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APPLIED DIFFERENTIAL CALCULUS
LECTURE 5: Fourier series and separation of variables: Heat equation.
PROBLEMS

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Problem 1 Consider the following model of heat equation

$$\begin{aligned}
\text{Partial Differential Equation (PDE)} &: \quad \frac{\partial^2 u}{\partial x^2}(x,t) \;=\; \frac{\partial u}{\partial t}(x,t)\,, \;\; t>0\,, \;\; 0<x<\pi/3\,,\\
\text{Boundary Conditions (BCs)} &: \quad \frac{\partial u}{\partial x}(0,t)=0\,, \;\; \frac{\partial u}{\partial x}(\pi/3,t)=0\,, \;\; t>0\,,\\
\text{Initial Condition (IC)} &: \quad u(x,0)=2x+1\,, \;\; 0\le x\le \pi/3\,.
\end{aligned}$$

Apply the separation of variables $u(x,t) = X(x)\,T(t) \not\equiv 0$. Then:

(i) Prove that $X(x)$ satisfies the boundary value problem

$$X'' + \lambda X = 0\,; \quad X'(0) = 0\,, \;\; X'(\pi/3) = 0\,,$$

and find the values of the separation constant $\lambda \geq 0$ yielding non-zero solutions.

(ii) Knowing that the solution $u(x,t)$ can be expressed as

$$u(x,t) \;=\; \sum_{n=0}^{\infty} A_n\, e^{-9n^2 t} \cos\left(3nx\right)\,, \quad \text{with} \;\; A_n \in \mathbb{R}\,,$$

find an approximate value for $u(\pi/6, 1/9)$, using only the first three terms of the series.

NOTE. The following result may be useful.

Given $L>0$ and $m\,, n \in \mathbb{N} \cup \{0\}$, we have $\displaystyle\int_0^L \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x \;=\; \begin{cases} 0\,, & m\neq n\\ L/2\,, & m=n\neq 0\\ L\,, & m=n=0\,. \end{cases}$

SOLUTION:

(i) By applying the separation of variables, we obtain $\dfrac{T'}{T} = \dfrac{X''}{X} = -\lambda$, where λ is the separation constant. Hence $X'' + \lambda X = 0$. In addition, by using the BCs we get

$$\frac{\partial u}{\partial x}(0,t) = X'(0)T(t) = 0 \implies X'(0) = 0\,, \text{ holding } \forall t \text{ and } T(t) \not\equiv 0\,;$$

$$\frac{\partial u}{\partial x}(\pi/3,t) = X'(\pi/3)T(t) = 0 \implies X'(\pi/3) = 0\,, \text{ holding } \forall t \text{ and } T(t) \not\equiv 0\,.$$

Let us now distinguish two cases.

Case 1. $\lambda = 0$

$X'' = 0 \implies X(x) = c_1 x + c_2$ with $c_1\,, c_2 \in \mathbb{R}$. As $X'(x) = c_1$, we have that $X'(0) = 0 = c_1 = X'(\pi/3)$. Hence, if $\boxed{\lambda = 0}$ then the function $X(x) = c_2 \neq 0$ is a nonzero solution of the problem.

Case 2. $\lambda > 0$

Let us take $\lambda = a^2$, with $a > 0$. The corresponding characteristic equation is then $r^2 + a^2 = 0 \implies r = \pm ia\,,\ i \in \mathbb{C}$. Hence

$$X(x) = c_1 \cos(ax) + c_2 \sin(ax)\,; \quad X'(x) = -ac_1 \sin(ax) + ac_2 \cos(ax)\,, \text{ with } c_1\,, c_2 \in \mathbb{R}\,.$$

Applying the BCs we get $X'(0) = 0 \implies c_2 = 0$; $X'(\pi/3) = 0 \implies -ac_1 \sin(a\pi/3) = 0$. Then, imposing $c_1 \neq 0$ yields $\sin(a\pi/3) = 0 \implies a\pi/3 = n\pi \implies a = 3n\,,\ n = 1, 2, 3, \ldots$. Finally, $\boxed{\lambda = (3n)^2 = 9n^2\,,\ \ n = 1, 2, 3, \ldots}$.

