $y(x) = u_1(x)\cos 2x + u_2(x)\sin 2x$ in the ODE yields

$$\begin{cases} u_1'\cos 2x + u_2'\sin 2x = 0, \\ -2u_1'\sin 2x + 2u_2'\cos 2x = \cos 2x. \end{cases}$$

We now multiply the first equation by $2\sin 2x$, the second one by $\cos 2x$ and add them, thereby getting $2u_2' = \cos^2 2x$. Then

$$u_2 = \int \frac{1+\cos 4x}{4}\,dx = \frac{x}{4} + \frac{\sin 4x}{16} + c_2.$$

Similarly, we obtain

$$u_1' = -u_2'\tan 2x = -\frac{1}{2}\sin 2x\cos 2x = -\frac{1}{4}\sin 4x \Longrightarrow u_1 = -\frac{1}{4}\int \sin 4x\,dx = \frac{1}{16}\cos 4x + c_1.$$

Therefore we have found

$$y(x) = \left(\frac{x}{4} + c_2\right) \sin 2x + \frac{\cos 2x \cos 4x + \sin 2x \sin 4x}{16} + c_1 \cos 2x$$
$$= \left(\frac{x}{4} + c_2\right) \sin 2x + \frac{\cos(4x - 2x)}{16} + c_1 \cos 2x = \left(\frac{x}{4} + c_2\right) \sin 2x + \left(c_1 + \frac{1}{16}\right) \cos 2x.$$

This is equivalent to the solution given by undetermined coefficients, from which we obtain the same solution of the IVP.

Problem 25 A mechanical fish is trying to extract a bar from a wall by applying a force per unit mass $\cos(\omega t)$ to it. The bar returns to its rest state with a restoring coefficient 34 per unit mass and the friction force per unit mass is minus 10 times the fish velocity. Assuming that the bar cannot be extracted, what is the oscillatory motion of the fish as time goes to infinity? For which value of ω is the displacement maximum? What is this maximum displacement?

SOLUTION:
Newton's second law yields

$$\frac{dv}{dt} = -34x - 10v + \cos(\omega t), \quad v = \frac{dx}{dt},$$

and therefore

$$x'' + 10x' + 34x = \cos(\omega t).$$

A solution of the homogeneous ODE, $x_h(t) = e^{rt}$, yields $r^2 + 10r + 34 = 0$, so that

$$r = -5 \pm \sqrt{25 - 34} = -5 \pm 3i \Longrightarrow x_h(t) = e^{-5t}(a \cos 3t + b \sin 3t).$$

The particular solution of the ODE is $x_p(t) = A \cos \omega t + B \sin \sin \omega t$. Then

$$\cos(\omega t) = x_p'' + 10x_p' + 34x_p = [(34 - \omega^2)A + 10B] \cos(\omega t) + [(34 - \omega^2)B - 10A] \sin(\omega t),$$

and

$$(34 - \omega^2)A + 10B = 1, \quad (34 - \omega^2)B - 10A = 0.$$

The solution of this system is

$$A = \frac{34 - \omega^2}{100 + (34 - \omega^2)^2}, \quad B = \frac{10}{100 + (34 - \omega^2)^2}.$$

Thus the general solution of the IVP is

$$x(t) = \frac{(34 - \omega^2) \cos(\omega t) + 10 \sin(\omega t)}{100 + (34 - \omega^2)^2} + e^{-5t}(a \cos 3t + b \sin 3t).$$

As $t \to \infty$, $x(t) \to x_p(t)$, which can be written as

$$x_p(t) = \frac{\sin(\omega t + \phi)}{\sqrt{100 + (34 - \omega^2)^2}}, \quad \tan \phi = \frac{34 - \omega^2}{10}.$$

Clearly, the maximum displacement occurs for $\sin(\omega t + \phi) = 1$:

$$x_M(\omega) = \frac{1}{\sqrt{100 + (34 - \omega^2)^2}} \Longrightarrow x_M(\sqrt{34}) = 0.1.$$

The maximum displacement, 0.1, is reached for the "resonant" frequency $\omega = \sqrt{34}$.

Problem 26 Find the solution of the following initial boundary value problem:

$$y'' - 2y' + 5y = \cos t, \quad y(0) = 0, \quad y'(0) = 0.$$

SOLUTION:
We insert a particular solution $y_p(t) = a\cos t + b\sin t$ in the ODE, thereby getting

$$(5a - a - 2b)\cos t + (5b - b + 2a)\sin t = \cos t \Longrightarrow (4a - 2b) = 1, \quad (4b + 2a) = 0.$$

We find $a = -2b = \frac{1}{5}t$, $b = -\frac{1}{10}$. Then

$$y_p(t) = \frac{1}{5}\cos t - \frac{1}{10}\sin t.$$

We now solve the homogeneous ODE using $y_h(t) = e^{rt}$, which yields $r^2 - 2r + 5 = 0$. Its solutions are $r = 1 \pm 2i$. We therefore obtain the general solution

$$y(t) = \frac{1}{5}\cos t - \frac{1}{10}\sin t + e^t(c_1\cos 2t + c_2\sin 2t).$$

The initial conditions produce

$$y(0) = 0 = \frac{1}{5} + c_1, \quad y'(0) = 0 = -\frac{1}{10} + c_1 + 2c_2 \Longrightarrow c_1 = -\frac{1}{5}, \quad c_2 = \frac{3}{20}.$$

Then we have found

$$y(t) = \frac{1}{5}(\cos t - e^t\cos 2t) + \frac{1}{20}(3e^t\sin 2t - 2\sin t).$$

