APPLIED DIFFERENTIAL CALCULUS
LECTURE 3: Systems of differential equations.
PROBLEMS

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Problem 1 Find the general solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

SOLUTION:
Eigenvalues of the matrix A: $\lambda_1 = 4$, $\lambda_2 = -1$, $\lambda_3 = 1$.
Eigenvectors: $V_{\lambda_1} = (1,1,1)$, $V_{\lambda_2} = (-1,0,1)$, $V_{\lambda_3} = (1,-2,1)$.
General solution:

$$\begin{aligned}
X_1(t) &= c_1\left[\frac{1}{6}e^{t} + \frac{1}{3}e^{4t} + \frac{1}{2}e^{-t}\right] + c_2\left[-\frac{1}{3}e^{t} + \frac{1}{3}e^{4t}\right] + c_3\left[\frac{1}{6}e^{t} + \frac{1}{3}e^{4t} - \frac{1}{2}e^{-t}\right]\,; \\
X_2(t) &= c_1\left[-\frac{1}{3}e^{t} + \frac{1}{3}e^{4t}\right] + c_2\left[\frac{2}{3}e^{t} + \frac{1}{3}e^{4t}\right] + c_3\left[-\frac{1}{3}e^{t} + \frac{1}{3}e^{4t}\right]\,; \\
X_3(t) &= c_1\left[\frac{1}{6}e^{t} + \frac{1}{3}e^{4t} - \frac{1}{2}e^{-t}\right] + c_2\left[-\frac{1}{3}e^{t} + \frac{1}{3}e^{4t}\right] + c_3\left[\frac{1}{6}e^{t} + \frac{1}{3}e^{4t} + \frac{1}{2}e^{-t}\right]\,;
\end{aligned}$$

with $\quad c_1\,, c_2\,, c_3\, \in \mathbb{R}$

Problem 2 Find the solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}, \quad \vec{X}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

SOLUTION:
Eigenvalues of the matrix A: $\lambda_1 = 4$, $\lambda_2 = 2$.
Eigenvectors: $V_{\lambda_1} = (1, 1)$, $V_{\lambda_2} = (1, 3)$.
Solution:

$$\begin{aligned} X_1(t) &= \frac{7}{2}e^{4t} - \frac{3}{2}e^{2t}\,; \\ X_2(t) &= \frac{7}{2}e^{4t} - \frac{3}{2}e^{2t}\,; \end{aligned}$$

Problem 3 Find the solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

SOLUTION:
Eigenvalues of the matrix A: $\lambda_1 = -1 + i$, $\lambda_2 = -1 - i$.
Eigenvectors: $V_{\lambda_1} = (2 + i, 5)$, $V_{\lambda_2} = (2 - i, 5)$.
Solution:

$$\begin{aligned} X_1(t) &= 5e^{-t}\sin(t) + 2e^{-t}\cos(t)\,; \\ X_2(t) &= 12e^{-t}\sin(t) - e^{-t}\cos(t)\,; \end{aligned}$$

Problem 4 Find the general solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

SOLUTION:
Eigenvalues of the matrix A: $\lambda_1 = 1 + i2$, $\lambda_2 = 1 - i2$.
Eigenvectors: $V_{\lambda_1} = (1 + i, 2)$, $V_{\lambda_2} = (1 - i, 2)$.
General solution:

$$\begin{aligned} X_1(t) &= c_1e^t(\sin(2t) + \cos(2t)) - c_2e^t\sin(2t)\,; \\ X_2(t) &= 2c_1e^t\sin(2t) + c_2e^t(\cos(2t) - \sin(2t))\,; \end{aligned}$$

with $\quad c_1\,, c_2 \in \mathbb{R}$

Problem 5 Find the solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

SOLUTION:
Eigenvalues of the matrix A: $\lambda_1 = -1 + i$, $\lambda_2 = -1 - i$.
Eigenvectors: $V_{\lambda_1} = (2 + i, 1)$, $V_{\lambda_2} = (2 - i, 1)$.
Solution:

$$\begin{aligned} X_1(t) &= e^{-t}(\cos(t) - 3\sin(t))\,; \\ X_2(t) &= e^{-t}(\cos(t) - \sin(t))\,; \end{aligned}$$

