APPLIED DIFFERENTIAL CALCULUS
LECTURE 5: Fourier series and separation of variables: Heat equation.
PROBLEMS

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Problem 1 Consider the following model of heat equation

$$\begin{aligned}
\text{Partial Differential Equation (PDE)} &: \quad \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial u}{\partial t}(x,t)\,, \;\; t>0\,, \;\; 0<x<\pi/3\,,\\
\text{Boundary Conditions (BCs)} &: \quad \frac{\partial u}{\partial x}(0,t)=0\,, \;\; \frac{\partial u}{\partial x}(\pi/3,t)=0\,, \;\; t>0\,,\\
\text{Initial Condition (IC)} &: \quad u(x,0) = 2x+1\,, \;\; 0\le x\le \pi/3\,.
\end{aligned}$$

Apply the separation of variables $u(x,t) = X(x)\,T(t) \not\equiv 0$. Then:

(i) Prove that $X(x)$ satisfies the boundary value problem

$$X'' + \lambda X = 0\,; \quad X'(0)=0\,, \;\; X'(\pi/3)=0\,,$$

and find the values of the separation constant $\lambda \ge 0$ yielding non-zero solutions.

(ii) Knowing that the solution $u(x,t)$ can be expressed as

$$u(x,t) = \sum_{n=0}^{\infty} A_n\, e^{-9n^2 t}\cos(3nx)\,, \quad \text{with} \;\; A_n \in \mathbb{R}\,,$$

find an approximate value for $u(\pi/6, 1/9)$, using only the first three terms of the series.

NOTE. The following result may be useful.

Given $L>0$ and $m\,,n \in \mathbb{N}\cup\{0\}$, we have $\displaystyle\int_0^L \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\,dx = \begin{cases} 0\,, & m\ne n\\ L/2\,, & m=n\ne 0\\ L\,, & m=n=0\,.\end{cases}$

Problem 2 Find the solution of the following problem.

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \;\; x \in (0,1)\,, \;\; t > 0\\
&\frac{\partial u}{\partial x}(0,t) = 0\,, \;\; u(1,t) = 0\,, \;\; t > 0\\
&u(x,0) = u_0(x) = \begin{cases} 1, \;\; x < 1/2, \\ 0, \;\; x \geq 1/2. \end{cases}
\end{aligned}$$

Problem 3 Let $f(x) = x$ be a function on the interval $(-1;,1)$.

1. Calculate the Fourier coefficients of $f(x)$ (*Hint.* Since $f(x)$ is odd, use only sine terms: $f(x) = \sum_{n=1}^{\infty} A_n \sin(\pi n x)$).
2. Is it possible to differentiate the resulting series term by term in order to obtain the Fourier expansion of the derivative $f'(x)$. Why or why not?
3. Since $f(x)$ is odd, $f'(x)$ is even and its Fourier expansion is $f'(x) = B_0 + \sum_{n=1}^{\infty} B_n \cos(\pi n x)$. Express the coefficients B_n of $f'(x)$ through the coefficients A_n of $f(x)$ using the correct formula for the Fourier series of $f'(x)$. Calculate B_n and B_0.

Problem 4 Solve the following initial value problem for the heat equation:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \;\; x \in (0,1), \;\; t > 0,\\
&\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0,\\
&u(x,0) = u_0(x) = \begin{cases} 1, \;\; x < 1/2, \\ 0, \;\; x \geq 1/2. \end{cases}
\end{aligned}$$

Hint: The boundary conditions correspond to insulated ends. Therefore a stationary solution is a constant equal to the average value of $u_0(x)$, which is $1/2$. Thus your solution should tend to $1/2$ as t tends to infinity.

Problem 5 Find the solution of the following initial boundary value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \;\; x \in (0,1)\,, \;\; t > 0,$$
$$\frac{\partial u}{\partial x}(0,t) = \cos(\pi t)\,, \;\; u(1,t) = 0\,, \;\; t > 0; \;\; u(x,0) = 0.$$

Problem 6 Solve the following initial boundary value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2, \;\; x \in (0,1)\,, \;\; t > 0,$$
$$\frac{\partial u}{\partial x}(0,t) = 0\,, \;\; u(1,t) = 0\,, \;\; t > 0,$$
$$u(x,0) = 0.$$