(ii) We need to calculate

$$u(\pi/6, 1/9) \approx A_0 + A_1 e^{-1} \cos(\pi/2) + A_2 e^{-4} \cos(\pi) = A_0 - \frac{A_2}{e^4}\,.$$

In order to compute the coefficients A_0 and A_2, let us apply the IC as to get

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos(3nx) = 2x + 1 \equiv f(x)\,.$$

In addition, using the orthogonality condition suggested in the note, we deduce that the coefficients A_n satisfy

$$A_0 = \frac{1}{L}\int_0^L f(x)dx = \frac{3}{\pi}\int_0^{\pi/3} (2x+1)dx = 1 + \pi/3\,,$$

$$(n \geq 1) \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos(3nx)dx = \frac{6}{\pi}\int_0^{\pi/3} (2x+1)\cos(3nx)dx$$

$$\implies A_2 = \frac{6}{\pi}\int_0^{\pi/3} (2x+1)\cos(6x)dx = \frac{1}{\pi}\left[(2x+1)\sin(6x) + \frac{1}{3}\cos(6x)\right]_0^{\pi/3} = 0\,.$$

Hence, the desired approximation is $\boxed{u(\pi/6, 1/9) \approx 1 + \pi/3}$.

Problem 2 Find the solution of the following problem.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \quad x \in (0,1)\,, \quad t > 0$$
$$\frac{\partial u}{\partial x}(0,t) = 0\,, \quad u(1,t) = 0\,, \quad t > 0$$
$$u(x,0) = u_0(x) = \begin{cases} 1, & x < 1/2, \\ 0, & x \geq 1/2. \end{cases}$$

SOLUTION:

Separation of variables, $u = X(x)T(t)$, yields $\frac{T'}{T} = \frac{X''}{X} = -\lambda$, from which we obtain the eigenvalue problem: $X'' + \lambda X = 0$, with boundary conditions $X'(0) = 0$, $X(1) = 0$. The general solution $X = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$ produces $c_2 = 0$ and then $\cos\sqrt{\lambda} = 0$. Therefore,

$$\sqrt{\lambda} = \left(n - \frac{1}{2}\right)\pi \Longrightarrow \lambda_n = \frac{(2n-1)^2\pi^2}{4}, \quad n = 1, 2, \ldots,$$

and the corresponding eigenfunctions are

$$X_n(x) = \cos\frac{(2n-1)\pi x}{2}.$$

We now solve the equation for $T(t)$: $T_n' = -\lambda_n T$, which results in $T_n = e^{-\lambda_n t}$. By using the superposition principle, we obtain the following general form of the solution:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(2n-1)^2\pi^2 t/4} \cos\frac{(2n-1)\pi x}{2}.$$

The initial condition yields:

$$\sum_{n=1}^{\infty} a_n \cos\frac{(2n-1)\pi x}{2} = \begin{cases} 1, & 0 < x < \frac{1}{2}, \\ 0, & \frac{1}{2} < x < 1. \end{cases}$$

The orthogonality condition for this cosine Fourier series yields

$$a_n = 2\int_0^{1/2} \cos\frac{(2n-1)\pi x}{2} dx = \frac{4}{(2n-1)\pi} \sin\frac{(2n-1)\pi x}{2}\bigg|_0^{1/2} = \frac{4}{(2n-1)\pi} \sin\frac{(2n-1)\pi}{4}.$$

Thus the solution is

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\frac{(2n-1)\pi}{4} e^{-(2n-1)^2\pi^2 t/4} \cos\frac{(2n-1)\pi x}{2}.$$

Problem 3 Let $f(x) = x$ be a function on the interval $(-1;,1)$.

1. Calculate the Fourier coefficients of $f(x)$ (*Hint.* Since $f(x)$ is odd, use only sine terms: $f(x) = \sum_{n=1}^{\infty} A_n \sin(\pi n x)$).

