APPLIED DIFFERENTIAL CALCULUS
LECTURE 4: Boundary value problems.
PROBLEMS

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Problem 1

Find the solution of the following boundary value problem:

$$y'' + 5y = 0\,; \quad y(0) = 1\,;\, y(\pi) = 0\,.$$

SOLUTION:
The general solution of the differential equation is:

$$y(x) = c_1 \cos(\sqrt{5}x) + c_2 \sin(\sqrt{5}x)\,, \quad \text{with} \quad c_1, c_2 \in \mathbb{R}\,.$$

The first boundary condition gives $c_1 = 1$ and the second gives $c_1 \cos(\sqrt{5}\pi) + c_2 \sin(\sqrt{5}\pi) = 0 \implies c_2 = -\cot(\sqrt{5}\pi)$, then, the problem has a unique solution:

$$y(x) = \cos(\sqrt{5}x) - \cot(\sqrt{5}\pi)\sin(\sqrt{5}x)\,.$$

Problem 2

Solve the boundary value problem:

$$y'' + y = 0\,; \quad y(0) = 1\,;\, y(\pi) = \alpha\,,$$

where α is a given real number.

SOLUTION:
The general solution of the differential equation is:

$$y(x) = c_1 \cos(x) + c_2 \sin(x)\,, \quad \text{with} \quad c_1, c_2 \in \mathbb{R}\,.$$

The first boundary condition gives $c_1 = 1$ and the second gives $-c_1 = \alpha$.
These two conditions on c_1 are incompatible if $\alpha \neq -1$, so the problem has no solution in that

case.
However, if $\alpha = -1$, then both boundary conditions are satisfied provided that $c_1 = 1$, regardless of the value of c_2. In this case there are infinitely many solutions of the form:

$$y(x) = \cos(x) + c_2 \sin(x)\,,$$

where c_2 is arbitrary.

Problem 3

Find the solution of the following boundary value problem:

$$y'' + 5y = 0\,; \quad y(0) = 0\,;\, y(\pi) = 0\,.$$

SOLUTION:
The general solution of the differential equation is:

$$y(x) = c_1 \cos(x) + c_2 \sin(x)\,, \quad \text{with} \quad c_1, c_2 \in \mathbb{R}\,.$$

The first boundary condition gives $c_1 = 1$ and the second gives $c_2 = 0$.
the problem has a unique solution: $y(x) = 0\,, \quad \forall x \in [0, \pi]\,.$

Problem 4

Solve the boundary value problem:

$$y'' + y = 0\,; \quad y(0) = 0\,;\, y(\pi) = 0\,.$$

SOLUTION:
The general solution of the differential equation is:

$$y(x) = c_1 \cos(x) + c_2 \sin(x)\,, \quad \text{with} \quad c_1, c_2 \in \mathbb{R}\,.$$

The first boundary condition gives $c_1 = 0$.
On the other hand, since $\sin(\pi) = 0$, the second boundary condition is also satisfied regardless of the value of c_2. Thus the solution of the problem is $y = c_2 \sin(x)$, where c_2 is arbitrary, therefore the problem have infinitely many solutions

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Problem 5 Given the following boundary value problem:

$$X'' + \lambda X = 0\,; \quad X'(0) = 0\,,\;\; X'(\pi/3) = 0\,,$$

find the values of the constant parameter $\lambda \geq 0$ yielding non-zero solutions.

SOLUTION:

Let us now distinguish two cases. **Case 1.** $\lambda = 0$

$X'' = 0 \implies X(x) = c_1 x + c_2$ with $c_1\,, c_2 \in \mathbb{R}$. As $X'(x) = c_1$, we have that $X'(0) = 0 = c_1 = X'(\pi/3)$. Hence, if $\boxed{\lambda = 0}$ then the function $X(x) = c_2 \neq 0$ is a nonzero solution of the problem.

Case 2. $\lambda > 0$

Let us take $\lambda = a^2$, with $a > 0$. The corresponding characteristic equation is then $r^2 + a^2 = 0 \implies r = \pm ia\,,\; i \in \mathbb{C}$. Hence

$$X(x) = c_1\cos(ax) + c_2\sin(ax)\,; \quad X'(x) = -ac_1\sin(ax) + ac_2\cos(ax)\,, \text{ with } c_1\,, c_2 \in \mathbb{R}\,.$$

Applying the BCs we get $X'(0) = 0 \implies c_2 = 0$; $X'(\pi/3) = 0 \implies -ac_1\sin(a\pi/3) = 0$. Then, imposing $c_1 \neq 0$ yields $\sin(a\pi/3) = 0 \implies a\pi/3 = n\pi \implies a = 3n\,,\; n = 1, 2, 3, \ldots$. Finally,

$$\boxed{\lambda = (3n)^2 = 9n^2\,,\;\; n = 1, 2, 3, \ldots}$$

Problem 6 Solve the following boundary value problem:

$$X'' + \lambda X = 0\,; \quad X'(0) = 0\,,\;\; X(1) = 0\,,$$

and find the values of the constant parameter $\lambda > 0$ yielding non-zero solutions.