Problem 6 Find the solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

SOLUTION:
Eigenvalues of the matrix A: $\lambda_1 = -2 + i$, $\lambda_2 = -2 - i$.
Eigenvectors: $V_{\lambda_1} = (1 - i, 1)$, $V_{\lambda_2} = (1 + i, 1)$.
Solution:

$$\begin{aligned} X_1(t) &= e^{-2t}(\cos(t) - 5\sin(t))\,; \\ X_2(t) &= -e^{-t}(3\sin(t) + 2\cos(2t))\,; \end{aligned}$$

Problem 7 Find the general solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

SOLUTION:
Eigenvalues of the matrix A: $\lambda_1 = \lambda_2 = 2$.
Eigenvector: $V = (-1, 1)$.
General solution:

$$\begin{aligned} X_1(t) &= c_1 e^{2t}(1 - t) - c_2 t e^{2t}\,; \\ X_2(t) &= c_1 t e^{2t} - c_2 e^{2t}(1 + t)\,; \\ \text{with} \quad c_1\,, c_2 &\in \mathbb{R} \end{aligned}$$

Problem 8 Find the general solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

SOLUTION:
Eigenvalues of the matrix A: $\lambda_1 = \lambda_2 = 1$.
Eigenvector: $V = (2, 1)$.
General solution:

$$\begin{aligned} X_1(t) &= c_1 e^t (2t+1) - 4c_2 t e^t \,; \\ X_2(t) &= c_1 t e^t - c_2 e^t (2t-1) \,; \\ \text{con} \quad c_1\,, c_2 \,\in \mathbb{R} & \end{aligned}$$

Problem 9 Find the general solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$$

SOLUTION:
Eigenvalues of the matrix A: $\lambda_1 = \lambda_2 = 0$.
Eigenvector: $V = (1, 2)$.
General solution:

$$\begin{aligned} X_1(t) &= c_1(4t+1) - 2c_2 t \,; \\ X_2(t) &= 8c_1 t + c_2(-4t+1) \,; \\ \text{con} \quad c_1\,, c_2 \,\in \mathbb{R} & \end{aligned}$$

Problem 10 Find the solution of the system of first-order linear ODEs $\vec{X}'(t) = A\,\vec{X}(t)$, with

$$A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

SOLUTION:
Eigenvalues of the matrix A: $\lambda_1 = \lambda_2 = -3$.
Eigenvector: $V = (1, 1)$.
Solution:

$$\begin{aligned} X_1(t) &= e^{-3t}(4t+3) \,; \\ X_2(t) &= e^{-3t}(4t+2) \,; \end{aligned} \tag{1}$$

Problem 11 Consider the system of differential equations $\overrightarrow{X}'(t) = A\overrightarrow{X}(t)$, with $A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$, satisfying the initial condition $\overrightarrow{X}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(a) Find the solution $\overrightarrow{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$.

(b) Solve the following initial value problem:

$$y'' - 6y' + 9y = 0\,; \qquad y(0) = 1\,, \;\; y'(0) = 6\,.$$

(c) Applying the change of variable $X_2(t) = y(t)$, prove that the system of differential equations is equivalent to the initial value problem given in (b). Then, compare the solutions obtained in (a) and (b).

NOTE. The following formula may be useful: $\mathcal{L}\left\{\frac{t^n e^{at}}{n!}\right\} = 1/(s-a)^{n+1}$ for $n = 0, 1, 2, ...$

SOLUTION:

(a) Let's calculate the eigenvalues and eigenvectors of the coefficient matrix A. Thus, solving the equation $|A - \lambda I| = 0$, we get the double eigenvalue $\lambda = 3$. An eigenvector associated with $\lambda = 3$ is $\overrightarrow{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which yields the first fundamental solution as

$$\overrightarrow{X}^1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}.$$

In addition, by solving the system $(A - 3I)\overrightarrow{w} = \overrightarrow{v}$, we get the second fundamental solution as

$$\overrightarrow{X}^2(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{3t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{3t}.$$

Hence, the general solution of the given system of differential equations is

$$\overrightarrow{X}(t) = C_1\overrightarrow{X}^1(t) + C_2\overrightarrow{X}^2(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + C_2 \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{3t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{3t} \right].$$

Finally, using the initial condition $\overrightarrow{X}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, we find the values $C_1 = 2$ and $C_2 = -3$, which gives the final general solution as

$$\overrightarrow{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} (2-3t)e^{3t} \\ (1+3t)e^{3t} \end{pmatrix}.$$