2. Is it possible to differentiate the resulting series term by term in order to obtain the Fourier expansion of the derivative $f'(x)$. Why or why not?

3. Since $f(x)$ is odd, $f'(x)$ is even and its Fourier expansion is $f'(x) = B_0 + \sum_{n=1}^{\infty} B_n \cos(\pi n x)$. Express the coefficients B_n of $f'(x)$ through the coefficients A_n of $f(x)$ using the correct formula for the Fourier series of $f'(x)$. Calculate B_n and B_0.

SOLUTION:

1. General formulas for the Fourier series coefficients are:

$$B_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\ dx,$$
$$B_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\pi n x/L)\ dx,$$
$$A_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\pi n x/L)\ dx.$$

In our case, $L = 1$ and only A_n survive; therefore we have:

$$A_n = 2\int_0^1 x \sin(\pi n x)\ dx = 2\left(-\left. \frac{x\cos(\pi n x)}{\pi n} \right|_0^1 + \int_0^1 \frac{\cos(\pi n x)}{\pi n}\ dx \right) = -2\frac{(-1)^n}{\pi n}.$$

Thus the expansion is:

$$f(x) = x = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{\pi n} \sin(\pi n x).$$

2. No, term by term differentiation does not yield the derivative $f'(x)$ because $f(x)$ is not continuous at $x = \pm 1$. Indeed, $f(1) = 1 \neq f(-1) = -1$.

3. In order to obtain correct formula for $f'(x)$, we have to include boundary terms. This can be done in the following way:

$$B_0 = \frac{1}{2L}\int_{-L}^{L} f'(x)\ dx = \frac{1}{2L}\left(f(L) - f(-L)\right) = \frac{1}{2}\left(f(1) - f(-1)\right) = 1,$$
$$B_n = \frac{1}{L}\int_{-L}^{L} f'(x)\cos(\pi n x/L)\ dx = \int_{-1}^{1} f'(x)\cos(\pi n x)\ dx$$
$$= f(x)\cos(\pi n x)|_{-1}^{1} + \pi n \int_{-1}^{1} f(x)\sin(\pi n x)\ dx = (-1)^n(f(1) - f(-1)) + \pi n A_n$$
$$= (-1)^n 2 + \pi n \frac{-2(-1)^n}{\pi n} = 0.$$

Note that the correct result $B_n = \delta_{n0}$ differs from that given by term by term differentiation: $\pi n A_n \neq 2(-1)^n + \pi n A_n$. The difference occurs because $f(x)$ has different values at the ends of interval. Thus the formula for the derivative $f'(x)$ is:

$$f'(x) = 1 \neq -2\sum_{n=1}^{\infty}(-1)^n \cos(\pi n x).$$

Problem 4 Solve the following initial value problem for the heat equation:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ x \in (0,1),\ t > 0,\\
&\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0,\\
&u(x,0) = u_0(x) = \begin{cases} 1,\ x < 1/2,\\ 0,\ x \geq 1/2.\end{cases}
\end{aligned}$$

Hint: The boundary conditions correspond to insulated ends. Therefore a stationary solution is a constant equal to the average value of $u_0(x)$, which is $1/2$. Thus your solution should tend to $1/2$ as t tends to infinity.

SOLUTION:
We will use *method of separation of variables.* Let $u(t,x) = T(t)X(x)$. Then after plugging it into the equation we obtain:

$$T'X = TX'' \Rightarrow \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

since left part depends only of t and right part only on x. For function $X(x)$ we have an eigenvalue problem:

$$\begin{aligned}
&X''(x) + \lambda X(x) = 0,\ x \in (0;1)\\
&X'(0) = X'(1) = 0.
\end{aligned}$$

The solution is:

$$\begin{aligned}
&\lambda_n = (\pi n)^2,\ n \in \mathbb{N} \cup \{0\},\\
&X_0(x) = 1,\\
&X_n(x) = \sqrt{2}\cos(\pi n x),\ x \in \mathbb{N}.
\end{aligned}$$