Problem 27 The vertical motion of a mass attached to a spring (suspended from a rigid support) is driven by an external forcing $f(t)$ and is described by the following second-order differential equation

$$y'' + 3y \;=\; f(t)\,,$$

neglecting any damping effect. Solve the equation by means of the Laplace transform, when the initial displacement and velocity are $y(0) = 1$ and $y'(0) = -2$, respectively, and the forcing is modelled by

$$f(t) \;=\; \frac{7}{2}\int_0^t e^{-2\tau}\,\sin\left(\,2(t-\tau)\,\right)\,d\tau\,.$$

SOLUTION:
Applying the Laplace transform $\mathcal{L}$ to the given equation yields

$$\mathcal{L}\{y''\} + 3\,\mathcal{L}\{y\} \;=\; \mathcal{L}\{f(t)\}\,,$$

where

$$\mathcal{L}\{f(t)\} \;=\; \frac{7}{2}\,\mathcal{L}\{e^{-2t}\}\,\mathcal{L}\{\sin 2t\} \;=\; \frac{1}{s+2}\,\frac{7}{s^2+4}$$

for $s > 0$, as the forcing $f(t)$ is proportional to the convolution of the functions e^{-2t} and $\sin 2t$. Now, letting $Y(s) = \mathcal{L}\{y\}$ and using properties of the Laplace transform, we get

$$s^2Y(s) - sy(0) - y'(0) + 3Y(s) \;=\; \frac{1}{s+2}\,\frac{7}{s^2+4}\,,$$

which, after imposing the initial conditions and reordering terms, yields

$$(s^2+3)Y(s) \;=\; \frac{1}{s+2}\,\frac{7}{s^2+4} + (s-2) \;=\; \frac{(s^2+3)(s^2-3)}{(s^2+4)(s+2)}$$

and

$$Y(s) \;=\; \frac{s^2-3}{(s^2+4)(s+2)}\,.$$

The last expression can be written, by using the partial fraction method, as

$$\begin{aligned} Y(s) \;&=\; \frac{1}{8}\left[\frac{7s-14}{s^2+4} + \frac{1}{s+2}\right] \;=\; \frac{1}{8}\left[7\,\frac{s}{s^2+4} \;-\; 7\,\frac{2}{s^2+4} \;+\; \frac{1}{s+2}\right] \\ &=\; \frac{1}{8}\left[\,7\,\mathcal{L}\{\cos 2t\} \;-\; 7\,\mathcal{L}\{\sin 2t\} \;+\; \mathcal{L}\{e^{-2t}\}\,\right]. \end{aligned}$$

Finally, the solution of the given equation is

$$\boxed{y(t) \;=\; \tfrac{7}{8}\,\cos 2t \;-\; \tfrac{7}{8}\,\sin 2t \;+\; \tfrac{1}{8}\,e^{-2t}}\,.$$

Problem 28 Solve the following initial value problem (IVP):

$$y'' - 2y' + 5y \;=\; e^{-3t}\,, \quad y(0) = 1\,,\;\; y'(0) = 0\,.$$

SOLUTION:
The given differential equation is a nonhomogeneous linear second-order ODE with constant coefficients. We find the general solution of the homogeneous ODE solving the caracteristic equation and we can find a particular solution applying undetermined coefficients. Finally, using the initial conditions, we get the solution of the IVP:

$$\boxed{y(t) \;=\; \frac{1}{20}e^{-3t} + \frac{19}{20}e^{t}\cos(2t) - \frac{2}{5}e^{t}\sin(2t)}\,.$$

Problem 29 Solve the following initial value problem (IVP):

$$y'' - 2y' + 10y \ = \ 1\,, \quad y(0) = 0\,, \quad y'(0) = 1\,.$$

SOLUTION:
The given differential equation is a nonhomogeneous linear second-order ODE with constant coefficients. We find the general solution of the homogeneous ODE solving the caracteristic equation and we can find a particular solution applying undetermined coefficients. Finally, using the initial conditions, we get the solution of the IVP:

$$\boxed{y(t) \ = \ \frac{1}{10} - \frac{1}{10}\,e^t\,\cos(3t) + \frac{11}{30}\,e^t\,\sin(3t)}\,.$$

Problem 30 Solve the initial value problem (IVP)

$$(IVP)\begin{cases} y'' - y' - 2y = 3e^{2x}\,, & x \in \mathbb{R} \\ y(0) = 0\,, \ y'(0) = 4 \end{cases}$$

SOLUTION:
The given differential equation is a nonhomogeneous linear second-order ODE with constant coefficients. We find the general solution of the homogeneous ODE solving the caracteristic equation and we can find a particular solution applying undetermined coefficients. Finally, using the initial conditions, we get the solution of the IVP:

$$\boxed{y(t) = te^{2t} + e^{2t} - e^{-t}.}$$

Problem 31 Solve the initial value problem

$$(IVP)\begin{cases} y'' + y' - 6y = 6\,, & x \in \mathbb{R} \\ y(0) = 1\,, \ y'(0) = 1 \end{cases}$$

SOLUTION:

The given differential equation is a nonhomogeneous linear second-order ODE with constant coefficients. We find the general solution of the homogeneous ODE solving the caracteristic equation and we can find a particular solution applying undetermined coefficients. Finally, using the initial conditions, we get the solution of the IVP:

$$y(x) = -1 + \frac{7}{5}e^{2x} + \frac{3}{5}e^{-3x}.$$