(b) The differential equation is a homogeneous linear second-order ODE with constant coefficients. Applying initial conditions we find the solution:

$$y(t) = e^{3t} + 3\,t\,e^{3t}.$$

(c) The second equation of the system is $X_2'(t) = X_1(t)+4X_2(t)$, which upon differentiating results in $X_2''(t) = X_1'(t)+4X_2'(t)$. The same equation also yields $X_1(t) = X_2'(t)-4X_2(t)$. On the other hand, the first equation of the system is $X_1'(t) = 2X_1(t)-X_2(t)$. Thus

$$X_2''(t) = 2X_1(t)-X_2(t)+4X_2'(t) = 2\left[X_2'(t)-4X_2(t)\right]-X_2(t)+4X_2'(t) = 6X_2'(t)-9X_2(t).$$

After changing $X_2(t)=y(t)$ we get $y''-6y'+9y = 0$, with the initial conditions $y(0)=X_2(0)=1$ and $y'(0)=X_2'(0)=X_1(0)+4X_2(0)=2+4=6$, which is exactly the problem in (b). Regarding the solutions, we can observe that $y(t)$ coincides with $X_2(t)$ and, since $X_1(t)=X_2'(t)-4X_2(t)$, we also have

$$X_1(t) = y'(t)-4y(t) = (6+9t)e^{3t}-4\,(1+3t)e^{3t} = (2-3t)e^{3t}.$$

Problem 12 Given the system of differential equations $\overrightarrow{X}'(t) = A\overrightarrow{X}(t)$, with $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, find $\overrightarrow{X}(t)$ and calculate $\lim_{t\to-\infty} \overrightarrow{X}(t)$.

SOLUTION:

The general solution of the given system is obtained in terms of the eigenvalues and eigenvectors of the coefficient matrix A. The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$ (real and different). Two associated eigenvectors are $\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\xi_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, respectively. Thus, the desired solution is given by

$$\boxed{\overrightarrow{X}(t) = c_1\,\xi_1\,e^{\lambda_1 t} + c_2\,\xi_2\,e^{\lambda_2 t} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}},$$

where c_1 and c_2 are constants. In addition, since $\lim_{t\to-\infty} e^{3t} = \lim_{t\to-\infty} e^{2t} = 0$ and c_1, c_2, ξ_1, ξ_2 do not depend on t, we get

$$\boxed{\lim_{t\to-\infty} \overrightarrow{X}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}.$$

Problem 13 Consider the system of differential equations $\overrightarrow{X}'(t) = A\overrightarrow{X}(t)$, with $A = \begin{bmatrix} 4 & 2 \\ 4 & -3 \end{bmatrix}$.

(a) Find the general solution $\overrightarrow{X}(t)$.

(b) Find <u>one</u> solution that is bounded for $t \longrightarrow +\infty$.

SOLUTION:

(a) The general solution of the system is obtained calculating the eigenvalues and eigenvectors of the matrix A. The eigenvalues are calculated by solving $|A-\lambda I| = 0$, which yields $\lambda_1 = -4$,

$\lambda_2 = 5$ (real and different). In addition, two associated eigenvectors are $\xi_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and $\xi_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, respectively. Hence, the general solution can be written as

$$\boxed{\overrightarrow{X}(t) \;=\; c_1\xi_1 e^{\lambda_1 t} + c_2\xi_2 e^{\lambda_2 t} \;=\; c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}},$$

where c_1 , c_2 are arbitrary constants.

(b) Considering the general solution of the system calculated in (a), we can get one particular solution that is bounded for $t \longrightarrow +\infty$ by setting for instance $c_1 = 1$ and $c_2 = 0$, which yields

$$\boxed{\overrightarrow{X}_p(t) \;=\; \xi_1 e^{\lambda_1 t} \;=\; \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4t}}.$$

Problem 14 i) Write the Ordinary Differential Equation (ODE) $x''+x = 0$ ($x(0) = x_0$, $x'(0) = v_0$) as a system of two first order ODEs for $x(t)$ and $v(t) = x'(t)$.

ii) Approximate the equation for $x(t)$ by a forward Euler scheme and that for $v(t)$ by a backward Euler scheme, both with time step $\Delta t = \tau$.

iii) Find the values of τ for which the solutions $x(t_n) \approx x_n = \lambda^n x_0$, $v(t_n) \approx v_n = \lambda^n v_0$ produce a stable scheme with $|\lambda| = 1$.