SOLUTION:

The general solution $X = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$ produces $c_2 = 0$ and then $\cos\sqrt{\lambda} = 0$. Therefore,

$$\sqrt{\lambda} = \left(n - \frac{1}{2}\right)\pi \implies \lambda_n = \frac{(2n-1)^2\pi^2}{4}, \quad n = 1, 2, \ldots,$$

and the corresponding solutions are:

$$X_n(x) = \cos\frac{(2n-1)\pi x}{2}.$$

Problem 7 Solve the following boundary value problem:

$$X'' + \lambda X = 0\,; \quad X(0) = 0\,,\ X'(1) = 0\,,$$

and find the values of the constant parameter $\lambda > 0$ yielding non-zero solutions.

SOLUTION:
The general solution of the ODE, $X = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$, produces $X(0) = c_1 = 0$, and $X'(1) = c_2 \sqrt{\lambda} \cos \sqrt{\lambda} = 0$.
Then $\sqrt{\lambda} = (2n-1)\pi/2$, $n = 1, 2, \ldots$.
And the corresponding solutions are:

$$X_n(x) = \sin\left(\frac{(2n-1)\pi}{2} x\right).$$

Problem 8 Solve the following boundary value problem:

$$X'' + \lambda X = 0\,; \quad X(0) = 0\,,\ X(1) = 0\,,$$

and find the values of the constant parameter $\lambda > 0$ yielding non-zero solutions.

SOLUTION:
The solution is $\lambda_n = n^2\pi^2$, $X_n(x) = \sin(n\pi x)$, $n = 1, 2, \ldots$. Why? The general solution of the ODE is $X = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$. $X(0) = c_1 = 0$ and $X(1) = c_2 \sin(\sqrt{\lambda}) = 0$, which yields $\sqrt{\lambda} = n\pi$, $n = 1, 2, \ldots$.

Problem 9 Find the eigenvalues and eigenfunctions of the following problem and expand $f(x) = e^x$ in terms of the eigenfunctions:

$$\begin{aligned} &u'' + \lambda u = 0\,, \quad x \in (0,1), \\ &u(0) = 0, \quad u'(1) = 0\,. \end{aligned}$$

SOLUTION:
The general solution of the ODE is $u(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$. The boundary conditions yield $c_1 = 0$ and $\sqrt{\lambda} \cos \sqrt{\lambda} = 0$. Then $\sqrt{\lambda} = (2n-1)\pi/2$, $n = 1, 2, \ldots$. We have

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4}, \quad u_n(x) = \sin\frac{(2n-1)\pi x}{2}, \quad n = 1, 2, \ldots.$$

The Fourier coefficients of $f(x) = e^x$ are

$$\begin{aligned} f_n &= 2\int_0^1 e^x \sin\frac{(2n-1)\pi x}{2}\,dx = 2\mathrm{Im}\int_0^1 \exp\left[\left(1 + i\frac{(2n-1)\pi}{2}\right)x\right]dx = 2\mathrm{Im}\frac{ee^{i(2n-1)\pi/2} - 1}{1 + i\frac{(2n-1)\pi}{2}} \\ &= 2\mathrm{Im}\frac{e^{in\pi}e^{-i\pi/2}e - 1}{1 + i\frac{(2n-1)\pi}{2}} = 2\mathrm{Im}\frac{-i(-1)^n e - 1}{1 + i\frac{(2n-1)\pi}{2}} = -2\mathrm{Im}\frac{[1 + (-1)^n ie][1 - i(n - \frac{1}{2})\pi]}{1 + \pi^2\left(n - \frac{1}{2}\right)^2} \\ &= -2\frac{(-1)^n e - \left(n - \frac{1}{2}\right)\pi}{1 + \pi^2\left(n - \frac{1}{2}\right)^2}. \end{aligned}$$

Then

$$f_n = \frac{(2n-1)\pi - (-1)^n 2e}{1+\pi^2\left(n-\frac{1}{2}\right)^2}.$$

Also, integrating twice by parts (always set $dv = e^x dx$, hence $v = e^x$),

$$\begin{aligned}
f_n &= 2\int_0^1 e^x \sin\frac{(2n-1)\pi x}{2}\,dx = 2e^x \sin\frac{(2n-1)\pi x}{2}\Big|_0^1 - (2n-1)\pi\int_0^1 e^x\cos\frac{(2n-1)\pi x}{2}\,dx \\
&= 2e(-1)^{n+1} - (2n-1)\pi e^x\cos\frac{(2n-1)\pi x}{2}\Big|_0^1 - \frac{(2n-1)^2\pi^2}{2}\int_0^1 e^x\sin\frac{(2n-1)\pi x}{2}\,dx \\
&= 2e(-1)^{n+1} + (2n-1)\pi - \frac{(2n-1)^2\pi^2}{2}\frac{f_n}{2}.
\end{aligned}$$

Then

$$\left[1+\frac{(2n-1)^2\pi^2}{4}\right]f_n = 2e(-1)^{n+1} + (2n-1)\pi \Longrightarrow f_n = \frac{(2n-1)\pi - (-1)^n 2e}{1+\pi^2\left(n-\frac{1}{2}\right)^2}.$$
