APPLIED DIFFERENTIAL CALCULUS
LECTURE 6: Fourier series and separation of variables: Wave equation.
PROBLEMS

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Problem 1 Consider the following model of wave equation.

$$\begin{aligned} \text{Partial Diff. Equation} &: \quad \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t)\,, \quad t>0\,, \quad 0<x<\pi\,; \\ \text{Boundary Conditions} &: \quad u(0,t) = 0\,, \ u(\pi,t) = 0\,, \quad t \geq 0\,; \\ \text{Initial Conditions} &: \quad \textbf{(i)}\ u(x,0) = 5\sin(2x) - 2\sin(5x)\,, \ \textbf{(ii)}\ \frac{\partial u}{\partial t}(x,0) = 0\,, \quad 0 \leq x \leq \pi\,. \end{aligned}$$

Using separation of variables plus condition (**ii**), the formal solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(nt) \sin(nx)\,, \quad \text{with } A_n \in \mathbb{R}\,.$$

Find the value $u(\pi/4, \pi/4)$.

Note. It can be useful $\displaystyle\int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) \mathrm{d}x = \begin{cases} 0\,, & m \neq n \\ L/2\,, & m = n \end{cases} \qquad (L>0\,;\ m,n \in \mathbb{N})$

SOLUTION:
Taking $t = 0$ in the formal solution we get

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx)\,, \quad \text{with } A_n \in \mathbb{R}\,.$$

On the other hand, observing that the initial condition (**i**) $u(x,0) = 5\sin(2x) - 2\sin(5x)$ is a linear combination of functions of the form $\sin(nx)$, with $n = 1, 2, 3, \ldots$, we can obtain A_n by simply equating coefficients of similar terms, namely

$$\sum_{n=1}^{\infty} A_n \sin(nx) = 5\sin(2x) - 2\sin(5x)$$

implies

$$A_1 = 0, \ A_2 = 5, \ A_3 = 0, \ A_4 = 0, \ A_5 = -2; \ A_n = 0 \ \forall \, n > 5 .$$

An alternative way to calculate A_n consists in fixing $m \in \mathbb{N}$ and using the identity suggested in the note of the text in what follows

$$5 \int_0^{\pi} \sin(2x) \sin(mx) \, \mathrm{d}x - 2 \int_0^{\pi} \sin(5x) \sin(mx) \, \mathrm{d}x = \sum_{n=1}^{\infty} A_n \int_0^{\pi} \sin(nx) \sin(mx) \, \mathrm{d}x .$$

Finally, the formal solution of the wave equation can be written as

$$u(x,t) = 5\cos(2t)\sin(2x) - 2\cos(5t)\sin(5x) ,$$

which yields

$$\boxed{u(\pi/4, \pi/4) = 5\cos(\pi/2)\sin(\pi/2) - 2\cos(5\pi/4)\sin(5\pi/4) = -1} .$$

Problem 2 Consider the following model of wave equation.

$$\begin{aligned} \text{Partial Diff. Equation} \ &: \ \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t) , \quad t > 0 , \quad 0 < x < \pi ; \\ \text{Boundary Conditions} \ &: \ u(0,t) = 0 , \ u(\pi,t) = 0 , \quad t \geq 0 ; \\ \text{Initial Conditions} \ &: \ (\mathbf{i}) \ u(x,0) = \sum_{k=1}^{4} k^2 \sin(kx) , \ (\mathbf{ii}) \ \frac{\partial u}{\partial t}(x,0) = 0 , \quad 0 \leq x \leq \pi . \end{aligned}$$

Using separation of variables and condition (**ii**), the formal solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(nt) \sin(nx) , \quad \text{with } A_n \in \mathbb{R} .$$

Find the coefficients A_n , $\forall \, n \geq 1$, and express $u(x,t)$ by means of a finite sum.

SOLUTION:
Taking $t = 0$ in the formal solution we get

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx) , \quad \text{with } A_n \in \mathbb{R} .$$

On the other hand, observing that the initial condition (**i**) $u(x,0) = \sum_{k=1}^{4} k^2 \sin(kx)$ is a linear combination of functions of the form $\sin(nx)$, with $n = 1, 2, \ldots$, we can obtain A_n by simply equating coefficients of similar terms, namely

$$\sum_{n=1}^{\infty} A_n \sin(nx) = \sum_{k=1}^{4} k^2 \sin(kx) = \sin(x) + 4\sin(2x) + 9\sin(3x) + 16\sin(4x)$$

implies

$$A_1 = 1 , \ A_2 = 4 , \ A_3 = 9 , \ A_4 = 16 ; \ A_n = 0 \ \forall \, n \geq 5 .$$

Finally, the solution of the wave equation can be written as the following finite sum

$$\boxed{u(x,t) = \sum_{n=1}^{4} A_n \cos(nt)\sin(nx) = \cos(t)\sin(x) + 4\cos(2t)\sin(2x) + 9\cos(3t)\sin(3x) + 16\cos(4t)\sin(4x)}.$$