Thus we have the following ODEs for T_n:

$$T_n' + (\pi n)^2 T_n = 0,\ n \in \mathbb{N} \cup \{0\},$$

with initial condition given by $T_n(0) = \int_0^1 X_n(x)u_0(x)\,dx$ that is:

$$\begin{aligned}
&T_0(0) = \frac{1}{2},\\
&T_n(0) = \int_0^{1/2} \sqrt{2}\cos(\pi n x)\,dx = \frac{\sqrt{2}}{\pi n}\sin(\pi n/2) = \begin{cases} 0,\ n = 2k,\ k \in \mathbb{N},\\ (-1)^{k-1}\dfrac{\sqrt{2}}{\pi(2k-1)},\ n = 2k-1,\ k \in \mathbb{N}.\end{cases}
\end{aligned}$$

Time dependent coefficients T_n now can be written as:

$$T_0(t) = \frac{1}{2},$$
$$T_{2k}(t) = 0, \; k \in \mathbb{N},$$
$$T_{2k-1}(t) = -(-1)^k \frac{\sqrt{2}}{\pi(2k-1)} e^{-(\pi(2k-1))^2 t}, \; k \in \mathbb{N}$$

and the solution of the initial problem is:

$$u(t,x) = \frac{1}{2} - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} e^{-\pi^2(2k-1)^2 t} \cos[(2k-1)\pi x].$$

This function indeed tends to $1/2$ as t tends to infinity.

Problem 5 Find the solution of the following initial boundary value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in (0,1), \quad t > 0,$$
$$\frac{\partial u}{\partial x}(0,t) = \cos(\pi t), \; u(1,t) = 0, \quad t > 0; \; u(x,0) = 0.$$

SOLUTION:
We find a function $U(x,t) = ax + b$ such that

$$\frac{\partial U}{\partial x}(0,t) = a = \cos \pi t, \quad U(1,t) = a + b = 0.$$

We find $a = \cos \pi t$, $b = -\cos \pi t$. Substituting $u(x,t) = (x-1)\cos \pi t + v(x,t)$ in the heat equation, we obtain

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = (x-1)\pi \sin \pi t, \quad \frac{\partial v}{\partial x}(0,t) = 0, \quad v(1,t) = 0, \quad v(x,0) = 1 - x.$$

The corresponding eigenvalues and eigenfunctions solve the problem $X'' + \lambda X = 0$, with boundary conditions $X'(0) = 0$, $X(1) = 0$. The general solution $X = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$ produces $c_2 = 0$ and then $\cos \sqrt{\lambda} = 0$. Therefore,

$$\sqrt{\lambda} = \left(n - \frac{1}{2}\right)\pi \Longrightarrow \lambda_n = \frac{(2n-1)^2\pi^2}{4}, \quad n = 1, 2, \ldots,$$

and the corresponding eigenfunctions are

$$X_n(x) = \cos \frac{(2n-1)\pi x}{2}.$$

We now substitute $v(x,t) = \sum_{n=1}^{\infty} A_n(t) \cos \frac{(2n-1)\pi x}{2}$ in the non-homogeneous heat equation, thereby obtaining:

$$A_n' + \lambda_n A_n = 2\pi \sin \pi t \int_0^1 (x-1) \cos \frac{(2n-1)\pi x}{2} dx = -\frac{8}{(2n-1)^2 \pi} \sin \pi t,$$

with initial condition:

$$A_n(0) = 2\int_0^1 (1-x)\cos\frac{(2n-1)\pi x}{2}dx = \frac{4(1-x)}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2}\Big|_{x=0}^1$$
$$+\frac{4}{(2n-1)\pi}\int_0^1 \sin\frac{(2n-1)\pi x}{2}dx = \frac{8}{(2n-1)^2\pi^2}.$$