SOLUTION:

i) First order linear ODEs are $x' = v$, $v' = -x$.

ii) The overall scheme is $x_{n+1} - x_n = v_n\tau$, $v_{n+1} - v_n = -x_{n+1}\tau$.

iii) Inserting $x_n = \lambda^n x_0$, $v_n = \lambda^n v_0$ in the scheme and dividing by λ^n, we get:

$$(\lambda - 1)x_0 = v_0\tau, \quad \text{and } (\lambda - 1)v_0 = -x_0\lambda\tau \Longrightarrow (\lambda - 1)^2 v_0 = -v_0\lambda\tau^2.$$

The last equation yields the eigenvalue problem $\lambda^2 - (2 - \tau^2)\lambda + 1 = 0$, with solutions

$$\lambda = \frac{2 - \tau^2 \pm \sqrt{(2 - \tau^2)^2 - 4}}{2} = \frac{2 - \tau^2 \pm \sqrt{\tau^2(\tau^2 - 4)}}{2} = 1 - \frac{\tau^2}{2} \pm i\frac{\tau}{2}\sqrt{4 - \tau^2}.$$

We find $|\lambda| = 1$ if the eigenvalues are complex conjugate or ± 1, which means $0 \le \tau^2 \le 4$, i.e., $0 \le \tau \le 2$.

Problem 15 Classify the equilibrium point and solve the following system of first order linear ODEs:

$$\begin{cases} x' = y, \\ y' = -5x - 4y. \end{cases} \quad \text{Find the solution that has the initial condition} \quad \begin{cases} x(0) = 2, \\ y(0) = -4. \end{cases}$$

SOLUTION:
The eigenvalues are given by:

$$\begin{vmatrix} -\lambda & 1 \\ -5 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 5 = 0, \Rightarrow \lambda_{1,2} = -2 \pm i.$$

As the real parts of both eigenvalues are negative, the equilibrium point is stable. As the λ-s are complex, the equilibrium point is a stable spiral point (stable focus). The eigenvectors can be calculated as solutions of the system:

$$\begin{cases} -\lambda v_1 + v_2 = 0, \\ -5v_1 + (-2 \pm i)v_2 = 0, \end{cases}$$

for the obtained eigenvalues. From the first equation it is clear the eigenvectors are

$$\mathbf{v}_{1,2} = \begin{pmatrix} 1 \\ \lambda_{1,2} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \pm i \end{pmatrix}.$$

Thus the solution is

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{(-2+i)t} \begin{pmatrix} 1 \\ -2+i \end{pmatrix} + c_2 e^{(-2-i)t} \begin{pmatrix} 1 \\ -2-i \end{pmatrix},$$

where c_1 and c_2 can be determined from the initial conditions:

$$\begin{aligned} & x(0) = c_1 + c_2 = 2, \\ & y(0) = (-2+i)c_1 + (-2-i)c_2 = -2(c_1+c_2) + i(c_1 - c_2) = -4, \\ & \Rightarrow \begin{cases} c_1 + c_2 = 2, \\ -2(c_1+c_2) + i(c_1 - c_2) = 4, \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = 2, \\ c_1 - c_2 = 0, \end{cases} \Rightarrow c_1 = c_2 = 1. \end{aligned}$$

Plugging these values into the formula for the solution, we obtain:

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = e^{-2t} \begin{pmatrix} e^{it} + e^{-it} \\ -2\left(e^{it} + e^{-it}\right) + i\left(e^{it} - e^{-it}\right) \end{pmatrix} = e^{-2t} \begin{pmatrix} 2\cos t \\ -4\cos t - 2\sin t \end{pmatrix}.$$

We have used here that

$$\frac{e^{it} + e^{-it}}{2} = \cos t,$$
$$\frac{e^{it} - e^{-it}}{2i} = \sin t.$$

Problem 16 Find the general solution of the following linear system of differential equations:

$$x' = 4x - y, \quad y' = 3x + y.$$

Classify the equilibrium solution $(0,0)$ and draw the phase portrait.