Problem 3 Find the values of ω for which the following initial boundary value problem for the wave equation has resonances:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \;\; x \in (0,1)\,, \;\; t > 0, \\
&u(0,t) = \cos(\omega t)\,, \;\; \frac{\partial u}{\partial x}(1,t) = 0\,, \;\; t > 0, \\
&u(x,0) = 0\,, \;\; \frac{\partial u}{\partial t}(x,0) = 0\,, \;\; x \in [0,1]\,.
\end{aligned}$$

SOLUTION:
The wave equation with homogeneous boundary conditions produces the following eigenvalue problem for $X(x)$ after separating variables by $u(x,t) = X(x)T(t)$:

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X'(1) = 0.$$

The general solution of the ODE, $X = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$, produces $X(0) = c_1 = 0$, and $X'(1) = c_2\sqrt{\lambda}\cos\sqrt{\lambda} = 0$. Then $\sqrt{\lambda} = (2n-1)\pi/2$, $n = 1, 2, \ldots$. The function $T(t)$ satisfies $T'' + \lambda T = 0$. Its solutions are sines and cosines of $\sqrt{\lambda}t = (2n-1)\pi t/2$. Then the natural frequencies of the string are $\sqrt{\lambda} = (2n-1)\pi/2$ and the resonant frequencies are

$$\omega = \sqrt{\lambda} = \left(n - \frac{1}{2}\right)\pi, \quad n = 1, 2, \ldots.$$

Problem 4 Solve the following initial boundary value problem:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - x, \;\; x \in (0,1)\,, \;\; t > 0, \\
&u(0,t) = 0\,, \;\; \frac{\partial u}{\partial x}(1,t) = 0\,, \;\; t > 0, \\
&u(x,0) = 0\,, \;\; \frac{\partial u}{\partial t}(x,0) = \begin{cases} 1, \; x < 1/2, \\ 0, \; x \geq 1/2. \end{cases}, \;\; x \in [0,1]\,.
\end{aligned}$$

SOLUTION:
A particular solution of the boundary value problem

$$U'' = x, \quad U(0) = 0, \quad U'(1) = 0,$$

is $U(x) = \frac{x^3}{6} - \frac{x}{2}$. Then $v(x,t) = u(x,t) - U(x)$ satisfies the homogeneous wave equation with homogeneous boundary conditions and different initial conditions:

$$v(x,0) = \frac{x}{2} - \frac{x^3}{6}, \quad \frac{\partial v}{\partial t}(x,0) = \begin{cases} 1, \; x < 1/2, \\ 0, \; x \geq 1/2. \end{cases}, \;\; x \in [0,1]\,.$$

The homogeneous wave equation with homogeneous boundary conditions yields the following eigenvalue problem for $X(x)$ after separating variables by $u(x,t) = X(x)T(t)$:

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X'(1) = 0.$$

The general solution of the ODE, $X = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$, produces $X(0) = c_1 = 0$, and $X'(1) = c_2\sqrt{\lambda}\cos\sqrt{\lambda} = 0$. Then $\sqrt{\lambda} = (2n-1)\pi/2$, and $X_n(x) = \sin[(2n-1)\pi x/2]$, $n = 1, 2, \ldots$. The function $T_n(t)$ satisfies $T_n'' + \lambda_n T_n = 0$. Its solutions are

$$T_n(t) = a_n \cos\frac{(2n-1)\pi t}{2} + b_n \sin\frac{(2n-1)\pi t}{2}.$$

The initial conditions produce

$$a_n = T_n(0) = 2\int_0^1 \left(\frac{x}{2} - \frac{x^3}{6}\right) \sin\frac{(2n-1)\pi x}{2} dx = \frac{32(-1)^{n+1}}{(2n-1)^4\pi^4},$$

$$b_n = \frac{2T_n'(0)}{(2n-1)\pi} = \frac{4}{(2n-1)\pi}\int_0^{1/2} \sin\frac{(2n-1)\pi x}{2} dx = \frac{8}{(2n-1)^2\pi^2}\left(1 - \cos\frac{(2n-1)\pi}{4}\right).$$

The solution is therefore

$$\begin{aligned} u(x,t) &= \frac{x^3}{6} - \frac{x}{2} + \frac{8}{(2n-1)^2\pi^2}\sum_{n=1}^{\infty}\left[\frac{4(-1)^{n+1}}{(2n-1)^2\pi^2}\cos\frac{(2n-1)\pi t}{2}\right. \\ &+ \left.\left(1 - \cos\frac{(2n-1)\pi}{4}\right)\sin\frac{(2n-1)\pi t}{2}\right]\sin\frac{(2n-1)\pi x}{2}. \end{aligned}$$