To find a particular solution, we insert: $A_n = a_n\cos\pi t + b_n\sin\pi t$ in the equation for A_n thereby getting

$$(\lambda_n + \pi b_n)\cos\pi t + (\lambda_n - \pi a_n)\sin\pi t = -\frac{8}{(2n-1)^2\pi}\sin\pi t \Longrightarrow$$
$$\lambda_n + \pi b_n = 0,\ \lambda_n - \pi a_n = -\frac{8}{(2n-1)^2\pi} \Longrightarrow a_n = \frac{\lambda_n}{\pi} + \frac{8}{(2n-1)^2\pi},\ b_n = -\frac{\lambda_n}{\pi}.$$

Adding a solution of the homogeneous ODE, we find

$$A_n(t) = \left(\frac{(2n-1)^2\pi}{4} + \frac{8}{(2n-1)^2\pi}\right)(\cos\pi t - e^{-(2n-1)^2\pi^2 t/4}) - \frac{(2n-1)^2\pi}{4}\sin\pi t + \frac{8e^{-(2n-1)^2\pi^2 t/4}}{(2n-1)^2\pi^2}.$$

Then

$$u(x,t) = (x-1)\cos\pi t + \sum_{n=1}^{\infty}\left[\left(\frac{(2n-1)^2\pi}{4} + \frac{8}{(2n-1)^2\pi}\right)(\cos\pi t - e^{-(2n-1)^2\pi^2 t/4})\right.$$
$$\left. - \frac{(2n-1)^2\pi}{4}\sin\pi t + \frac{8e^{-(2n-1)^2\pi^2 t/4}}{(2n-1)^2\pi^2}\right]\cos\frac{(2n-1)\pi x}{2}.$$

Problem 6 Solve the following initial boundary value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2,\ \ x \in (0,1)\,,\ \ t > 0,$$
$$\frac{\partial u}{\partial x}(0,t) = 0\,,\ \ u(1,t) = 0\,,\ \ t > 0,$$
$$u(x,0) = 0.$$

SOLUTION:

A particular solution of the boundary value problem

$$U'' = 2, \quad U'(0) = 0, U(1) = 0,$$

is $U(x) = x^2 - 1$. Then $v(x,t) = u(x,t) - U(x)$ satisfies the homogeneous heat equation with homogeneous boundary conditions and different initial condition: $v(x,0) = 1 - x^2$. We find the following eigenvalue problem for $X(x)$ after separating variables by $u(x,t) = X(x)T(t)$:

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(1) = 0.$$

The general solution of the ODE, $X = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$, produces $X'(0) = \sqrt{\lambda}c_2 = 0$, and $X(1) = c_1\sin\sqrt{\lambda} = 0$. Then $\sqrt{\lambda} = n\pi$, $n = 1, 2, \ldots$. The function $T(t)$ satisfies $T' + \lambda T = 0$. Its solution is $T = e^{-\lambda t} = e^{-n^2\pi^2 t}$. Using now the superposition principle, we obtain

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t}\sin(n\pi x) \Longrightarrow v(x,0) = 1 - x^2 = \sum_{n=1}^{\infty} A_n\sin(n\pi x).$$

Then

$$A_n = 2\int_0^1 (1-x^2)\sin(n\pi x)dx = -\frac{2(1-x^2)}{n\pi}\cos(n\pi x)\Big|_0^1 - \frac{4}{n\pi}\int_0^1 x\cos(n\pi x)dx$$
$$= \frac{2}{n\pi} - \frac{4x}{n^2\pi^2}\sin(n\pi x)\Big|_0^1 + \frac{4}{n^2\pi^2}\int_0^1 \sin(n\pi x)dx = \frac{2}{n\pi} + \frac{4}{n^3\pi^3}[1-\cos(n\pi)].$$

We therefore find

$$u(x,t) = (x^2-1) + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(1 + \frac{2[1-(-1)^n]}{n^2\pi^2}\right)\frac{e^{-n^2\pi^2 t}}{n}\sin(n\pi x).$$
