APPLIED DIFFERENTIAL CALCULUS
LECTURE 3: Systems of differential equations.
PROBLEMS

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Problem 1 Find the general solution of the system of first-order linear ODEs $\vec{X}'(t) = A\vec{X}(t)$, with

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Problem 2 Find the solution of the system of first-order linear ODEs $\vec{X}'(t) = A\vec{X}(t)$, with

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}, \quad \vec{X}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Problem 3 Find the solution of the system of first-order linear ODEs $\vec{X}'(t) = A\vec{X}(t)$, with

$$A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Problem 4 Find the general solution of the system of first-order linear ODEs $\vec{X}'(t) = A\vec{X}(t)$, with

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

Problem 5 Find the solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Problem 6 Find the solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Problem 7 Find the general solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

Problem 8 Find the general solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

Problem 9 Find the general solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$$

Problem 10 Find the solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Problem 11 Consider the system of differential equations $\overrightarrow{X}'(t) = A\overrightarrow{X}(t)$, with $A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$, satisfying the initial condition $\overrightarrow{X}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(a) Find the solution $\overrightarrow{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$.

(b) Solve the following initial value problem:

$$y'' - 6y' + 9y = 0\,; \qquad y(0) = 1\,, \quad y'(0) = 6\,.$$

(c) Applying the change of variable $X_2(t) = y(t)$, prove that the system of differential equations is equivalent to the initial value problem given in (b). Then, compare the solutions obtained in (a) and (b).

NOTE. The following formula may be useful: $\mathcal{L}\left\{\frac{t^n e^{at}}{n!}\right\} = 1/(s-a)^{n+1}$ for $n = 0, 1, 2, ...$

Problem 12 Given the system of differential equations $\overrightarrow{X}'(t) = A\,\overrightarrow{X}(t)$, with $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, find $\overrightarrow{X}(t)$ and calculate $\lim_{t \to -\infty} \overrightarrow{X}(t)$.

Problem 13 Consider the system of differential equations $\overrightarrow{X}'(t) = A\overrightarrow{X}(t)$, with $A = \begin{bmatrix} 4 & 2 \\ 4 & -3 \end{bmatrix}$.

(a) Find the general solution $\overrightarrow{X}(t)$.

(b) Find <u>one</u> solution that is bounded for $t \longrightarrow +\infty$.

Problem 14 i) Write the Ordinary Differential Equation (ODE) $x''+x=0$ ($x(0)=x_0$, $x'(0)=v_0$) as a system of two first order ODEs for $x(t)$ and $v(t)=x'(t)$.

ii) Approximate the equation for $x(t)$ by a forward Euler scheme and that for $v(t)$ by a backward Euler scheme, both with time step $\Delta t = \tau$.

iii) Find the values of τ for which the solutions $x(t_n) \approx x_n = \lambda^n x_0$, $v(t_n) \approx v_n = \lambda^n v_0$ produce a stable scheme with $|\lambda| = 1$.

Problem 15 Classify the equilibrium point and solve the following system of first order linear ODEs:

$$\begin{cases} x' = y, \\ y' = -5x - 4y. \end{cases} \quad \text{Find the solution that has the initial condition} \quad \begin{cases} x(0) = 2, \\ y(0) = -4. \end{cases}$$

Problem 16 Find the general solution of the following linear system of differential equations:

$$x' = 4x - y, \quad y' = 3x + y.$$

Classify the equilibrium solution $(0, 0)$ and draw the phase portrait.

Problem 17 Solve the following system of first order linear ODEs, classify the equilibrium point $(0, 0)$ and draw its phase portrait, indicating explicitly any real eigendirections that may be relevant.

$$\begin{cases} x_1' = 7x_1 + 6x_2 \\ x_2' = 2x_1 + 6x_2 \end{cases}$$