SOLUTION:
Firstly, we write this system of ODEs as a vector equation,

$$\underline{x}' = \underline{\underline{A}}\,\underline{x}, \quad \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \underline{\underline{A}} = \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix}.$$

To solve it, we set $\underline{x} = e^{\lambda t}\underline{v}$, thereby obtaining the eigenvalue problem $\underline{\underline{A}}\underline{v} = \lambda \underline{v}$. The eigenvalue equation is $\lambda^2 - 5\lambda + 7 = 0$. Then

$$\lambda = \frac{5 \pm \sqrt{25-28}}{2} = \frac{5}{2} \pm i\frac{\sqrt{3}}{2}.$$

As the eigenvalues are complex conjugate, with positive real part, the equilibrium solution is an unstable spiral point (or unstable focus).

To find the solution, we write

$$\underline{x}(t) = e^{5t/2}\left(\underline{a}\,\cos\frac{t\sqrt{3}}{2} + \underline{b}\,\sin\frac{t\sqrt{3}}{2}\right),$$

insert it in the vector ODE and obtain

$$\left[\left(\underline{\underline{A}} - \frac{5}{2}\underline{\underline{I}}\right)\underline{a} - \frac{\sqrt{3}}{2}\underline{b}\right]\cos\frac{t\sqrt{3}}{2} + \left[\left(\underline{\underline{A}} - \frac{5}{2}\underline{\underline{I}}\right)\underline{b} + \frac{\sqrt{3}}{2}\underline{a}\right]\sin\frac{t\sqrt{3}}{2} = \underline{0}.$$

Then

$$\underline{a} = \frac{\sqrt{3}}{2}\left(\underline{\underline{A}} - \frac{5}{2}\underline{\underline{I}}\right)^{-1}\underline{b} = \frac{\sqrt{3}}{2}\begin{pmatrix} \frac{3}{2} & -1 \\ 3 & -\frac{3}{2} \end{pmatrix}^{-1}\underline{b} = \frac{\sqrt{3}}{2}\begin{pmatrix} -2 & \frac{4}{3} \\ -4 & 2 \end{pmatrix}\underline{b} = \sqrt{3}\begin{pmatrix} -b_1 + \frac{2}{3}b_2 \\ -2b_1 + b_2 \end{pmatrix}.$$

Inserting this in the solution, we obtain the general solution

$$\underline{x}(t) = e^{5t/2}\begin{pmatrix} b_1(\sin\frac{t\sqrt{3}}{2} - \sqrt{3}\cos\frac{t\sqrt{3}}{2}) + \frac{2}{\sqrt{3}}b_2\cos\frac{t\sqrt{3}}{2} \\ -2b_1\sqrt{3}\cos\frac{t\sqrt{3}}{2} + b_2(\sin\frac{t\sqrt{3}}{2} + \sqrt{3}\cos\frac{t\sqrt{3}}{2}) \end{pmatrix},$$

in which b_1 and b_2 are arbitrary real numbers.

Problem 17 Solve the following system of first order linear ODEs, classify the equilibrium point $(0,0)$ and draw its phase portrait, indicating explicitly any real eigendirections that may be relevant.

$$\begin{cases} x_1' = 7x_1 + 6x_2 \\ x_2' = 2x_1 + 6x_2 \end{cases}$$

SOLUTION:

i) The coefficient matrix has trace 13 and determinant 30. Therefore the characteristic polynomial is $\lambda^2 - 13\lambda + 30$ and thus the eigenvalues are $\frac{13}{2} \pm \frac{7}{2}$, i.e., 3 and 10. The components of the eigenvector corresponding to 3 satisfy: $4v_1 + 6v_2 = 0$, so that $v_1 = 3$ and $v_2 = -2$. Similarly, the components of the eigenvector corresponding to 10 satisfy: $-3v_1 + 6v_2 = 0$, so that $v_1 = 2$ and $v_2 = 1$. Then the general solution of the system of ODEs is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a\, e^{3t} \begin{pmatrix} 3 \\ -2 \end{pmatrix} + b\, e^{10t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

ii) As the eigenvalues are both positive, the origin is an unstable node. The eigenvectors provide the directions of exit for trajectories. The ordinary eigendirection $(3, -2)$ corresponds to the eigenvalue 3 and the extraordinary eigendirection $(2, 1)$ corresponds to the eigenvalue 10. The phase portrait is depicted in the figure.