APPLIED DIFFERENTIAL CALCULUS
LECTURE 6: Fourier series and separation of variables: Wave equation.
PROBLEMS

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Problem 1 Consider the following model of wave equation.

$$\begin{aligned}
\text{Partial Diff. Equation} &: \quad \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t)\,, \quad t>0\,, \quad 0<x<\pi\,;\\
\text{Boundary Conditions} &: \quad u(0,t) = 0\,,\ \ u(\pi,t) = 0\,, \quad t \geq 0\,;\\
\text{Initial Conditions} &: \quad \textbf{(i)}\ \ u(x,0) = 5\sin(2x) - 2\sin(5x)\,, \ \ \textbf{(ii)}\ \ \frac{\partial u}{\partial t}(x,0) = 0\,, \quad 0 \leq x \leq \pi\,.
\end{aligned}$$

Using separation of variables plus condition (**ii**), the formal solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(nt)\sin(nx)\;, \quad \text{with } A_n \in \mathbb{R}\,.$$

Find the value $u(\pi/4, \pi/4)$.

Note. It can be useful $\displaystyle\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x = \begin{cases} 0\,, & m \neq n \\ L/2\,, & m = n \end{cases} \qquad (L>0\,;\ m,n\in\mathbb{N})$

Problem 2 Consider the following model of wave equation.

$$\text{Partial Diff. Equation} \quad : \quad \frac{\partial^2 u}{\partial x^2}(x,t) \;=\; \frac{\partial^2 u}{\partial t^2}(x,t)\,, \quad t>0\,, \quad 0<x<\pi\,;$$

$$\text{Boundary Conditions} \quad : \quad u(0,t) \;=\; 0\,,\;\; u(\pi,t) \;=\; 0\,, \quad t \geq 0\,;$$

$$\text{Initial Conditions} \quad : \quad (\mathbf{i})\;\; u(x,0) \;=\; \sum_{k=1}^{4} k^2 \sin(kx)\,, \;\; (\mathbf{ii})\;\; \frac{\partial u}{\partial t}(x,0) \;=\; 0\,, \quad 0 \leq x \leq \pi\,.$$

Using separation of variables and condition (**ii**), the formal solution can be written as

$$u(x,t) \;=\; \sum_{n=1}^{\infty} A_n \cos(nt) \sin(nx)\,, \quad \text{with } A_n \in \mathbb{R}\,.$$

Find the coefficients A_n , $\forall\, n \geq 1$, and express $u(x,t)$ by means of a finite sum.

Problem 3 Find the values of ω for which the following initial boundary value problem for the wave equation has resonances:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \;\; x \in (0,1)\,, \;\; t>0, \\
&u(0,t) = \cos(\omega t)\,, \;\; \frac{\partial u}{\partial x}(1,t) = 0\,, \;\; t>0, \\
&u(x,0) = 0\,, \;\; \frac{\partial u}{\partial t}(x,0) = 0\,, \;\; x \in [0,1]\,.
\end{aligned}$$

Problem 4 Solve the following initial boundary value problem:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - x, \;\; x \in (0,1)\,, \;\; t>0, \\
&u(0,t) = 0\,, \;\; \frac{\partial u}{\partial x}(1,t) = 0\,, \;\; t>0, \\
&u(x,0) = 0\,, \;\; \frac{\partial u}{\partial t}(x,0) = \begin{cases} 1, \;\; x < 1/2, \\ 0, \;\; x \geq 1/2. \end{cases}, \;\; x \in [0,1]\,.
\end{aligned}$$
